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MODEL THEORY OF HILBERT SPACES WITH A DISCRETE GROUP ACTION

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Abstract. In this paper we study expansions of infinite dimensional Hilbert spaces with a unitary representation of a discrete countable group. When the group is finite, we prove the theory of the corresponding expansion, regardless if it is existentially closed, has quantifier elimination, is $\aleph_0$-categorical, $\aleph_0$-stable and SFB. On the other hand, when the group involved is countably infinite, the theory of the Hilbert space expanded by the representation of this group is $\aleph_0$-categorical up to perturbations. Additionally, when the expansion is model complete, we prove that it is $\aleph_0$-stable up to perturbations.

1. Introduction

In this paper we work on model theoretic aspects of the expansion of a Hilbert space by a unitary representation of a countable discrete group. A *unitary representation* of a group G in a Hilbert space H is defined as an action of G on H by elements of the group of unitary maps, denoted by $U(H)$. In other words, a representation is given by a homomorphism $\pi : G \to U(H)$, where the action of $g \in G$ on $v \in H$ is denoted by $\pi(g)v$.

We treat Hilbert spaces as continuous structures in the language $\mathcal{L} = \{0, -, \dot{2}, \frac{x+y}{2}, e^{i\theta} : \theta \in 2\pi\mathbb{Q}\}$, which allows to axiomatize Hilbert spaces as a universal theory (the proof is a small modification of the argument in [3], the proof in [3] deals with real Hilbert spaces where one omits the family $\{e^{i\theta} : \theta \in 2\pi\mathbb{Q}\}$ and for the complex case, one usually only includes $i = e^{i\pi/2}$) and in this language the theory has quantifier elimination. To deal with expansions by a group of unitary maps, we add a unary function symbol for each element g of the group and we interpret it as $\pi(g)$.

There are several papers that deal with similar expansions. For instance, expansions of a Hilbert space with a single automorphism were studied in [8], showing that the existentially closed models correspond to expansions by a unitary map with spectrum S^1. Moreover, it is also proved that the expansion is superstable but not $\aleph_0$-stable. In [9] it is proved that if G is *amenable* and countable, then a Hilbert space H expanded by a countable number of copies of the left regular representation of G is existentially closed. It is also proved that this class of expansions has a model companion which is existentially axiomatizable. Furthermore, when G is countable, this model companion is superstable. The theory of a Hilbert space expanded by a single unitary operator with countable spectrum is treated in [1], where it is proved that the expansion eliminates quantifiers and is $\aleph_0$-stable. Both papers [1, 8] relied heavily on tools from spectral theory like the spectral decomposition theorem.

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In this paper, when the group involved is non abelian, we cannot use the spectral decomposition theorem to describe the action. Instead, we rely on tools from representation theory when dealing with finite groups or tools from C^*-algebras theory when the group is not finite. We denote the theory of a Hilbert space expanded by the unitary representation π of a group G as IHS_π.

We first consider expansions corresponding to actions of finite groups. The main tool we use are basic ideas from representation theory (see [14], which we review in Section 2 below). In this context, existentially closed expansions (that played a crucial role in the literature of similar expansions) can be understood as those with a richer presence of irreducible representations, other representations will have instead irreducible pieces with finite multiplicity. We prove that the theory of any such expansion is $\aleph_0$-categorical and $\aleph_0$-stable. We also define a natural notion of *independence* and prove it coincides with non-forking, which allows us to prove some "geometric" results associated to the theory of the expansion. For example, we show the expansion is *non-multidimensional*. We also show that the associated theory of *Belles Paires* of IHS_π is $\aleph_0$-categorical and thus the theory IHS_π is strongly finitely based (SFB)(see Fact 3.19 and the discussion before for more details).

Then we deal with the case when G is countable infinite. To analyze these expansions, we need to switch to new tools. Instead of using tools from representation theory, we need to consider the C^*-algebra generated by the unitary maps from the representation and use consequences of Voiculescu's theorem (see [11, Theorem II.5.8] and Section 2 below) to prove that the theory IHS_π is $\aleph_0$-categorical up to perturbations, and when the theory IHS_π is model complete, it is $\aleph_0$-stable up to perturbations (see Definition 2.26 below).

This paper is organized as follows. In Section 2 we give some basic tools from representation theory of finite groups, and basic background on operator theory including the notion of spectrum of a unitary operator, some ideas from C^*-algebras like Voicolescu's Theorem, and some model-theoretic applications to perturbations. In Section 3 we consider the case where G is finite, we prove the corresponding expansions $\aleph_0$-categorical and $\aleph_0$-stable and give a natural characterization of non-forking independence and show the theory is SFB. In Section 4, we deal with the case where G is infinite and prove that the theory IHS_π is $\aleph_0$-categorical up to perturbations. Finally, we show that when the theory IHS_π is model complete, then IHS_π is $\aleph_0$-stable up to perturbations.

We will assume the reader is familiar with continuous logic, all background needed can be found in [6, 7], some basic knowledge of perturbations will also be helpful, the corresponding background can be found in [2]. We will assume no prior knowledge of representation theory. The necessary background on this subject and on operator theory will be introduced in Section 2.

2. Background on operator theory and representation theory

In this section, we first review results from representations of finite groups, our main focus is on irreducible representations and projections onto sums of isomorphic irreducible representations; these sums play the role of basic blocks that will help us describe the theory IHS_π. We then introduce some technical tools from operator theory and C^*-algebras concerning Voiculescu's theorem.

We start by introducing the language used to treat a Hilbert space expanded by a unitary group representation as a metric structure.

DEFINITION 2.1. — Let G be a group and let H be a Hilbert space. A *unitary representation* π of G on H is a homomorphism $\pi\colon G \to U(H)$. We define

$$\mathcal{L} = \left\{0, -, \dot{2}, \frac{x+y}{2}, e^{i\theta} : \theta \in 2\pi\mathbb{Q}\right\}$$

to be the language of Hilbert spaces and

$$\mathcal{L}_\pi := \mathcal{L} \cup \{\pi(g) : g \in G\},$$

as the representation language, where each $\pi(g)$ is a unary function with modulus of uniform continuity $\Delta(\varepsilon) = \varepsilon$. We denote the theory of infinite dimensional Hilbert spaces by IHS. The language $\mathcal{L}$ defined above is the one presented in [3] enriched by multiplication by the family of complex scalars $\{e^{i\theta} : \theta \in 2\pi\mathbb{Q}\}$. We denote by $\mathrm{IHS}_\pi := \mathrm{Th}(H, \pi)$ the theory of the infinite-dimensional Hilbert space H expanded by the unitary representation π of G in the language $\mathcal{L}_\pi$. Note that the theory IHS_π includes information like "each $\pi(g)$ is a unitary map" and "for all $g_1, g_2 \in G$, $\pi(g_1 \cdot g_2) = \pi(g_1)\pi(g_2)$ as functions".

2.1. **Representation theory of finite groups on linear groups.** In this subsection we recall some results about representations of finite groups from [14]. These results will be useful to prove that the theory IHS_π is $\aleph_0$-categorical and $\aleph_0$-stable, where π is any unitary representation of a finite group G on an infinite dimensional Hilbert space. In this subsection, G will always stand for a finite group and V for a finite dimensional vector space.

DEFINITION 2.2 ([14, Definition 1.1]). — A *linear representation* of G in V is a homomorphism π from G into $\mathrm{GL}(V)$. When V has dimension n, the representation is said to have *degree* n.

Now, we introduce the *left regular representation* of G. As we will later see, this representation is especially rich respect to other representations.

DEFINITION 2.3 ([14, Example 1.2.b]). — Suppose that V has dimension $|G|$ with basis $\{e_g\}_{g\in G}$ indexed by the elements of G (if we add to V a Hilbert space structure, it is denoted by $\ell_2(G)$). For all $h \in G$, we denote by $\lambda_G(h)$ the linear map sending each e_g to e_{hg}; this defines a linear representation of G, which is called the *left regular representation* of G and it is denoted by λ_G.

DEFINITION 2.4 ([14, Section 1.3]). — Let $\pi : G \to \mathrm{GL}(V)$ be a linear representation and let W be a vector subspace of V. Assume that W is *invariant* under the action of G. Then, the restriction maps $\{\pi(g)\restriction_W\}_{g\in G}$ are automorphisms of W satisfying for all $g_1, g_2 \in G$

$$\pi(g_1 g_2)\restriction_W = \pi(g_1)\restriction_W \pi(g_2)\restriction_W.$$

Thus, $\pi\restriction_W : G \to \mathrm{GL}(W)$ is a linear representation of G on W and it is called a *subrepresentation* of V. Additionally, if W has no non-proper and non-trivial subrepresentation, it is called an *irreducible representation*.

FACT 2.5 ([14, Theorem 2]). — *Every linear representation is a direct sum of irreducible representations.*

Fact 2.6 ([14, Corollary 1, Theorem 4]). — *The number of irreducible representations W_i isomorphic to a given irreducible representation W is independent of the chosen decomposition.*

These last two facts will allow us to understand the theory of a Hilbert space expanded by a unitary representation of a group. An arbitrary unitary representation can be split into a direct sum of its irreducible subrepresentations and we can count the number of times each irreducible representation appears in the sum. We call this number (which can be a natural number or ∞ when dealing with infinite dimensional Hilbert spaces) the *multiplicity* of the given irreducible representation. Part of our work in Section 3 is to show that the multiplicity of each irreducible representation can be recovered from the theory of the expansion.

Remark 2.7. — Let H be a Hilbert space of infinite dimension and let π be a homomorphism from G to $U(H)$. Studying such a representation can always be reduced to the study of representations of G in finite dimensional subspaces. To do this reduction, take some $x \in H$ and consider the finite dimensional subspace generated by $\{\pi(g)x\}_{g\in G}$. In this space we could use the theory for representations of finite dimension and then we wrap these spaces together to understand the action of G all over H.

Fact 2.8 ([14, Corollary 1, Proposition 5]). — *Every irreducible unitary representation W of G is contained in the left regular representation of G with multiplicity equal to its degree. In particular, there are only finitely many irreducible unitary representations of G.*

This fact suggests that if we take the direct sum of countably many left regular representations of G, then the structure obtained should be existentially closed, which indeed is the case even in the larger setting of amenable groups:

Fact 2.9 ([9, Theorem 2.5 and Theorem 2.8]). — *Let S be a countable and amenable group. Then the model $(\infty\ell_2(S), \infty\lambda_S) := \oplus_{n\geqslant 1}(\ell_2(S), \lambda_S)$ (countable copies of the representation $(\ell_2(S), \lambda_S)$) is existentially closed and its theory has quantifier elimination.*

The theory of the model described in Fact 2.9 is the *model companion* of the theory of a Hilbert space expanded by any unitary representation of S. The proof provided in [9] uses Hulanicki's theorem. In this paper, we will give a different proof of this result in Corollary 3.8 when the underlying group G is finite.

