CONFLUENTES
MATHEMATICI

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Tome 15 (2023), p. 137–147.
https://doi.org/10.5802/cml.96

Confluentes Math.
15, (2023) 137-147

MODEL COMPLETENESS FOR FINITELY RAMIFIED HENSELIAN FIELDS VALUED IN A $\mathbb{Z}$-GROUP AND FOR PSEUDO-ALGEBRAICALLY CLOSED FIELDS

JAMSHID DERAKHSHAN AND ANGUS MACINTYRE

Abstract. We prove that the theory of a finitely ramified Henselian valued field of characteristic zero with perfect residue field of positive characteristic whose value group is a $\mathbb{Z}$-group is model-complete in the language of rings if the theory of its residue field is model-complete in the language of rings. This extends results of Ax–Kochen [4], Macintyre [15], Ziegler [22], Basarab [5], and Prestel–Roquette [17].

We also prove that the theory of a perfect pseudo-algebraically closed (PAC) field K such that the absolute Galois group $\mathrm{Gal}(K)$ is pro-cyclic is model-complete in the language of rings if and only if every finite algebraic extension of K is generated by elements that are algebraic over the prime subfield of K.

From these we deduce that every infinite algebraic extension of the field of p-adic numbers $\mathbb{Q}_p$ with finite ramification is model-complete in the language of rings.

Our proofs of model completeness for Henselian fields use only basic model-theoretic and algebraic tools including Cohen's structure theorems for complete local rings and basic results on coarsenings of valuations. These enable us to obtain short proofs of model completeness in the language of rings without any need to extend the ring language.

Our proofs on PAC fields use the model theory of the absoute Galois group dual to the field and elementary invariants given by Cherlin–van den Dries–Macintyre [8] for the theory of PAC fields generalizing Ax's results for the pseudo-finite case [3].

1. Introduction

Model completeness for Henselian valued fields in characteristic zero has been studied for a long time. Ax and Kochen [4, Theorem 15, p. 453] proved model completeness for the theory of p-adically closed fields in a language extending the language of valued fields with a cross section. Ziegler [22] proved relative model completeness for Henselian fields with residue field of characteristic zero. Macintyre [15] proved quantifier elimination for the theory of the field of p-adic numbers $\mathbb{Q}_p$ in the Macintyre language which is the language of rings augmented by predicates for sets of n-powers for all $n \geqslant 1$, which implies model completeness of $\mathbb{Q}_p$ in the ring language.

It is not straightforward to extend this result to a finite extension of $\mathbb{Q}_p$ as in general use of constants gives results weaker than model completeness (e.g consider Presburger arithmetic without a constant for the least positive element 1). The extension to a finite extension of $\mathbb{Q}_p$ was (almost) done in the work of Prestel and Roquette [17, Theorem 5.1, pp. 86] with a slight defect that their language is augmented by constants (relating to p-basis elements) and it is not quite trivial to do without these constants. However, one can dispense with them. We have given a new approach in [9] where we define "finite-by-Presburger" Abelian groups

2020 *Mathematics Subject Classification:* 03C10, 12E30, 12J20, 12L12, 13J15.
Keywords: model theory, model completeness, Henselian valued fields, pseudo-algebraically closed fields.

and prove their model completeness, and deduce model completeness of a finite extension of $\mathbb{Q}_p$ in the ring language. Examples of finite-by-Presburger groups are Presburger arithmetic and the groups of multiplicative congruence classes $K^*/1+\mathcal{M}_K^n$ where $n \geqslant 1$, K is a valued field whose value group is a $\mathbb{Z}$-group, and $\mathcal{M}_K$ is the valuation ideal of K. (These were studied in the case of $\mathbb{Q}_p$ by Hasse).

In the general case of Henselian valued fields with finite ramification Basarab [5, Theorem 2.3.1, p. 195] proved model completeness provided each of the residue rings and the residue field are model-complete.

In this paper we give another approach using basic algebraic and model theoretic results and prove model completeness in the ring language in a general context.