Let us return to tools from representation theory. Let T be a linear transformation over V, and let be B a basis of V. If $[a_{\mathrm{ij}}]_B$ is the matrix representation of T in the basis B, then the *trace* $\mathrm{Tr}(T) := \sum_{i=1}^n a_{\mathrm{ii}}$, is independent of the choice of B.

Definition 2.10 ([14, Definition 2.1]). — Let π be a linear representation of G in V. For each $g \in G$, the map $\pi(g)$ is a linear transformation over V and we denote the trace of $\pi(g)$ by $\chi_\pi(g) := \mathrm{Tr}(\pi(g))$. This complex valued function $\chi_\pi : G \to \mathbb{C}$ is called the *character* of π.

Characters are important in representation theory since they determine the irreducible representations. Indeed, two representations having the same characters are isomorphic (see [14, Corollary 2, Theorem 4]), meaning that there is a bijective linear transformation between the representations preserving the action of G. Additionally, in our context, characters play an important role since we will use them (see Fact 2.11 below) for defining projections from a linear representation onto its irreducible representations.

Given the group G, by Fact 2.8 there are finitely many irreducible unitary representations $W_1, \dots, W_k$ (modulo isomorphism) of G. Let $\chi_1, \dots, \chi_k$ be their characters and let $n_1, \dots, n_k$ be their degrees.

Let V be a finite dimensional vector space with a linear representation of G. Write $V = U_1 \oplus \cdots \oplus U_m$ as a decomposition of V into a direct sum of irreducible representations of G. For each $i = 1, \dots, k$ we denote by V_i the direct sum of those irreducible pieces among $U_1, \dots, U_m$ that are isomorphic to W_i. Then, we can write $V = V_1 \oplus \cdots \oplus V_k$, a new decomposition of V into sums of irreducible subrepresentations of V that belong to distinct classes of isomorphism.

Fact 2.11 ([14, Theorem 8]).

(1) *The decomposition $V \cong V_1 \oplus \cdots \oplus V_k$ does not depend on the initially chosen decomposition of V into irreducible representations of G in V.*
(2) *If $1 \leqslant i \leqslant k$, the projection p_i of V onto V_i associated to the decomposition in (1) is given by $p_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g)^* \pi(g)$ (and may be identically 0 when W_i is not represented in π).*

Remark 2.12. — Consider now an infinite dimensional Hilbert space H and a representation π of G in H and consider the expansion of H in the language $\mathcal{L}_\pi$ that includes a symbol for each $\pi(g)$ for $g \in G$. For each $1 \leqslant i \leqslant k$ we let $P_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g)^* \pi(g)$, then the function P_i is definable in $\mathcal{L}_\pi$.

2.2. **Operator theory and C*-algebras.**

Let H be a Hilbert space and let T be a bounded linear operator on H.

Definition 2.13. — The *spectrum* of a linear operator T, denoted by $\sigma(T)$, is defined as the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not bijective}\}.$$

The spectrum can be divided into three different types:

- $\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq 0\}$; if $\lambda \in \sigma_p(T)$ we call λ a *punctual eigenvalue* of T.
- $\sigma_c(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) = 0 \quad \text{and} \quad \overline{\operatorname{Im}(T - \lambda I)} = H\}$; if $\lambda \in \sigma_c(T)$ we call λ an *approximate eigenvalue* of T.
- $\sigma_r(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) = 0 \quad \text{and} \quad \overline{\operatorname{Im}(T - \lambda I)} \neq H\}$; if $\lambda \in \sigma_r(T)$ we call λ a *residual eigenvalue* of T.

The *punctual spectrum* is the collection of punctual eigenvalues, the *continuous spectrum* is the collection of approximate eigenvalues, and the *residual spectrum* is the collection of residual eigenvalues.

Fact 2.14 ([12, Corollary 6.10.11]). — *Let T be a normal operator over a Hilbert space H. Then T has no residual eigenvalues. Thus, the spectrum of a normal operator is divided only into two pieces*

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T).$$

Next, we introduce the concept of a C*-algebra along with the concept of a representation of a C*-algebra. These two are fundamental to understand the key concept that we will use in this paper: approximately unitarily equivalence between algebras.

Definition 2.15 (Basics of [11]). — A *Banach algebra* $\mathcal{A}$ is a complex normed algebra which is complete (as a topological space) and satisfies $\|AB\| \leqslant \|A\|\|B\|$ for all $A, B \in \mathcal{A}$.

Definition 2.16 (Basics of [11]). — A *C*-algebra* $\mathcal{A}$ is Banach $*$-algebra (a Banach algebra with an involutive operation $*$) with the additional condition that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$.

Example 2.17 ([11, Example 1.1]). — The algebra of all bounded operators $B(H)$ on a Hilbert space H is a C*-algebra with the usual operation of adjoint $-^*$. This result follows from the equality:

$$\|A^*A\| = \sup_{\|x\|=\|y\|=1} |\langle A^*Ax, y\rangle| = \sup_{\|x\|=\|y\|=1} |\langle Ax, Ay\rangle| = \|A\|^2.$$

Example 2.18. — The subalgebra of compact operators in $B(H)$ on a Hilbert space H, denoted by $K(H)$, also defines a C*-algebra.

Definition 2.19 (Basics of [11]). — Let $\mathcal{A}$ be a C*-algebra and let H be a Hilbert space. A map $\pi : \mathcal{A} \to B(H)$ is said to be a *$*$-representation* of $\mathcal{A}$ if π is a homomorphism of $*$-algebras which commutes with the involution.

Definition 2.20 ([11, Section II.4]). — Let $\mathcal{A}$ be a C*-algebra and let π_1 and π_2 be representations of $\mathcal{A}$ on Hilbert spaces H_1 and H_2 respectively. The representations π_1 and π_2 are called *Approximately Unitarily Equivalent* (AUE) if there is a sequence of unitary operators $\{\mathcal{O}_k\}_{k\in\mathbb{N}}$ with $\mathcal{O}_k : H_1 \to H_2$ such that

$$\pi_2(A) = \lim_{k\to\infty} \mathcal{O}_k \pi_1(A) \mathcal{O}_k^* \quad \text{for all } A \in \mathcal{A}.$$

where convergence is in the sense of the operator norm topology.

There is a strong connection between two families of operators being approximately unitarily equivalent and the structures (the Hilbert spaces expanded with the C*-algebras) being elementary equivalent.

Remark 2.21. — Let $\mathcal{A}$ be a C*-algebra and let π_1 and π_2 be representations of $\mathcal{A}$ on separable infinite Hilbert spaces H_1 and H_2 respectively. Assume the representations π_1 and π_2 are AUE. Then we have $(H_1, \pi_1) \equiv (H_2, \pi_2)$.

Proof. — Since H_1 and H_2 are separable, we may assume $H_1 = H_2$. Since the representations π_1 and π_2 are AUE there is a sequence $\{\mathcal{O}_k\}_{k\in\mathbb{N}}$ of unitary operators satisfying

$$\pi_2(A) = \lim_{k\to\infty} \mathcal{O}_k \pi_1(A) \mathcal{O}_k^* \quad \text{for all } A \in \mathcal{A}. \tag{2.1}$$

Let $\mathcal{F}$ be a non-principal ultrafilter over $\mathbb{N}$ and consider the ultrapowers

$$\prod_{k,\mathcal{F}} (H_1, \pi_1) \quad \text{and} \quad \prod_{k,\mathcal{F}} (H_2, \pi_2).$$

First define $\Phi\colon \prod_{k,\mathcal{F}} H_1 \to \prod_{k,\mathcal{F}} H_2$ as the map $\Phi([(v_k)_k]_\mathcal{F}) = [(v_k)_k]_\mathcal{F}$ induced by the identification $H_1 = H_2$ as Hilbert spaces. We extend the function Φ to the maps in the representation by defining, for each $A \in \mathcal{A}$, the correspondence $\Phi[(\pi_1(A))_k]_{\mathcal{F}} = [(\mathcal{O}_k \pi_1(A) \mathcal{O}_k^*)_k]_{\mathcal{F}}$. For a fixed index $k \in \mathbb{N}$, the map $\mathcal{O}_k$ is a unitary transformation so the k^{th} component of the map Φ that sends $\pi_1(A)$ to $\mathcal{O}_k \pi_1(A) \mathcal{O}_k^*$ is an isomorphism of representations of $\mathcal{A}$. Moreover, by Equation (2.1), for each $A \in \mathcal{A}$ we have $\pi_2(A) = \lim_{k\to\infty} \mathcal{O}_k \pi_1(A) \mathcal{O}_k^*$, so the representation $\prod_{k,\mathcal{F}}(H_2, \pi_2)$ is isomorphic to $\prod_{k,\mathcal{F}}(H_2, \mathcal{O}_k \pi_1 \mathcal{O}_k^*)$ and thus $\prod_{k,\mathcal{F}}(H_1, \pi_1) \equiv \prod_{k,\mathcal{F}}(H_2, \pi_2)$. □

Voiculescu's Theorem states that two separable C*-algebras, say $\mathcal{A}$ and $\mathcal{B}$, are AUE if there is a completely positive definite map $\Phi\colon \mathcal{A} \to \mathcal{B}$ (see [11, p. 65]) such that $K(H) \cap \mathcal{A} \subseteq \ker(\Phi)$ (see [11, Theorem II.5.3]). This last requirement is problematic in our context. Since we are considering not only the regular representations (which have no compact operators in the generated C*-algebra when the group is infinite), but arbitrary representations of discrete groups, it is likely that they may include compact operators. Thus, requiring the condition $\Phi(K(H) \cap \mathcal{A}) = 0$ is too restrictive. However, there are some consequences of Voiculescu's theorem which include a finer control on the behavior of compact operators under Φ that will work better in our setting.

Definition 2.22 ([11, p. 34]). — Let $\mathcal{A}$ be a C^*-algebra and let (H, π) be a representation of $\mathcal{A}$. We say that π is non-degenerate if $\pi(\mathcal{A})H$ is dense in H.

Fact 2.23 ([11, Lemma II.5.7]). — *Let $\mathcal{A}$ be a C^*-subalgebra of the algebra of compact operators and let π_1 and π_2 non-degenerate representations of $\mathcal{A}$ on separable Hilbert spaces H_1 and H_2 respectively. Then, π_1 and π_2 are unitarily equivalent, i.e. there exists $U : H_1 \to H_2$ unitary such that*

$$\|\pi_2(A) - U\pi_1(A)U^*\| = 0$$

if and only if rank($\pi_1(A)$) =rank($\pi_2(A)$) for all $A \in \mathcal{A}$.

Fact 2.24 ([11, Theorem II.5.8]). — *Let $\mathcal{A}$ a separable C*-algebra, and let π_1 and π_2 non-degenerate representations of $\mathcal{A}$ on separable Hilbert spaces. Then, π_1 and π_2 are AUE, if and only if rank($\pi_1(A)$) =rank($\pi_2(A)$) for all $A \in \mathcal{A}$.*

Model theoretic consequences of the previous fact go back to unpublished work by C. Ward Henson, who pointed out to the first author of the paper that this result characterizes, up to elementary equivalence, expansions of Hilbert spaces with a self adjoint operator (in the form of the Weyl–von Neumann–Berg theorem).

Lemma 2.25. — *Let $\mathcal{A}$ be a separable C*-algebra. Then, every $*$-representation π of $\mathcal{A}$ on a separable Hilbert space H, can be written as $\pi_c \oplus \pi_c^\perp$, where π_c is composed only by compact operators and $\pi_c^\perp$ has no compact operators. Moreover, we can write $H = H_c \oplus H_c^\perp$, and the operators in π_c and $\pi_c^\perp$ act on H_c and $H_c^\perp$ respectively.*

Proof. — It folows from the proof of [11, Corollary II.5.9]. □

We now introduce perturbations in our setting. The reader may want to check [2] for an introduction to the subject. The reader may also want to check [4] for the theory of perturbations applied to expansions of Hilbert spaces with a single unitary map.

Definition 2.26. — Let G be a countable group, let π be a unitary representation of G in an infinite dimensional Hilbert space H, and let $\mathrm{IHS}_\pi = \mathrm{Th}(H, \pi)$. Fix $\{g_n\}_{n\in\mathbb{N}}$ an enumeration of G. Let (H_1, π_1) and (H_2, π_2) be models of IHS_π of the same density character and let $\varepsilon \geqslant 0$. We define an *ε-perturbation* between (H_1, π_1) and (H_2, π_2) to be an isometric isomorphism of Hilbert spaces $U : H_1 \cong H_2$ which also satisfies

$$\sum_{n\geqslant 0} \frac{1}{2^n} \left\| U\pi_1(g_n)U^{-1} - \pi_2(g_n) \right\| \leqslant \varepsilon.$$

The set of all ε-perturbations will be denoted by $\mathrm{Pert}_\varepsilon((H_1, \pi_1), (H_2, \pi_2))$ or simply by $\mathrm{Pert}_\varepsilon(H_1, \pi_1)$ if $(H_2, \pi_2) = (H_1, \pi_1)$.

Assume now that (H_1, π_1) is saturated and strongly homogeneous and let $A \subset H_1$ be small and $p, q \in S_n(A)$. Given $\varepsilon \geqslant 0$, we say that $d_{\mathrm{pert}}(p, q) \leqslant \varepsilon$ if there is a unitary map U on H_1 and realizations $\vec{a} \models p$, $\vec{b} \models q$ in H_1 such that $U \in \mathrm{Pert}_\varepsilon(H_1, \pi_1)$, $U\restriction_A = \mathrm{id}_A$ and $\|U(\vec{a}) - \vec{b}\| \leqslant \varepsilon$. We say IHS_π is $\aleph_0$-stable up to perturbations if for any $A \subset H_1$ separable, the density character of $(S_1(A), d_{\mathrm{pert}})$ is countable.

We say that (H_1, π_1) is $\aleph_0$-saturated over (H_0, π_0) up to perturbations if for any $(H_2, \pi_2) \succeq (H_0, \pi_0)$ with $\dim(H_2 \cap H_1^\perp) \leqslant \aleph_0$ and every $\varepsilon > 0$ there is an *ε-perturbation* between (H_2, π_2) and an elementary substructure of (H_1, π_1) that fixes (H_0, π_0) pointwise.

We say the models (H_1, π_1) and (H_2, π_2) are *approximately isomorphic* if for each $\varepsilon > 0$ there exists an ε-peturbation between them. The theory IHS_π is called *$\aleph_0$-categorical up to perturbations* if each pair of separable models of IHS_π are approximately isomorphic.

Remark 2.27. — Fix a theory IHS_π. Assume that for every separable $(H_0, \pi_0) \models \mathrm{IHS}_\pi$ there is a separable $(H_1, \pi_1) \succeq (H_0, \pi_0)$ which is $\aleph_0$-saturated over (H_0, π_0) up to perturbations. We claim that if this is the case, then IHS_π is $\aleph_0$-stable up to perturbations. Indeed, by Löwenheim–Skolem any separable subset A of a model of IHS_π is a subset of a separable model $(H_0, \pi_0) \models \mathrm{IHS}_\pi$. Then, if $(H_1, \pi_1) \succeq (H_0, \pi_0)$ is separable and $\aleph_0$-saturated over (H_0, π_0), the density character of $(S_1(H_0), d_{\mathrm{pert}})$ is countable and thus so is the case for $(S_1(A), d_{\mathrm{pert}})$.

3. Hilbert spaces expanded by a representation of a finite group G

In this section we will study unitary representations of a finite group G on a separable infinite dimensional Hilbert space H. Let $\mathcal{L}_\pi$ be as in Definition 2.1. Let $\vec{a} \in H^n$, we will write $\mathrm{tp}(\vec{a})$ for the type in the sense of Hilbert spaces and $\mathrm{tp}_\pi(\vec{a})$ for the type in the extended language $\mathcal{L}_\pi$. Similarly, we write $\mathrm{qftp}_\pi(\vec{a})$ for the quantifier free type of the tuple in the extended language.

We will show that for any such representation π, the theory $\mathrm{IHS}_\pi = \mathrm{Th}(H, \pi(g) : g \in G)$ is $\aleph_0$-categorical, has quantifier elimination and is $\aleph_0$-stable.

Since G is finite, by Fact 2.8 there are finitely many irreducible representations of G and all of them are finite dimensional. Let $W_1, \ldots, W_k$ be a list of these representations and, after reorganizing the list if necessary, we may assume there is $m \leqslant k$ such that $W_1, \ldots, W_m$ are the irreducible representations of G having infinitely many copies in $(H, \pi(g) : g \in G)$. Notice that, since H is infinite dimensional and separable, the number m can not be zero, and the number of copies of any W_i

for $1 \leqslant i \leqslant k$ is at most $\aleph_0$. We can also define $H^{\text{fin}} = V_{m+1} \oplus \cdots \oplus V_k$ as the direct sum of the remaining irreducible representations, the ones appearing only finite many times in H, say with multiplicity $d_{m+1}, \ldots, d_k$ respectively. Then, if $m+1 \leqslant i \leqslant k$, we have $V_i \cong W_i^{d_i}$. Notice that k could be equal to m and thus H^{fin} could be the zero subspace. Finally write

$$H \cong \bigoplus_{t=1}^{\infty} W_1 \bigoplus \cdots \bigoplus_{t=1}^{\infty} W_m \bigoplus H^{\text{fin}}$$

Definition 3.1. — For each $m+1 \leqslant i \leqslant k$ we call $V_i = \bigoplus_{t=1}^{d_i} W_i$ a *finite component* of H. On the other hand, for each $1 \leqslant i \leqslant m$ we call $H_i = \bigoplus_{t=1}^{\infty} W_i$ an *infinite component* of H. Finally by a *component* we mean a finite component or an infinite component. Also, by H^{inf} we mean the sum $H_1 \oplus \cdots \oplus H_m$.

Proposition 3.2. — *The projections from H onto each of its components are definable in the theory* IHS_π.

Proof. — Fix $1 \leqslant i_0 \leqslant k$ and take n_{i_0} and $\chi_{i_0}(g)$ as in Fact 2.11 (2). Then (see Remark 2.12) the function $P_{i_0} \colon H \to H$, defined as

$$P_{i_0} v = \frac{n_{i_0}}{|G|} \sum_{g \in G} \chi_{i_0}(g)^* \pi(g) v \tag{3.1}$$

is a definable function in the language $\mathcal{L}_\pi$. Given $v \in H$, we can write v as the sum

$$\sum_{t=1}^{\infty} \sum_{1 \leqslant i \leqslant m} v_i^{(t)} + \sum_{m+1 \leqslant i \leqslant k} v_i \in H^{\text{inf}} \oplus H^{\text{fin}},$$

where for $1 \leqslant i \leqslant m$ the vector $v_i^{(t)}$ denotes the projection of v on the t^{th} copy of W_i in H^{inf} and for $m+1 \leqslant i \leqslant k$ the vector v_i denotes de projection of v on V_i in H^{fin}. We will prove that the equation defined in (3.1) defines the projection onto H_{i_0}. We will do the proof for some infinite component H_{i_0} of H, the proof for a finite component of H is analogous. By definition

$$\begin{aligned} P_{i_0} v &= \frac{n_{i_0}}{|G|} \sum_{g \in G} \chi_{i_0}(g)^* \pi(g) \left(\sum_{t=1}^{\infty} \sum_{1 \leqslant i \leqslant m} v_i^{(t)} + \sum_{m+1 \leqslant i \leqslant k} v_i \right) \\ &= \sum_{t=1}^{\infty} \sum_{1 \leqslant i \leqslant m} \frac{n_{i_0}}{|G|} \sum_{g \in G} \chi_{i_0}(g)^* \pi(g) v_i^{(t)} + \sum_{m+1 \leqslant i \leqslant k} \frac{n_{i_0}}{|G|} \sum_{g \in G} \chi_{i_0}(g)^* \pi(g) v_i. \end{aligned}$$

Notice that we can restrict $\pi(g)$ to each copy of the irreducible subrepresentations W_i of π

$$\pi(g) v_i^{(t)} = \pi(g)\restriction_{W_i^{(t)}} v_i^{(t)} \quad \text{and} \quad \pi(g) v_i = \pi(g)\restriction_{W_i} v_i,$$

then using the projection p_{i_0} given by Fact 2.11(2), we could write $P_{i_0} v$ as

$$P_{i_0} v = \sum_{t=1}^{\infty} \sum_{1 \leqslant i \leqslant m} p_{i_0} v_i + \sum_{m+1 \leqslant i \leqslant k} p_{i_0} v_i = \sum_{t=1}^{\infty} v_{i_0}^{(t)}.$$

We can conclude that the function P_{i_0} is the projection of H onto the infinite component H_{i_0} of H and it is definable. ☐

Theorem 3.3. — *Let π be a unitary representation of G on a Hilbert space H such that $H = H^{\mathrm{inf}}$. Then the theory IHS_π is $\aleph_0$-categorical.*

Proof. — First note that the theory IHS_π includes as part of the information that $(\pi(g) : g \in G)$ is a representation of G by unitary maps.