Recall that an ordered Abelian group is called a *$\mathbb{Z}$-group* if it is elementarily equivalent to $\mathbb{Z}$ as an ordered Abelian group. Recall also that the *ramification index* e of a valued field K with valuation v and residue characteristic $p > 0$ is defined to be the cardinality of the set $\{\gamma : 0 < \gamma \leqslant v(p)\}$. If $e < \infty$, the field K is called *finitely ramified*.

If $e = 0$ or $e = 1$, then K is called *unramified*. For example, an extension of $\mathbb{Q}_p$ got by adjoining an eth root of p has ramification index e, whereas an extension of $\mathbb{Q}_p$ got by adjoining roots of unity of order prime to p is unramified. The field obtained by adjoining all roots of unity of order prime to p to $\mathbb{Q}_p$ is in fact the *maximal unramified extension of* $\mathbb{Q}_p$, and is denoted by $\mathbb{Q}_p^{nr}$. Kochen [13] established the basic model theory of this field, including decidability.

Theorem 1.1. — *Let K be a finitely ramified Henselian valued field of characteristic zero with perfect residue field of positive characteristic p. Suppose the value group of K is a $\mathbb{Z}$-group and the theory of the residue field of K is model-complete in the language of rings. Then the theory of K is model-complete in the language of rings.*

This extends the results of Ax and Kochen [4, Theorem 15, p. 453], Basarab [5, Theorem 2.3.1, p. 195], and Prestel and Roquette [17, Theorem 5.1, pp. 86] cited above.

The proof of Theorem 1.1 uses the technique of coarsening a valuation (used by Ax–Kochen [4] and Kochen [13]), the Ax–Kochen–Ershov Theorem (cf. van den Dries' paper [20, Theorem 7.2, p. 143]), and Cohen's structure theorem for complete local fields (cf. Serre's book [19]).

In [13, Theorem 1, p. 409] Kochen proves that any two unramified ω-pseudo-complete Henselian valued fields of cardinality $\aleph_1$ with normalized cross-sections are isomorphic if and only if their residue fields and value groups are isomorphic. We can deduce from Theorem 1.1 the following closely related result.

Corollary 1.2. — *The theory of the maximal unramified extension $\mathbb{Q}_p^{nr}$ of $\mathbb{Q}_p$ is model-complete in the language of rings.*

Recall that a field K is called *pseudo-algebraically closed* if every absolutely irreducible variety defined over K has a K-rational point, see [3] and [10]. Denote the absolute Galois group of a field K by $\mathrm{Gal}(K)$, and write $Abs(K)$ for the field of absolute numbers in K.

Theorem 1.3. — *Let K be a perfect pseudo-algebraically closed field such that $\mathrm{Gal}(K)$ is pro-cyclic. Let k denote the prime subfield of K. Then the theory of K*

in the language of rings is model-complete if and only if

$$K^{alg} = K \otimes_{Abs(K)} k^{alg}, \tag{1.1}$$

that is, every finite algebraic extension of K is generated by elements that are algebraic over k.

In the proof of Theorem 1.3, we use the elementary invariants given by Cherlin–van den Dries–Macintyre [8] for the theory of PAC fields (using a model theory for $\mathrm{Gal}(K)$ dual to that of K) generalizing Ax's work for the pseudo-finite case [3].

We can apply the preceding results to certain infinite algebraic extensions of $\mathbb{Q}_p$.

Theorem 1.4. — *Let K be an infinite algebraic extension of $\mathbb{Q}_p$ with finite ramification. Then the theory of K in the language of rings is model-complete.*

By the Lang–Weil estimates or the theorem of André Weil on the Riemann hypothesis for curves over finite fields, any infinite algebraic extension K of $\mathbb{F}_p$ is pseudo-algebraically closed (see for [10, Corollary 11.2.4]). For such a K, $\mathrm{Gal}(K)$ is pro-cyclic (see [10, Chapter 1]). Thus Theorem 1.4 follows from Theorem 1.3 and Theorem 1.1.