By Proposition 3.2, for each $1 \leqslant i \leqslant k$ the projection P_i onto H_i is definable in the theory IHS_π. Fix some $1 \leqslant i \leqslant m$, and consider the following scheme indexed by n. To simplify the notation, the indexes t, s will range over $\{1, \ldots, n\}$ and the indexes j_1, j_2 and j will range over $\{1, \ldots, n_i\}$ (recall that n_i is the degree of W_i):

$$\inf_{\bar{v}_t} \max\left\{ \max_{t,j}\left\{ \left| \|v_t^j\| - 1 \right|, \left\| P_i v_t^j - v_t^j \right\| \right\}, \max_{t<s, j_1 \leqslant j_2} \left| \left\langle v_t^{j_1}, v_s^{j_2} \right\rangle \right|, \max_{t, j_1 < j_2} \left| \left\langle v_t^{j_1}, v_t^{j_2} \right\rangle \right| \right\} = 0. \quad (3.2)$$

In the sentence above $\bar{v}_t$ denotes the set of vectors $\{v_t^1, \ldots, v_t^{n_i}\}$. The n^{th} sentence in the Scheme (3.2) indicates that there are n collections of n_i vectors $\{v_t^1, \ldots, v_t^{n_i}\}_{t=1}^n$ which are almost orthonormal and almost invariant under the projection P_i. Observe that this scheme belongs to IHS_π for any $1 \leqslant i \leqslant m$, that is, for any of the irreducible representations W_i of G appearing in H. Additionally, the sentence

$$\sup_v \max_{m+1 \leqslant i \leqslant k} \|P_i(v)\| = 0, \quad (3.3)$$

indicates that the irreducible representation W_i for $i \geqslant m+1$ are not represented in (H, π).

Now let $(K, \rho) \models \mathrm{IHS}_\pi$ be separable. Since K is a model of the theory IHS, we have that for all $n \in \mathbb{N}$ there is $\varepsilon > 0$, such that for $j_1 \leqslant j_2$ and j, and $s < t$ as above if there are vectors in K satisfying

$$\left| 1 - \|v_t^j\|_2 \right| < \varepsilon \quad \text{and} \quad \left| \left\langle v_t^{j_1}, v_s^{j_2} \right\rangle \right| < \varepsilon$$

then (after applying Gram–Schmidt and taking a different family) there is a family of exact witnesses for the property described in the equation. Thus we may assume the set $\{v_t^1, \ldots, v_t^{n_i}\}$ is orthonormal to the set $\{v_s^1, \ldots, v_s^{n_i}\}$ when $t \neq s$. Now, for all $n \in \mathbb{N}$ and each $1 \leqslant i \leqslant m$, we shall prove there are n orthogonal copies of W_i in K. For each t consider the vectors

$$w_t^1 := \frac{P_i v_t^1}{\|P_i v_t^1\|}, \ldots, w_t^{n_i} := \frac{P_i v_t^{n_i}}{\|P_i v_t^{n_i}\|}.$$

It follows from Scheme (3.2), that the set $\{w_t^1, \ldots, w_t^{n_i}\}$ forms a basis for a copy of the irreducible representation W_i. Furthermore if $s < t$ the set $\{w_t^1, \ldots, w_t^{n_i}\}$ is orthogonal to the set $\{w_s^1, \ldots, w_s^{n_i}\}$. Thus, for any irreducible representation W_i appearing in H^{inf}, actually we can find a countable number copies of it in K.

Moreover, by Proposition 3.2 we can build a first order sentence axiomatizing that any vector $v \in K$ (or any other model of IHS_π) can be written as $v = \sum_{1 \leqslant i \leqslant m} P_i v$. Therefore, since both (H, π) and (K, ρ) are separable, the multiplicity of each W_i in K is $\aleph_0$, and K is only composed of copies of W_i for $1 \leqslant i \leqslant m$. Since both K and H contain $\aleph_0$ copies of each irreducible representation $W_1, \ldots, W_m$, and no copy of W_i for $m+1 \leqslant i \leqslant k$, the two representations are isomorphic. □

We can extend Theorem 3.3 to a general unitary representation π of G acting on a separable Hilbert space H:

Corollary 3.4. — *Let π be a representation of G on an infinite dimensional Hilbert space. Then the theory IHS_π is $\aleph_0$-categorical.*

Proof. — Following the notation of Definition 3.1 associated to the model (H,π), we get a new axiomatization by keeping Scheme (3.2) and replacing the sentence (3.3) for a collections of sentences describing the dimension and multiplicity of each irreducible representation W_i in H^{fin}. Fix some $m+1 \leqslant i \leqslant k$, as in Theorem 3.3, to simplify the notation, the indexes s,t will range over $\{1,\dots,d_i\}$, and the index j will range over $\{1,\dots,n_i\}$. Consider the following sentence from IHS_π:

$$\inf_{\bar v_t}\sup_{\|v\|=1}\max\left\{\max_{t,j}\left\{\left|\|v_t^j\|-1\right|,\ \left\|P_iv_t^j-v_t^j\right\|\right\},\ \max_{s<t,j}\left|\left\langle v_t^j,v_s^j\right\rangle\right|,\right.$$
$$\left.\left\|P_iv-\sum_{t=1}^{d_i}\sum_{j=1}^{n_i}\left\langle v_t^j,P_iv\right\rangle v_t^j\right\|\right\}=0. \quad (3.4)$$

In the sentence, $\bar v_t$ denotes the set of vectors $\{v_t^1,\dots,v_t^{n_i}\}$, and recall from the beginning of the section that for each $1+m\leqslant i\leqslant k$ the multiplicity of W_i in H^{fin} is d_i, and n_i is the dimension of W_i. As we shall prove below, the sentence in (3.4) axiomatizes the presence of exactly d_i copies of W_i in H^{fin}.

Let (K,ρ) be a separable model of IHS_π, and let $\varepsilon>0$. We can assume that for each t, the family $\left\{v_t^1,\dots,v_t^{n_i}\right\}$ given by the sentence (3.4) satisfies

$$\left|\|P_iv_t^1\|-1\right|<\varepsilon,\dots,|\|P_iv_t^{n_i}\|-1|<\varepsilon.$$

Furthermore, using that $K\models\mathrm{IHS}$ is $\aleph_0$-saturated (or by choosing ε sufficiently small), there is subspace K_i of K with dimension n_id_i invariant under the action of π with $P_i(K_i)=K_i$. Thus by Fact 2.11 there are at least d_i different copies of W_i in K. Moreover, that is the exact number of copies of W_i in K. Suppose there is one more copy of W_i with vector basis $w_1,\dots,w_{n_i}$, then using sentence (3.4) the projection of each w_j in the orthogonal complement of K_i is arbitrary small and thus it should be zero.

It also follows from the proof of Theorem 3.3 that $\dim(H_i)=\dim(K_i)=\aleph_0$ for all $1\leqslant i\leqslant m$, so we get $\dim(H_i)=\dim(K_i)$ for all $1\leqslant i\leqslant k$ and thus the structures are isomorphic. □

Remark 3.5. — Let (K,ρ) be a model of IHS_π. Let $C\subseteq K$ be a closed subspace. We denote by Pr_C the orthogonal projection from H onto C. Let $\bar a$ and $\bar b$ be tuples of K^n, if for each $1\leqslant i\leqslant k$ we write P_i for the projection on the i^{th} component of K, then

$$\mathrm{qftp}_\pi(\bar a/C)=\mathrm{qftp}_\pi(\bar b/C)\text{ implies }\mathrm{Pr}_C(P_i(a_j))=\mathrm{Pr}_C(P_i(b_j)).$$

Proof. — By Proposition 3.2 each P_i is quantifier free definable, so for each $1\leqslant i\leqslant k$ and each $1\leqslant j\leqslant n$ we have $\mathrm{qftp}_\pi(\bar a/C)=\mathrm{qftp}_\pi(\bar b/C)$ implies $\mathrm{qftp}(P_i(a_j)/C)=\mathrm{qftp}(P_i(b_j)/C)$. It follows from basic results on IHS (see for example [6, Lemma 15.1]) that $\mathrm{qftp}(P_i(a_j)/C)=\mathrm{qftp}(P_i(b_j)/C)$ implies $\mathrm{Pr}_C(P_i(a_j))=\mathrm{Pr}_C(P_i(b_j))$. □

Proposition 3.6. — *Let π be a unitary representation of a finite group G on an infinite dimensional Hilbert space K. Then the theory IHS_π has quantifier elimination.*

Proof. — Let (K,π) be a separable model of IHS_π. We will prove that if $\bar{a}$ and $\bar{b}$ are arbitrary n-tuples of K such that $\mathrm{qftp}_\pi(\bar{a}) = \mathrm{qftp}_\pi(\bar{b})$ then $\mathrm{tp}_\pi(\bar{a}) = \mathrm{tp}_\pi(\bar{b})$. Fix $1 \leqslant i_0 \leqslant k$ and define $K_{i_0} := P_{i_0}(K)$. Additionally, let us assume without loss of generality that $\bar{a}$ and $\bar{b}$ are non-trivial arbitrary n-tuples of K_{i_0} and define

$$A_{i_0} := \bigcup_{g\in G}\{\pi(g)a_1,\dots,\pi(g)a_n\} \quad \text{and} \quad B_{i_0} := \bigcup_{g\in G}\{\pi(g)b_1,\dots,\pi(g)b_n\}.$$

Thus, by assumption $\langle A_{i_0}\rangle$ and $\langle B_{i_0}\rangle$ (the span of the corresponding sets) are non-trivial subspaces of K_{i_0}.

Claim 3.7. — *There is an $\mathcal{L}_\pi$-isomorphism sending $\langle A_{i_0}\rangle$ to $\langle B_{i_0}\rangle$.*

Proof. — Since $\langle A_{i_0}\rangle$ and $\langle B_{i_0}\rangle$ are subspaces of K_{i_0} closed under the action of π, by Fact 2.11 (2) we can decompose these finite dimensional vector spaces as the direct sum of copies of the irreducible representation W_{i_0}. Notice that $\mathrm{qftp}_\pi(\bar{a}) = \mathrm{qftp}_\pi(\bar{b})$ implies

$$\langle \pi(h)a_i, \pi(g)a_j\rangle = \langle \pi(h)b_i, \pi(g)b_j\rangle \quad \text{for all } 1 \leqslant i,j \leqslant n \quad \text{and} \quad g,h \in G, \tag{3.5}$$

and recall that the linear independence of one vector from others is a quantifier free sentence expressible in the language of Hilbert spaces. Then, these inner product equations imply that if we take a subset of A_{i_0} which forms a basis for $\langle A_{i_0}\rangle$ then the corresponding subset in B_{i_0} forms a basis of $\langle B_{i_0}\rangle$, hence $\dim\langle A_{i_0}\rangle = \dim\langle B_{i_0}\rangle$. Then we can write

$$\langle A_{i_0}\rangle \cong U_1^A \oplus \cdots \oplus U_\ell^A \quad \text{and} \quad \langle B_{i_0}\rangle \cong U_1^B \oplus \cdots \oplus U_\ell^B,$$

for some $1 \leqslant \ell$, where each term of the sum is a copy of W_{i_0} where the sum for $\langle B_{i_0}\rangle$ comes from the map sending A_{i_0} to B_{i_0}. In fact, for all $1 \leqslant j \leqslant n$ and all $1 \leqslant \ell' \leqslant \ell$ we have that

$$\left\|\mathrm{Pr}_{U_{\ell'}^A} a_j\right\| = \left\|\mathrm{Pr}_{U_{\ell'}^B} b_j\right\|$$

is implied by $\mathrm{qftp}_\pi(\bar{a}) = \mathrm{qftp}_\pi(\bar{b})$ and the way we constructed the sum. Now, take any of the copies of W_{i_0} in $\langle A_{i_0}\rangle$, namely $U_{\ell'}^A$. As above, if we choose a basis of $U_{\ell'}^A$ from $\langle A_{i_0}\rangle$ its corresponding subset of $\langle B_{i_0}\rangle$ give us a basis for $U_{\ell'}^B$ and the action by the maps $(\pi(g) : g \in G)$ is compatible with this correspondence.

Then the bijection $\langle A_{i_0}\rangle \to \langle B_{i_0}\rangle$ induced by the bijection $A_{i_0} \to B_{i_0}$ is an $\mathcal{L}_\pi$-isomorphism between $\langle A_{i_0}\rangle$ and $\langle B_{i_0}\rangle$. ☐

Notice that, if we start with $\bar{a}$ and $\bar{b}$ as general tuples of K, we can apply, for each $i \leqslant k$, the above construction to obtain a map $f_i : \langle P_i(\bar{a})\rangle \to \langle P_i(\bar{b})\rangle$ that respects the action of π. Putting the f_i together into a single function f one constructs a $\mathcal{L}_\pi$-isomorphism f between the closed subspaces

$$A := \left\langle \bigcup_{g\in G}\{\pi(g)a_1,\dots,\pi(g)a_n\}\right\rangle \quad \text{and} \quad B := \left\langle \bigcup_{g\in G}\{\pi(g)b_1,\dots,\pi(g)b_n\}\right\rangle.$$

sending $\bar{a}$ to $\bar{b}$. Now we extend f to an $\mathcal{L}_\pi$ isomorphism ϕ over K. Observe that we can write $K \cong A \oplus A^\perp$ and $K \cong B \oplus B^\perp$. Moreover, both $A^\perp$ and $B^\perp$ are models of $T_{\pi'}$, where π' is the restriction of π to $A^\perp$, by Corollary 3.4 we have $A^\perp \cong B^\perp$ as $\mathcal{L}_{\pi'}$-structures. Thus, there is a $\mathcal{L}_\pi$-isomorphism $\phi\colon K \to K$ sending $\bar{a}$ to $\bar{b}$. □

Corollary 3.8. — *Consider the representation formed by the sum of countable copies of the left regular representation, denoted by $\infty\lambda_G$, which acts on the separable Hilbert space $\infty\ell_2(G)$. Then, for any unitary representation π, we can embed a separable model (H, π) of IHS_π into $(\infty\ell_2(G), \infty\lambda_G)$ and the structure $(\infty\ell_2(G), \infty\lambda_G)$ is existentially closed.*

Proof. — Let (H, π) be as in the hypothesis. By Fact 2.8 all irreducible representations of G appear in $(\infty\ell_2(G), \infty\lambda_G)$ with infinite multiplicity. In particular, each of the irreducible representations used in (H, π) (with finite or infinite multiplicity) appears in $(\infty\ell_2(G), \infty\lambda_G)$ and thus we can embed the structure (H, π) into $(\infty\ell_2(G), \infty\lambda_G)$.

Now we show the structure $(\infty\ell_2(G), \infty\lambda_G)$ is existentially closed. Assume that $(H, \pi) \geqslant (\infty\ell_2(G), \infty\lambda_G)$ is a separable superstructure. As we mentioned above, we can find a copy $(\infty\ell_2(G)', \infty\lambda'_G)$ extending (H, π) and thus also extending $(\infty\ell_2(G), \infty\lambda_G)$. By quantifier elimination, $(\infty\ell_2(G), \infty\lambda_G) \preccurlyeq (\infty\ell_2(G)', \infty\lambda'_G)$ and thus all existential witnesses in (H, π) have an approximate witness inside $\infty\ell_2(G)$. □

This gives another proof that $\mathrm{Th}(\infty\ell_2(G), \infty\lambda_G)$ is the model companion of the theory of representations of G (compare with [9, Theorem 2.8]).

Theorem 3.9. — *Let π be a unitary representation of a finite group G in an infinite dimensional Hilbert space. Then the theory IHS_π is $\aleph_0$-stable.*

Proof. — Let (K, ρ) be a separable model of IHS_π, by Corollary 3.4 we can decompose $K = K^{\mathrm{inf}} \oplus K^{\mathrm{fin}}$ as in Definition 3.1, then

$$K \cong (K_1 \oplus \cdots \oplus K_m) \oplus (V_{m+1} \oplus \cdots \oplus V_k).$$

To show that the theory IHS_π is $\aleph_0$-stable, it is sufficent to take $E \subseteq K$ countable such that $\overline{E} = K$ and prove that the density character $\|(S_1(E), d)\| \leqslant \aleph_0$. Let us consider $(\widehat{K}, \widehat{\rho}) \succcurlyeq (K, \rho)$ the superstructure defined as

$$\widehat{K} := K \bigoplus W_1^0 \oplus \cdots \oplus W_m^0,$$

where $W_1^0, \ldots, W_m^0$ are copies of the irreducible representations $W_1, \ldots, W_m$ in K^{inf} respectively and $\widehat{\rho}$ is the direct sum of the homomorphism from G to $W_1^0 \oplus \cdots \oplus W_m^0$ with ρ.

Let $(F, \tau) \succcurlyeq (K, \rho)$ be an arbitrary separable elemental superstructure. By $\aleph_0$-categoricity of the theory IHS_π, we can write $F \cong F^{\mathrm{inf}} \oplus F^{\mathrm{fin}}$ and notice that $K^{\mathrm{fin}} = F^{\mathrm{fin}}$, then

$$F \cong F_1 \oplus \cdots \oplus F_m \bigoplus F^{\mathrm{fin}}$$

and each of the spaces F_i with $1 \leqslant i \leqslant m$ has dimensions $\aleph_0$. Now, for each $1 \leqslant i \leqslant k$ let us define $\mathrm{Pr}^i_{\overline{E}} : F \to F_i \cap \overline{E}$, as the projection of F onto $F_i \cap \overline{E}$ defined by $\mathrm{Pr}^i_{\overline{E}} = \mathrm{Pr}_{\overline{E}} P_i$, where $\mathrm{Pr}_{\overline{E}}$ is the orthogonal projection onto the closed subspace $\overline{E}$, and $P_i : F \to F_i$ is the projection on the i^{th} component of F (defined as in Proposition 3.2). Now, take $v \in F$, by Proposition 3.6 the theory IHS_π has quantifier elimination and the type $\mathrm{tp}_\pi(v/\overline{E})$ is determined by the elements $\mathrm{Pr}_{\overline{E}}(v)$

and the types $\{\mathrm{tp}_\pi(P_i v - \mathrm{Pr}^i_{\overline{E}}\, v) : 1 \leqslant i \leqslant m\}$, i.e. the types of elements orthogonal to $\overline{E}$ that lie in the different components of F^{inf}. Since $\overline{E} = K$ and $K^{\mathrm{fin}} = F^{\mathrm{fin}}$, and for all $m+1 \leqslant i \leqslant k$ we have $\mathrm{Pr}^i_{\overline{E}}\, v \in K_i$. Thus we only need to realize in $(\widehat{K}, \widehat{\rho})$ the types $\{\mathrm{tp}_\pi(P_i v - \mathrm{Pr}^i_{\overline{E}}\, v) : 1 \leqslant i \leqslant m\}$ in a space orthogonal to $\overline{E}$.

Fix an index $1 \leqslant i \leqslant m$ and set $w_i := P_i v - \mathrm{Pr}^i_{\overline{E}}\, v$. If $w_i = 0$ there is nothing to prove. Otherwise $\langle \bigcup_{g\in G}\{\tau(g)w_i\}\rangle$ is a copy of W_i. Then there exists an isomorphism

$$f_i \colon \left\langle \bigcup_{g\in G}\{\tau(g)w_i\}\right\rangle \longrightarrow W_i^0 \subseteq \widehat{K},$$

respecting the action of the operators induced by τ and $\widehat{\rho}$. Repeating the argument for all $1 \leqslant i \leqslant m$, we define

$$\widehat{v} := \big(\mathrm{Pr}_{\overline{E}}(v) + (f_1(w_1) + \cdots + f_m(w_m)\big) \in \widehat{K}.$$

Then we have $\mathrm{tp}_\pi(\widehat{v}/\overline{E}) = \mathrm{tp}_\pi(v/\overline{E})$ and the new realization of the type belongs to $(\widehat{K}, \widehat{\rho})$ which is separable, hence $\|(S_1(E), d)\| \leqslant \aleph_0$. □

Now we characterize algebraic closure in models of IHS_π and we will also give a natural description of non-forking. We work in (H, π) a κ-saturated and κ-strongly homogeneous model of IHS_π for some uncountable inaccessible cardinal κ. We say that $A \subset H$ is *small* if $|A| < \kappa$, and if $C \subseteq H$ is a closed subspace, we denote by Pr_C the orthogonal projection of H onto C. Additionally, if $A \subseteq H$ we write $\mathrm{acl}(A)$ and $\mathrm{dcl}(A)$ for the algebraic and definable closures of A in the language $\mathcal{L}_\pi$, respectively, and $\mathrm{cl}(A)$ for the topological closure.

Proposition 3.10. — *Let $A \subset H$ be small. Then*

$$\mathrm{acl}(A) = \mathrm{cl}\big(\langle\{\pi(g)(a) : a \in A, g \in G\} \cup H^{\mathrm{fin}}\rangle\big).$$

Proof.

“$\supseteq$”. — Notice that, for each $m+1 \leqslant i \leqslant k$ the component V_i of H is finite-dimensional, and by Proposition 3.2 the projection $P_i : H \to V_i$ is definable in the theory IHS_π. Then, the unitary ball of V_i is definable over $\emptyset$ and compact and thus algebraic over $\emptyset$. Hence $H^{\mathrm{fin}} \subseteq \mathrm{acl}(A)$. Clearly we also have $\{\pi(g)(a) : a \in A\} \subseteq \mathrm{acl}(A)$ and thus the containment follows.