Notes (Added 20 June 2023). —

(1) After we had put this paper on arXiv:1603.08598 on 28 March 2016, a number of related results were proved. Sylvy Anscombe and Franziska Jahnke [2] have proved model completeness for Cohen rings which apply to unramified Henselian valued fields with not necessarily perfect residue field relative to the value group and residue field.

We are grateful to the anonymous referee for suggesting that using this work of Anscombe and Jahnke [2] the proof of Theorem 1.1 in this paper will go through in the case when the residue field is not necessarily perfect. For this one can use the relative embedding [2, Theorem 6.4] instead of the Cohen structure theorem for complete local rings in our proof of Theorem 1.1. Then the proof goes through in the more general context as it stands.

In a very recent preprint [1] Anscombe–Dittman–Jahnke prove relative model completeness results for finitely ramified Henselian fields up to value group and (not necessarily perfect) residue field which they derive from mixed characteristic Ax–Kochen type results.

Originally, our main interest was in the case of infinite algebraic extensions of $\mathbb{Q}_p$, so we had dealt only with perfect residue fields. However, because of the appearance of [1], we now prefer to keep the exposition of this paper restricted to perfect residue fields.

(2) Jochen Koenigsmann has informed us that Corollary 1.2 implies that there is a complete recursive axiomatization of the theory of $\mathbb{Q}_p^{nr}$ (either in the language of rings or in the language of rings expanded by a predicate for the valuation ring). These axioms state that the field is Henselian, the value group is a $\mathbb{Z}$-group, $v(p)$ is a minimal element of the value group, and the residue field is algebraically closed of characteristic p. This result is related to Koenigsmann's ideas for a complete axiomatization for the maximal Abelian extension of $\mathbb{Q}_p$. See Koenigsmann's survey paper in the Proceedings of the International Congress of Mathematicians, Rio 2018 [14].

2. Definability of valuations

Let K be a valued field. We shall denote the valuation on K by v_K or v, the ring of integers of K by $\mathcal{O}_K$, the valuation ideal by $\mathcal{M}_K$, and the value group by Γ_K or Γ. We denote the residue field by k.

Assume throughout that K has characteristic zero and residue characteristic $p > 0$. We take the smallest convex subgroup Δ of Γ_K containing $v(p)$ and consider the quotient Γ_K/Δ with the ordering coming from convexity of Δ (see [18]). K carries a valuation which is the composition of v_K with the canonical surjection

$$\Gamma_K \to \Gamma_K/\Delta.$$

This valuation will be denoted by $\dot{v} : K \to \Gamma_K/\Delta \cup \{\infty\}$ and is called the *coarse valuation corresponding to* v. We denote the valued field $(K, \dot{v})$ by $\dot{K}$.

The valuation ring of $\mathcal{O}_{\dot{K}}$ of $\dot{v}$ is the set

$$\{x \in K : \exists\, \delta \in \Delta\ (v(x) \geqslant \delta)\}.$$

It is also the smallest overring of $\mathcal{O}_K$ in which p becomes a unit, or the localization of $\mathcal{O}_K$ with respect to the multiplicatively closed set $\{p^m : m \in \mathbb{N}\}$.

The maximal ideal $\mathcal{M}_{\dot{K}}$ of $\dot{v}$ is the set

$$\{x \in K : \forall\, \delta \in \Delta\ (v(x) > \delta)\}.$$

Clearly $\mathcal{M}_{\dot{K}} \subseteq \mathcal{M}_K$.

The residue field of K with respect to the coarse valuation $\dot{v}$ has characteristic zero, and is called the *core field* of K corresponding to v. It is denoted by K°.

The core field carries a valuation v_0 defined by

$$v_0(x + \mathcal{M}_{\dot{K}}) = v(x).$$

The valuation v_0 has value group Δ, valuation ring $\mathcal{O}_K/\mathcal{M}_{\dot{K}}$, maximal ideal $\mathcal{M}_K/\mathcal{M}_{\dot{K}}$, and residue field k.

The residue degree of K is defined to be the dimension over $\mathbb{F}_p$ of the residue field k. We note that $(\mathcal{O}_K/\mathcal{M}_{\dot{K}})/(\mathcal{M}_K/\mathcal{M}_{\dot{K}}) \cong \mathcal{O}_K/\mathcal{M}_K \cong k$.

Lemma 2.1. — *The ramification index and residue degree of K and the core field K° are the same.*

Proof. — See [17, pp. 27]. □

We insert here the definition of, and a basic lemma about, ω-pseudo-convergence (to be used this later in the proof of Theorem 1.1).

A sequence $\{a_n\}_{n\in\omega}$ of elements of a valued field is called *ω-pseudo-convergent* if for some integer n_0, we have $v(a_m - a_n) > v(a_n - a_k)$ for all $m > n > k > n_0$. An element $a \in K$ is called a pseudo-limit of the sequence $\{a_n\}$ if for some integer n_0 we have $v(a - a_n) > v(a - a_k)$ for all $n > k > n_0$. The field K is called ω-pseudo-complete if every ω-pseudo-convergent sequence of length ω has a pseudo-limit in the field. We shall use the following lemma.

Lemma 2.2. — *An $\aleph_1$-saturated valued field is ω-pseudo-complete.*

Proof. — Obvious. □

We shall need the following result on existential definability of valuation rings.

Lemma 2.3. — *Let K be a Henselian valued field of characteristic zero, residue characteristic $p > 0$, and ramification index $e > 0$.*

(1) *Let $n > e$ be an integer that is not divisible by p. Then $\mathcal{O}_K$ is existentially definable by the formula $\exists\, y\ (1 + px^n = y^n)$.*
(2) *The maximal ideal $\mathcal{M}_K$ is existentially definable, i.e. $\mathcal{O}_K$ is universally definable.*

Proof. — (1) proved in [6, Lemma 1.5,pp. 4] under the assumption of a finite residue field but the same proof goes through in the more general case as follows. Let $x \in \mathcal{O}_K$. Let $f(y) := y^n - px^n - 1$. Then $v(f(1)) > 2v(f'(1))$, so f has a root in K by Hensel's Lemma. Conversely, suppose $1 + px^n$ is an nth power. If $v(x) < 0$, then $v(px^n) < 0$, and so $v(y) < 0$, hence $nv(y) = e + nv(x)$, thus n divides e, contradiction to the choice of n.

To prove (2) it suffices to show that

$$\mathcal{M}_K = \left\{x \in K : x^e p^{-1} \in \mathcal{O}_K\right\}. \tag{$*$}$$

Indeed, by (1) $\mathcal{O}_K$ is existentially definable, say by an existential $\mathcal{L}_{rings}$-formula $\Phi(x)$, then by $(*)$ $\mathcal{M}_K$ is definable by the existential formula $\Psi(x) := \Phi(x^e p^{-1})$.

To prove $(*)$, suppose that $x \in \mathcal{M}_K$. Then $ev(x) - e \geqslant 0$, thus since $v(p) = e$, we deduce that $x^e p^{-1} \in \mathcal{O}_K$. Conversely, suppose that $x \in K$ satisfies the condition $x^e p^{-1} \in \mathcal{O}_K$. Then $ev(x) - e \geqslant 0$, hence $ev(x) \geqslant e$, so $v(x) \geqslant 1$. □

Note. — The existential definitions above are uniform once one fixes p and a finite bound on the ramification index e. In particular, for any extension K of $\mathbb{Q}_p$ with ramification index e, the valuation ring of K is defined by an existential formula of the language of rings that depends only on p and e, and not K.