“$\subseteq$”. — Now, set $E := \mathrm{cl}(\langle\{\pi(a) : a \in A, g \in G\} \cup H^{\mathrm{fin}}\rangle)$ and suppose $v \notin E$, then $\|\mathrm{Pr}_{E^\perp}(v)\| > 0$. By hypothesis $E^\perp \subset H^{\mathrm{inf}}$, so there exists $1 \leqslant j \leqslant m$ such that $\|P_j(\mathrm{Pr}_{E^\perp} v)\| > 0$. Since A is a small subset of H, so is E. Since H_j is large, the subspace $H_j \cap E^\perp$ has infinite dimension. Thus, we can find a sequence $\{v_t^j\}_{t=1}^\infty \subset H_j \cap E^\perp$ of orthogonal vectors with norm $\|P_j(\mathrm{Pr}_{E^\perp} v)\|$, such that

$$\mathrm{tp}\left(v_t^j + \sum_{i=1, i\neq j}^{m} P_i(\mathrm{Pr}_{E^\perp} v) + \mathrm{Pr}_E\, v/A\right) = \mathrm{tp}(v/A) \quad \text{for all} \quad t \geqslant 1.$$

The elements of the sequence $\{v_t^j + \sum_{i=1,i\neq j}^m P_i(\mathrm{Pr}_{E^\perp} v) + \mathrm{Pr}_E\, v\}_{t=1}^\infty$ are at the same positive distance one from the other and thus $v \notin \mathrm{acl}(A)$. □

Observation 3.11. — *It follows from the previous proof that for $A \subset H$ be small we have* $\mathrm{dcl}(A) = \mathrm{cl}(\langle\{\pi(g)(a) : a \in A, g \in G\}\rangle)$.

For the rest of this section, whenever $A \subset H$, we write $\overline{A}$ for the algebraic closure of A in the language $\mathcal{L}_\pi$. To deal with non-forking, we introduce an abstract notion of independence and then show it coincides with non-forking. Our approach follows the argument for a single unitary operator presented in [1]. A similar characterization was used in [9, Section 3] to show that that IHS_π is superstable when G is countable and the expansion is existentially closed. Instead of repeating again all of the steps, we will prove the key steps that make the arguments work.

Definition 3.12. — Let $(H, \pi) \models \mathrm{IHS}_\pi$ be κ-saturated and κ-strongly homogeneous. Let $\vec{a} = (a_1, \ldots, a_n) \in H^n$, and let A, B and $C \subset H$ be small. We say that $\vec{a}$ is *$*$-independent from B over C* if for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant k$ we have $\mathrm{Pr}_{\overline{B \cup C}}(P_i(a_j)) = \mathrm{Pr}_{\overline{C}}(P_i(a_j))$. If $\vec{a}$ is $*$-independent from B over C we write $\vec{a} \mathop{\smile\!\!\!\!\!|}^*_C B$. Additionally, if all finite subsets $\vec{a}$ of A are such that $\vec{a} \mathop{\smile\!\!\!\!\!|}^*_C B$, we say that A is *$*$-independent from B over C* and we write $A \mathop{\smile\!\!\!\!\!|}^*_C B$.

Lemma 3.13. — *Let $C \subseteq H$ be such that $C = \overline{C}$ and let $v \in H$. Then, for all $1 \leqslant i \leqslant k$ we have $P_i(\mathrm{Pr}_C(v)) = \mathrm{Pr}_C(P_i(v))$.*

Proof. — Notice that for all $v \in H$ we can write $v = v_1 + v_2$, where $v_1 \in C$ and $v_2 \in C^\perp$. Let $g \in G$, then for all $c \in C$ we have

$$\pi(g)c \in C \quad \text{and} \quad \langle \pi(g)c, \pi(g)v_2 \rangle = \langle c, v_2 \rangle = 0.$$

Since $\pi(g)$ acts in C as a bijection, we obtain $\pi(g)v_2 \in C^\perp$. Thus,

$$\mathrm{Pr}_C \pi(g)v = \pi(g)v_1 = \pi(g) \mathrm{Pr}_C v,$$

this means that Pr_C and $\pi(g)$ commute for all $g \in G$. Since the projection P_i is a linear combination of the operators $\{\pi(g)\}_{g \in G}$ (see Proposition 3.2), it follows that Pr_C commutes with the projection P_i for all $1 \leqslant i \leqslant k$. □

The previous result will allow us to characterize independence over closed sets without using the projections P_i over the components, just as was done in [9, Section 3]:

Corollary 3.14. — *Let $\vec{a} = (a_1, \ldots, a_n) \in H^n$, and let $B, C \subset H$ be small. Then, $\vec{a} \mathop{\smile\!\!\!\!\!|}^*_C B$ if and only if for each $1 \leqslant j \leqslant n$, we have $\mathrm{Pr}_{\overline{B \cup C}}\, a_j = \mathrm{Pr}_{\overline{C}}\, a_j$.*

Proof. — Suppose that $\vec{a} \mathop{\smile\!\!\!\!\!|}^*_C B$. Then by Lemma 3.13 for all $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant k$ we have

$$\mathrm{Pr}_{\overline{B \cup C}}\, P_i a_j = \mathrm{Pr}_{\overline{C}}\, P_i a_j \quad \text{if and only if} \quad P_i \mathrm{Pr}_{\overline{B \cup C}}\, a_j = P_i \mathrm{Pr}_{\overline{C}}\, a_j.$$

Also, for each $v \in H$ we have that $v = \sum_{i=1}^k P_i v$. Then, for all $1 \leqslant j \leqslant n$, we have the following equivalence

$$P_i \mathrm{Pr}_{\overline{B \cup C}}\, a_j = P_i \mathrm{Pr}_{\overline{C}}\, a_j \quad \text{for each } 1 \leqslant i \leqslant k \text{ if and only if } \mathrm{Pr}_{\overline{B \cup C}}\, a_j = \mathrm{Pr}_{\overline{C}}\, a_j. \quad \square$$

Proposition 3.15 (Triviality). — *Let $\vec{a} = (a_1, \ldots, a_n) \in H^n$ and $\vec{b} = (b_1, \ldots, b_\ell) \in H^\ell$, and let $C \subset H$ be small. Then, $\vec{a} \mathop{\smile\!\!\!\!\!|}^*_C \vec{b}$ if and only if for all $1 \leqslant i \leqslant k$, $1 \leqslant j_1 \leqslant n$ and $1 \leqslant j_2 \leqslant \ell$ we have $P_i(a_{j_1}) \mathop{\smile\!\!\!\!\!|}^*_C P_i(b_{j_2})$.*

Proof. — Assume that $\vec{a} \mathop{\smile\!\!\!\!\!\!|}^*_C \vec{b}$, then for all $1 \leqslant i \leqslant k$ and all $1 \leqslant j_1 \leqslant n$ we have $P_i(\mathrm{Pr}_{\overline{\{b_1,\ldots,b_\ell\}\cup C}}(a_{j_1})) = P_i(\mathrm{Pr}_{\overline{C}}(a_{j_1}))$. Since

$$\overline{C} \subseteq \overline{C \cup \{P_i b_{j_2}\}} \subseteq \overline{C \cup \{b_{j_2}\}} \subseteq \overline{C \cup \{b_1, \ldots, b_\ell\}},$$

we have $P_i(\mathrm{Pr}_{\overline{C\cup\{P_i b_{j_2}\}}}(a_{j_1})) = P_i(\mathrm{Pr}_{\overline{C}}(a_{j_1}))$. By Lemma 3.13, we have $P_i(a_{j_1}) \mathop{\smile\!\!\!\!\!\!|}^*_C P_m(b_{j_2})$.

Now, assume for all $1 \leqslant i \leqslant k, 1 \leqslant j_1 \leqslant n$ and $1 \leqslant j_2 \leqslant \ell$ we have that $P_i a_{j_1} \mathop{\smile\!\!\!\!\!\!|}^*_C P_i b_{j_2}$. Let us write $P_i a_{j_1} = \mathrm{Pr}_{\overline{C}}(P_i a_{j_1}) + \mathrm{Pr}_{\overline{C}^\perp}(P_i a_{j_1})$. By hypothesis the projection of $P_i a_{j_1}$ over $\mathrm{Pr}_{\overline{C}^\perp}(P_i b_{j_2})$ is equal to 0. Then $\mathrm{Pr}_{\overline{C}^\perp}(P_i a_{j_1})$ is orthogonal to $P_i b_{j_2}$. Similarly, for any $g \in G$ we will obtain that $\mathrm{Pr}_{\overline{C}^\perp}(P_i a_{j_1})$ is orthogonal to $\pi(g)(P_i b_{j_2})$. This implies that $\mathrm{Pr}_{\overline{C}^\perp}(P_i a_{j_1}) \perp \overline{C \cup \{P_i b_{j_2}\}}$. The above orthogonality relation holds for each b_{j_2} with $1 \leqslant j_2 \leqslant \ell$, making the projection of $\mathrm{Pr}_{\overline{C}^\perp}(P_i a_{j_1})$ on $\overline{C \cup \{P_i b_1, \ldots, P_i b_\ell\}}$ equal to 0. Then we obtain

$$\mathrm{Pr}_{\overline{C}}(P_i a_{j_1}) = \mathrm{Pr}_{\overline{C\cup\{P_i b_1,\ldots,P_i b_\ell\}}}(P_i a_{j_1}).$$

Observe that the i^{th} component H_i of H is closed under the action of the operators $\{\pi(g)\}_{g\in G}$ and subspace projections. Then

$$\mathrm{Pr}_{\overline{C\cup\{P_i b_1,\ldots,P_i b_\ell\}}}(P_i a_{j_1}) = \mathrm{Pr}_{\overline{C\cup\{b_1,\ldots,b_\ell\}}}(P_i a_{j_1}).$$

Thus, $\mathrm{Pr}_{\overline{C}}(P_i a_{j_1}) = \mathrm{Pr}_{\overline{C\cup\{b_1,\ldots,b_\ell\}}}(P_i a_{j_1})$ for all $1 \leqslant i \leqslant k$ and $1 \leqslant j_1 \leqslant n$. □

THEOREM 3.16. — *Let $(K,\pi) \models \mathrm{IHS}_\pi$ be κ-saturated. Then the notion $*$-independence agrees with non-forking and non-forking is trivial.*

Proof. — It is enough to show that $*$-independence satisfies finite character, local character, transitivity, symmetry, invariance, existence, and stationarity. We check finite character, the other properties can be easily checked using the approach from [1].