Corollary 2.4. — *Suppose that $K_1 \subseteq K_2$ is an extension of Henselian valued fields of characteristic 0 and residue characteristic $p > 0$, and whose value groups are $\mathbb{Z}$-groups. Suppose that the index of ramification of both K_1 and K_2 is e where $0 < e < \infty$. Then*

$$\mathcal{O}_{K_2} \cap K_1 = \mathcal{O}_{K_1}.$$

Proof. — By Lemma 2.3, $\mathcal{O}_{K_1}$ and $\mathcal{O}_{K_2}$ are existentially definable by the same formula, and $\mathcal{M}_{K_1}$ and $\mathcal{M}_{K_2}$ are existentially definable by the same formula. Thus

$$\mathcal{O}_{K_1} \subseteq \mathcal{O}_{K_2} \cap K_1.$$

For the other direction, suppose there is $\beta \in K_1 \cap \mathcal{O}_{K_2}$ such that $\beta \notin \mathcal{O}_{K_1}$. Then $\beta^{-1} \in \mathcal{O}_{K_1}$, so $\beta^{-1} \in \mathcal{O}_{K_2}$. Thus β is a unit in $\mathcal{O}_{K_2}$. From $\beta \notin \mathcal{O}_{K_1}$ we deduce that $\beta^{-1} \in \mathcal{M}_{K_1}$, hence $\beta^{-1} \in \mathcal{M}_{K_2}$, contradiction. □

Remark 2.5. — Corollary 2.4 also follows from Lemma 2.3 (2) together with Prestel's Characterization Theorem [16, p. 1260].

3. Proof of Theorem 1.1

Proof. — We now give the proof of Theorem 1.1. Let $K_1 \to K_2$ be an embedding of models of $Th(K)$. By Corollary 2.4, this is an embedding of valued fields. Thus there is a natural inclusion of the residue field (resp. value group) of K_1 into the residue field (resp. value group) of K_2. We make a series of reductions.

Step 1. — *We may assume that K_1 and K_2 are $\aleph_1$-saturated:* Indeed, let U be any non-principal ultrafilter on a large enough index set. Since K_1^U is an elementary substructure of K_2^U and K_i is an elementary substructure of K_i^U for $i = 1, 2$; we deduce that K_1 is an elementary substructure of K_2.

Step 2. — *It suffices to prove that the core field K_1° is an elementary substructure of the core field K_2°:* since the coarse valued fields $\dot{K}_1$ and $\dot{K}_2$ have characteristic zero residue fields K_1° and K_2° respectively, and divisible torsion-free abelian value groups, and the theory of divisible torsion-free abelian groups is model-complete in the extension of the language of groups by predicates $D_n(x)$ expressing that x is n-divisible, (see Zakon's paper [21]), we can apply the Ax–Kochen–Ershov theorem for equicharacteristic zero Henselian valued fields to deduce that the embedding of K_1 in K_2 is elementary provided the embedding of K_1° into K_2° is elementary.

Step 3. — *We prove the embedding of K_1° into K_2° is elementary.* Since the fields K_1 and K_2 are $\aleph_1$-saturated, by Lemma 2.2 they are ω-pseudo-complete. Thus the valued fields K_1° and K_2° are also ω-pseudo-complete (since the map $\Gamma \to \Gamma/\Delta$ is order-preserving). However, these fields are valued in Δ which is canonically isomorphic to $\mathbb{Z}$. Thus K_1° and K_2° are Cauchy complete. By Lemma 2.1, the ramification index of K_1° and K_2° is the same as the ramification index of K_1 and K_2 which equals the ramification index of K which is e.

By the structure theorem for complete fields with ramification index e (see [19, Theorem 4,pp.37]), K_1° and K_2° are respectively finite extensions of degree e, obtained by adjoining a uniformizing element, of the fields $W(k_1)$ and $W(k_2)$ which are fraction fields of the rings of Witt vectors of k_1 and k_2 respectively, where k_1 and k_2 are the residue fields of K_1° and K_2° (which coincide with the residue fields of (K_1, v_{K_1}) and (K_2, v_{K_2}) respectively).

Thus $K_1^\circ = W(k_1)(\pi)$ for some uniformizing element $\pi \in K_1^\circ$. The element π is the root of a polynomial

$$E(x) := x^e + c_1 x^{e-1} + \cdots + c_e$$

that is Eisenstein over $W(k_1)$. So

$$c_j \in \mathcal{M}_{W(k_1)}$$

for all j and

$$c_e \in \mathcal{M}_{W(k_1)} - \mathcal{M}^2_{W(k_1)}.$$

Claim 3.1. — *$E(x)$ is Eisenstein over $W(k_2)$ and $K_2^\circ = W(k_2)(\pi_1)$.*

Proof. — A coefficient c_j is in the maximal ideal $\mathcal{M}_{W(k_1)}$ if and only if it is divisible by p in the ring $W[k_1]$. Since the ring $W[k_2]$ is a ring extension of $W[k_1]$, if c_j is p-divisible in $W[k_1]$, then it is p-divisible in $W[k_2]$. Similarly if $p^{-1}c_e \in W[k_1]$ is a unit, then it is also a unit in $W[k_2]$.

Since K_1° and K_2° have the same ramification index e, an Eisenstein polynomial over $W[k_1]$ remains Eisenstein over $W[k_2]$, and π_1 remains a uniformizer in K_2°. Thus $W(k_2)(\pi_1)$ has dimension e over $W(k_2)$. But

$$W(k_2) \subseteq W(k_2)(\pi_1) \subseteq K_2^\circ$$

and K_2° is totally ramified with ramification index e over $K_2^\sharp$, hence has dimension e over $W(k_2)$. Therefore $W(k_2)(\pi_1) = K_2^\circ$. $\square$

Claim 3.2. — *The embedding of $W(k_1)$ in $W(k_2)$ is elementary.*

Proof. — This follows from the Ax–Kochen theorem for mixed characteristic Henselian valued fields, see for example van den Dries' paper [20, Theorem 7.2, p. 143]. Alternatively, one can give a direct proof as follows. Since k_1 and k_2 are residue fields of K_1 an K_2 for the valuation v_K, the embedding of k_1 in k_2 is elementary. Since K_1 and K_2 are $\aleph_1$-saturated, the fields k_1 and k_2 are also $\aleph_1$-saturated. Given any finitely many elements $a_1, \ldots, a_m$ from $W(k_1)$, there is an isomorphism from $W(k_1)$ to $W(k_2)$ fixing $a_1, \ldots, a_m$ since elements of $W(k_1)$ and $W(k_2)$ can be represented in the form $\sum_i c_i p^i$, where c_i are from the residue field. The countable subfields of k_1 and k_2 form a back-and-forth system. This induces a back-and-forth system between $W(k_1)$ and $W(k_2)$, and it follows that the embedding of $W(k_1)$ into $W(k_2)$ is elementary $\square$

Step 4. — *We prove that the embedding of K_1° into K_2° is elementary.* This will complete the proof of Theorem 1.1. We interpret $K_i^\sharp(\pi_1)$ inside $K_i^\sharp$ (for $i = 1, 2$) in the usual way as follows. We identify $K_i^\sharp(\pi_1)$ with $(K_i^\sharp)^e$. On the e-tuples we define addition as the usual addition on vector spaces and multiplication by

$$(x_1, \ldots, x_e) \times (y_1, \ldots, y_e) = \left(x_1 I_e + x_2 M_\pi + \cdots + x_e M_\pi^{e-1}\right)(y_1, \ldots, y_e)^T$$

where I_e is the identity $e \times e$-matrix and M_π is the $e \times e$-matrix of multiplication by π. Note that M_π depends uniformly only on the coefficients $c_1, \ldots, c_e$ of $E(x)$. x^2 In Steps 3 and 4 we had $K_1^\circ = K_1^\sharp(\pi_1)$ and $K_2^\circ = K_2^\sharp(\pi_1)$. Step 3 and Claim 3.1 imply that the embedding

$$K_1^\sharp(\pi_1) \to K_2^\sharp(\pi_1)$$

is elementary. Thus $K_1^\circ \to K_2^\circ$ is elementary. The proof is complete. $\square$

4. Model completeness for pseudo-algebraically closed fields and proof of Theorem 1.3

Given a field K, the field of absolute numbers of K is defined by $Abs(K) := k^{alg} \cap K$, where k is the prime subfield of K. By a result of Ax [3], two perfect pseudo-algebraically closed fields K_1 and K_2 whose absolute Galois groups are isomorphic to $\widehat{\mathbb{Z}}$ are elementarily equivalent if and only if $Abs(K_1) = Abs(K_2)$. In other words, the theory of a such a field is determined by its absolute numbers $Abs(K)$ (equivalently by the polynomials $f \in k[x]$ that are solvable in K).