Finite character: let $\vec{a} = (a_1, \ldots, a_n) \in K^n$ be a finite tuple, and let $B, C \subset K$ be small. We prove that if $\vec{a} \mathop{\smile\!\!\!\!\!\!|}^*_C B_0$ for all finite $B_0 \subseteq B$ then $\vec{a} \mathop{\smile\!\!\!\!\!\!|}^*_C B$. Note that if $\vec{a} \mathop{\smile\!\!\!\!\!\!|}^*_C B_0$, then $\mathrm{Pr}_{\overline{B_0\cup C}}(a_j) = \mathrm{Pr}_{\overline{C}}(a_j)$ for all $1 \leqslant j \leqslant n$. If this happens for all finite $B_0 \subseteq B$ then $\mathrm{Pr}_{\overline{B\cup C}}(a_j) = \mathrm{Pr}_{\overline{C}}(a_j)$ for all $1 \leqslant j \leqslant n$ as desired.

Finally, triviality of forking follows from the previous result and Proposition 3.15. □

There are some easy applications of our characterization of non-forking, among them:

PROPOSITION 3.17. — *The theory IHS_π is non-multidimensional.*

Proof. — It suffices to prove that any non-algebraic stationary type is not orthogonal to a type over $\emptyset$. Let $a \in H$, let $C \subseteq H$ be small and algebraically closed. Consider $p = \mathrm{tp}(a/C)$ and $q = \mathrm{tp}(a - \mathrm{Pr}_C(a)/\emptyset)$. Then, if p is non-algebraic, we have $a - \mathrm{Pr}_C(a) \neq 0$ and clearly $a - \mathrm{Pr}_C(a) \not\mathop{\smile\!\!\!\!\!\!|}^*_C a$. □

We can classify the models of IHS_π in terms of the density character of the irreducible representations that appear in the model. This gives a classification of models of IHS_π in terms of finitely many cardinals. We will now study more "geometric complexity" aspects of the theory. For this we need:

Definition 3.18. — Let (H,π) be a separable model of IHS_π, and let $K \subseteq H$ be a closed subspace, which is invariant under the action of π and is such that $(H,\pi) \succcurlyeq (K,\pi\restriction_K)$. Let P be the predicate on H that measures the distance to the subspace K. Then, if (H,π) is $\aleph_0$-saturated over $(K,\pi\restriction_K)$ and $(K,\pi\restriction_K)$ is $\aleph_0$-saturated, we call the pair of structures $((H,\pi),P)$ in the language $\mathcal{L}_\pi \cup \{P\}$ a *belle paire*. We write $T_{\pi P}$ for the theory of belles paires of models of IHS_π. Sometimes instead of writing $((H,\pi),P)$ we will abuse notation and write $((H,\pi),K)$ for the same structure.

Belle paires were first defined in first order by Poizat in [13]. There are many applications of belles paires, among them the work of the first named author of the paper with Ben Yaacov and Henson around the notion of the topology of convergence of canonical bases [5]. Recall from [5] that a stable theory is SFB *(strongly finitely based)* if the topology of convergence of canonical bases coincides with the distance topology on the space of types over models. This notion is a reasonable continuous analogue to the notion of 1-basedness for stable first order theories, for more details see [5]. In this paper we will need belle paires for the following result:

Fact 3.19 ([5, Theorem 3.10]). — *Let T be any stable continuous theory. Then T_P is $\aleph_0$-categorical if and only if T is $\aleph_0$-categorical and SFB.*

Our next goal is to show that IHS_π has SFB. We already know IHS_π is $\aleph_0$-categorical and stable, so by Fact 3.19, it remains to prove that $T_{\pi P}$ is $\aleph_0$-categorical. In order to show this, let (H,π), (K,ρ) be separable models of IHS_π, by Corollary 3.4, we have a complete description of these models in terms of the invariant subspaces, so we can write

$$H = H^{\mathrm{inf}} \oplus H^{\mathrm{fin}} \quad \text{where} \quad H^{\mathrm{fin}} = V_{m+1} \oplus \cdots \oplus V_k,$$

and

$$K = K^{\mathrm{inf}} \oplus K^{\mathrm{fin}} \quad \text{where} \quad K^{\mathrm{fin}} = V_{m+1} \oplus \cdots \oplus V_k.$$

Notice that the finite dimensional components of the models are isomorphic as representations of G in finite dimensional Hilbert spaces.

Lemma 3.20. — *Let $(H,\pi) \succcurlyeq (K,\rho)$ and assume $((H,\pi),K)$ is a separable* belle paire *of models of IHS_π. Write $K^\perp$ for the subspace $K^\perp \cap H$ of H, which is invariant under the action of π. Then,*

$$\left(K^\perp, \pi\restriction_{K^\perp}\right) \cong \left(H^{\mathrm{inf}}, \pi\restriction_{H^{\mathrm{inf}}}\right).$$

Proof. — Recall that the predicate $P(v) = \min_{w\in K}\|v-w\|$ measures the distance to K. Then $\mathrm{Pr}_K(v) = \operatorname{argmin} P(v)$ (the projection of v in the subspace K) is definible in the extended language $\mathcal{L}_\pi \cup \{P\}$ (see for example [10, Proposition 2.4] for a proof). It is also easy to see that the distance from a vector v to $K^\perp$ is given by $\sqrt{\|v\|^2 - P(v)^2}$ and so we can quantify over $K^\perp$ and we get that the projection over $K^\perp$ is definable as well. The space $Pr_K(H)^\perp = K^\perp$ only has copies of the irreducible representations of G appearing in H^{inf}. Moreover, since $(H,\pi) \succcurlyeq (K,\rho)$ is an $\aleph_0$-saturated extension, each irreducible representation appearing in H^{inf} also appears in $K^\perp$ and has dimension equal to $\aleph_0$. From this we get the desired isomorphism. □

Remark 3.21. — Notice that since we can quantify over $K^{\perp}$, we can express that $\dim(H_i) \cap K^{\perp} \geqslant \aleph_0$ for $1 \leqslant i \leqslant m$ as a scheme of sentences that belong $T_{\pi P}$.

THEOREM 3.22. — *The theory of* belle paires *of* IHS_π *is* $\aleph_0$*-categorical.*

Proof. — Let $((K_1, \rho), K_2)$ and $((H_1, \pi), H_2)$ be two separable models of the theory $T_{\pi P}$. Then, by Lemma 3.20 and Remark 3.21 we have that $(K_2^{\perp}, \rho\restriction_{K_2^{\perp}})$ and $(H_2^{\perp}, \pi\restriction_{H_2^{\perp}})$ are isomorphic as representations. Additionally, the models $(K_2, \rho\restriction_{K_2})$ and $(H_2, \pi\restriction_{H_2})$ are isomorphic as they are separable models of IHS_π. Then, the expansions $((K_1, \rho), K_2)$ and $((H_1, \pi), H_2)$ are also isomorphic. ☐

COROLLARY 3.23. — *The theory* IHS_π *has* SFB*.*

Proof. — The result follows from Fact 3.19 and Theorem 3.22. ☐

One can change perspective and follow the ideas from [15] and consider actions by compact groups instead of finite groups and generalize results of G-actions to that setting. A natural starting point would be:

QUESTION 3.24. — *Assume G is a compact group. Can one characterize again the existentially closed expansions in terms of the left regular representations? Do irreducible representations play the same role in this setting as they did for finite groups?*

4. Hilbert spaces expanded by a representation of infinite groups

In this section G will denote a discrete infinite countable group and we will fix $\{g_n\}_{n\in\mathbb{N}}$ an enumeration of G. Additionally, H will be an infinite dimensional Hilbert space, and $\pi : G \to U(H)$ will denote a unitary representation of G. In this setting, we first give some examples where the theory IHS_π is either $\aleph_0$-categorical or only $\aleph_0$-categorical up to perturbations. Then, we prove the general result for IHS_π, which states that regardless of the nature of G or π, the theory IHS_π is $\aleph_0$-categorical up to perturbations. Finally, we prove that when we also assume that IHS_π is model complete, then IHS_π is $\aleph_0$-stable up to perturbations.

Example 4.1. — Suppose that $\pi : G \to U(H)$ has finite image. Then, the isomorphism $G/\ker(\pi) \cong \mathrm{Im}(\pi)$, implies that the unitary irreducible representations of π are in correspondence with the irreducible representations of the group $G/\ker(\pi)$. In this case, we can apply the results from the previous section and by Theorem 3.3 the theory IHS_π is $\aleph_0$-categorical.

On the other hand, having nonempty continuous spectrum (see Definition 2.13 and the corresponding notation) in one of the operators belonging to the representation of G over H allows us to construct two separable non-isomorphic models.

PROPOSITION 4.2. — *Let (H_1, π_1) be a separable model of* IHS_π*. Suppose that there is $g \in G$ such that $\sigma(\pi_1(g)) \setminus \sigma_{\mathrm{p}}(\pi_1(g)) \neq \emptyset$. Then the theory* IHS_π *is not* $\aleph_0$*-categorical.*

Proof. — Let $g \in G$ be as in the hypothesis. By Fact 2.14 there is $\lambda \in \sigma_c(\pi(g))$ and thus we can find a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq H$ of normal vectors such that

$$\lim_{n\to\infty} \|\pi_1(g)v_n - \lambda v_n\|_2 = 0.$$

Let $\mathcal{F}$ be a non-principal ultrafilter over $\mathbb{N}$ and define $\mathcal{M} := \Pi_{n,\mathcal{F}}(H_1, \pi_1)$. Then, the element $[(v_n)_n]$ is normal and satisfies

$$\pi^{\mathcal{M}}(g)[(v_n)_n] = [(\pi_1(g)v_n)_n] = \lambda[(v_n)_n].$$

Thus, $\lambda \in \sigma_p(\pi^{\mathcal{M}}(g))$. By Löwenheim–Skolem there exists a separable model (H_2, π_2) of IHS_π where λ is in the punctual spectrum of $\pi_2(g)$. Then the representations (H_1, π_1) and (H_2, π_2) of G are not isomorphic, so the theory is not $\aleph_0$-categorical. □

Modulo perturbations, we get a simpler picture that does not depend on the spectrum of the operators $\pi(g_n)$. We start with a technical lemma:

Lemma 4.3. — *Let (H, π) be a model of IHS_π, and for each $g_n \in G$ define $U_n := \pi(g_n)$. Also, let $\mathcal{A}$ be the C*-algebra generated by $\{U_n\}_{n\in\mathbb{N}}$. Then all the operators in $\mathcal{A}$ are definable in the language $\mathcal{L}_\pi$.*

Proof. — We denote by $\mathcal{A}_0$ the $*$-algebra generated by $\{U_n\}_{n\in\mathbb{N}}$. Since the product of U_k with U_m is in $\{U_n\}_{n\in\mathbb{N}}$, any element $T \in \mathcal{A}_0$ can be expressed as $T = \sum_{i=1}^n \lambda_i U_i$ and thus it is definable in $\mathcal{L}_\pi$. Observe that if we take the topological closure of $\mathcal{A}_0$ in the operator topology, we obtain $\mathcal{A}$. Since the topology in $B(H)$ is the generated by the norm:

$$\|T\| = \sup_{\|x\|\leqslant 1} \|Tx\| \quad \text{where } T \in B(H),$$

if $T \in \mathcal{A}$, then T is the limit of sums of the form $T_m = \sum_{i=1}^m \lambda_i U_i$. Thus, we can write $T = \sum_{i=1}^\infty \lambda_i U_i$ which satisfy for every x, y in the unit ball of H the following

$$\big|\|T_m x - y\| - \|Tx - y\|\big| \leqslant \|Tx - T_m x\| \leqslant \|T - T_m\|.$$

Hence the sequence $\{\|T_m x - y\|\}_{m\in\mathbb{N}}$ converges uniformly to $\|Tx - y\|$ in the unit ball of H. Thus, the function $T : H \to H$ is definable. □

For the next results, recall Definition 2.26 and Remark 2.27.

Theorem 4.4. — *The theory IHS_π is $\aleph_0$-categorical up to perturbations.*

Proof. — Let (H_1, π_1) and (H_2, π_2) be separable models of IHS_π. For each $n \in \mathbb{N}$ define $U_n := \pi_1(g_n)$ and $V_n := \pi_2(g_n)$, also we denote by ϕ the $*$-morphism induced by the assignment $\phi(U_n) = V_n$. Let $\mathcal{A}$ and $\mathcal{B}$ be the C*-algebras generated by $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ respectively, then the extension Φ of ϕ to $\mathcal{A}$ is a representation of $\mathcal{A}$ in $B(H_2)$ with image equal to $\mathcal{B}$. We are dealing with two representations of $\mathcal{A}$, first $\mathrm{id}_{\mathcal{A}}$ the representation that sends each $T \in \mathcal{A}$ to itself, and Φ, the representation induced by ϕ. If we prove that for all $T \in \mathcal{A}$ we have that $\mathrm{rank}(T) = \mathrm{rank}(\Phi(T))$, then by Fact 2.24 the representaions $\mathrm{id}_{\mathcal{A}}$ and Φ are approximately unitarily equivalent.

Recall that in Lemma 4.3 we proved that any $T \in \mathcal{A}$ is definable in the language $\mathcal{L}_\pi$. We will now prove that the rank of the operator T is coded in the theory IHS_π.

Case 1: Suppose rank$(T) = m$. — Then, the sentence

$$\inf_{v_1 \dots v_m} \sup_{v} \max\Big\{ \max_{i \leqslant m} |\|Tv_i\| - 1|,\ \max_{i<j\leqslant m} |\langle Tv_i, Tv_j\rangle|, \\ \big\|Tv - (\mathrm{Pr}_{Tv_1} v + \dots + \mathrm{Pr}_{Tv_m} v)\big\| \Big\} = 0 \quad (4.1)$$

is part of the theory IHS_π, and it axiomatizes $\mathrm{rank}(T) = m$.

Case 2: Suppose rank$(T) = \aleph_0$. — Consider the following scheme indexed by $m \in \mathbb{N}^{>0}$

$$\inf_{v_1 \dots v_m} \max\Big\{ \max_{i \leqslant m} |\|Tv_i\| - 1|,\ \max_{i<j\leqslant m} |\langle Tv_i, Tv_j\rangle| \Big\} = 0. \quad (4.2)$$

Then, the scheme (4.2) is part of the theory IHS_π, and axiomatizes $\mathrm{rank}(T) = \infty$.

The sentence (4.2) and the scheme (4.1) imply that if $T \in \mathcal{A}$ and $m \in \{0, 1, \dots, \aleph_0\}$ is such that $\mathrm{rank}(T) = m$, then $\mathrm{rank}(\Phi(T)) = m$, because $\Phi(T)$ is the interpretation of T in (H_2, π_2) which models IHS_π. Hence $\mathrm{id}_\mathcal{A}$ and Φ are AUE, implying that (H_1, π_1) and (H_2, π_2) are approximately isomorphic, thus the theory IHS_π is $\aleph_0$-categorical up to perturbations. □

Theorem 4.5. — *Assume the theory* IHS_π *is model-complete. Then the theory* IHS_π *is* $\aleph_0$*-stable up to perturbations.*

Proof. — Let (H, π) be a separable model of IHS_π and let $\mathcal{A}$ be the C*-algebra generated by $\{\pi(g_n)\}_{n\in\mathbb{N}}$. By Lemma 2.25 we can write

$$(H, \mathcal{A}) \cong \big(H_c \oplus H_c^\perp, \mathcal{A}_c \oplus \mathcal{A}_c^\perp\big),$$

where the subalgebra $\mathcal{A}_c$ are composed only by compact operators, and the subalgebra $\mathcal{A}_c^\perp$ has no compact operators. Also, $\mathcal{A}_c$ and $\mathcal{A}_c^\perp$ act over H_c and $H_c^\perp$ respectively. The algebra $\mathcal{A}_c$ is the topological closure of the $*$-algebra generated by the family

$$\{\mathrm{Pr}_{H_c} \pi(g_n)\restriction_{H_c}\}_{n\in\mathbb{N}}.$$

In the same way, the algebra $\mathcal{A}_c^\perp$ is the topological closure of the $*$-algebra generated by the family

$$\big\{\mathrm{Pr}_{H_c^\perp} \pi(g_n)\restriction_{H_c^\perp}\big\}_{n\in\mathbb{N}}.$$

Now, if we write $\pi_c^\perp$ for the restriction of π to $H_c^\perp$, we can define the representation $(H_1, \tau) = \bigoplus_{i\in\omega}(H_c^\perp, \pi_c^\perp)$. This representation is a Hilbert space with an action where all operators have rank $\aleph_0$. Finally let $(H \oplus H_1, \pi \oplus \tau)$ be the representation coming from the direct sum.

Claim 1: $(H \oplus H_1, \pi \oplus \tau) \models \mathrm{IHS}_\pi$. — Let $\mathcal{B}$ the C*-algebra generated by the operators $\{\pi \oplus \tau(g_n)\}_{n\in\mathbb{N}}$. We denote by ϕ the $*$-morphism induced by the assignment $\phi(\pi(g_n)) = \pi \oplus \tau(g_n)$. The extension Φ of ϕ to $\mathcal{A}$ obtained by linearity and continuity is a representation of $\mathcal{A}$ in $B(H \oplus H_1)$ whose image is $\mathcal{B}$. In this setting we have again two representations of $\mathcal{A}$, first $\mathrm{id}_\mathcal{A}$ the representation that sends each $T \in \mathcal{A}$ to itself, and Φ. The subalgebra of compact operators of both algebras $\mathcal{A}$ and $\mathcal{B}$, appear in the copy of (H_c, π_c) inside each sum, and by Fact 2.23 they are isomorphic; the non-compact operators, which appear in $(H_c^\perp, \pi_c^\perp)$ and in (H_1, τ) all have rank $\aleph_0$. It follows by Fact 2.24 that the representations (H, π) and $(H \oplus H_1, \pi \oplus \tau)$ are AUE and thus satisfy the same theory IHS_π.

Since IHS_π is model complete, we have

$$(H, \pi) \preccurlyeq (H \oplus H_1, \pi \oplus \tau).$$

We will prove that the elementary superstructure $(H \oplus H_1, \pi \oplus \tau)$ is $\aleph_0$-saturated up to perturbations over (H, π) and thus, since (H, π) was any separable model of IHS_π, this shows that IHS_π is $\aleph_0$-stable up to perturbations.

Let $(K, \rho) \succcurlyeq (H, \pi)$ be an elementary separable superstructure. As in Claim 1, we also have that $(K \oplus H_1, \rho \oplus \tau) \models \mathrm{IHS}_\pi$ and since IHS_π is model-complete, $(K, \rho) \preccurlyeq (K \oplus H_1, \rho \oplus \tau)$. By construction of $(K \oplus H_1, \rho \oplus \tau)$, we can write $K \oplus H_1 \cong H \oplus H^\perp$ and $\rho \oplus \tau \cong \pi \oplus \rho'$, where $\rho' = \rho\restriction_{H^\perp}$. By construction, the C*-algebras induced by the representations τ and ρ' over H_1 and $H^\perp$ respectively, are free of compact operators. We get again using Fact 2.24 that these two representations are approximately unitarily equivalent and so for every $\varepsilon > 0$ there is a unitary map $\mathcal{O}_\varepsilon : H_1 \to H^\perp$ such that $\lim_{\varepsilon \to 0} \|\pi_2(g) - \mathcal{O}_\varepsilon \pi_1(g) \mathcal{O}_\varepsilon^*\| = 0$ for each $g \in G$. Then for all $\varepsilon > 0$ we have the following diagram

$$\begin{array}{ccc} (H \oplus H_1, \pi \oplus \tau) & \xleftarrow{\ \mathrm{id} \oplus \mathcal{O}_\varepsilon^*\ } & \left(H \oplus H^\perp, \pi \oplus \rho'\right) \\ \uparrow \preccurlyeq & & \uparrow \preccurlyeq \\ (H, \pi) & \xrightarrow[\preccurlyeq]{\qquad\qquad} & (K, \rho) \end{array}$$

where the map id is the identity map over H. $\square$

Observation 4.6. — *Let G be a finite group, and let π be a representation of G on an infinite dimensional Hilbert space H. Following the notation of Section 3 we can write $H = H^{\mathrm{fin}} \oplus H^{\mathrm{inf}}$. Using the notation from Theorem 4.5, we have $H_c = H^{\mathrm{fin}}$ and $H_c^\perp = H^{\mathrm{inf}}$.*

Example 4.7. — Let G be a countable amenable group, and consider $\mathrm{Th}(\infty\ell_2(G), \infty\lambda_G)$. By [9, Theorem 2.8] this theory is the model companion of the theory of G-representations and thus it is model complete. By Theorem 4.5 we get that $\mathrm{Th}(\infty\ell_2(G), \infty\lambda_G)$ is $\aleph_0$-stable up to perturbations; it was already known by [9, Section 3] that it is superstable.

The special case where $G = \mathbb{Z}$ was considered in [8] and the model companion was characterized as the collection of expansions $(H, \tau(n) : n \in \mathbb{Z})$ where the spectrum of $\tau(1)$ is S^1. In [4] it was proved that this expansion is $\aleph_0$-stable up to perturbations, a special case of Theorem 4.5.

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