Elementary invariants for pseudo-algebraically closed fields were given by Cherlin–van den Dries–Macintyre in [8] (see also [7]) in terms of the language for profinite groups. In this case one has to preserve the degree of imperfection and the co-elementary theory defined as follows.

The language CSIS for complete stratified inverse systems is a language with infinitely many sorts indexed by $\mathbb{N}$, each sort is equipped with the group operation. The n^{th} sort describes properties for the set of groups in the inverse system which have cardinality n. The language has in addition symbols for the connecting canonical maps between the groups in different sorts.

Given any profinite group G, the set of finite quotients of G with the canonical maps between them is a stratified inverse system. A coformula is a formula of the language CSIS. A profinite group cosatisfies a cosentence if the associated stratified inverse system satisfies the cosentence. A cosentence or coformula has a translation to the language of fields. For details see [8].

For any field K, the Galois diagram of K is defined to be the theory

$$\big\{\exists\, \bar{x}, \bar{y}, \bar{z}, \bar{t}\ \ \big(\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{t}) \wedge \delta(\bar{x}, \bar{y}, \bar{z}, \bar{t})\big) : \\ \exists\, \bar{a}, \bar{b}, \bar{c}, \bar{d} \in Abs(K)\ \ \big(K \models \varphi'\left(\bar{a}, \bar{b}, \bar{c}\right) \wedge \delta(\bar{a}, \bar{b}, \bar{c}, \bar{d})\big)\big\}$$

where $\delta(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ describes the isomorphism type of the field generated by $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, and φ is a coformula and φ' its "translation" into the language of rings (cf. [8]).

The theorem of Cherlin–ven den Dries–Macintyre extending Ax's result is the following.

Theorem 4.1 ([8, Theorem 6, p. 53]). — *Two pseudo-algebraically closed fields K and L are elementarily equivalent if and only if K and L have the same characteristic and same degree of imperfection, and $\Delta(K) = \Delta(K)$.*

To prove Theorem 1.3 we also need the following results.

Theorem 4.2 ([10]). — *Infinite finitely generated fields are Hilbertian.*

Proof. — See [10, Theorem 13.4.2, pp. 242]. □

Theorem 4.3 ((Jarden)). — *If L is a countable Hilbertian field, then the set*

$$\{\sigma \in Gal(L) : Fix(\sigma) \text{ is pseudofinite}\}$$

has measure 1.

Proof. — See [11, pp. 76] or [10, Theorem 18.6.1,pp. 380]. □

Proof of Theorem 1.3. The condition 1.1 implies that every finite algebraic extension $K(\alpha)$ of K is generated by elements algebraic over k, and thus by the primitive element theorem, by a single algebraic element α.

Now all this is part of the theory $Th(K)$. For example, the unique extension of K of dimension n is generated by a root α of some polynomial f over k. Fix the minimum polynomial f of α. Then we just say that some root of f generates the unique extension of K of dimension n. This will be true for any L with $L \equiv K$. It follows that any embedding $L \to K_1$ of models of $Th(K)$ is regular, and thus elementary (cf. Cherlin–van den Dries–Macintyre [8] or Jarden–Kiehne [12]).

Conversely, Suppose that 1.1 does not hold. We shall prove that $Th(K)$ is not model-complete. Since $K^{alg} \neq K \otimes_{Abs(K)} k^{alg}$, there is some finite algebraic extension $K(\alpha)$ that is not included in any $K(\beta)$, where β is algebraic over k.

Now consider such a field $K(\alpha)$ of minimal dimension d over K. $K(\alpha)$ is normal cyclic over K, so d is a prime p, otherwise, $d = p_1^{k_1} \dots p_r^{k_r}$, where $n > 1$, and each of the degree $p_j^{k_j}$ extensions is included in some $K(\beta)$ that is algebraic over k, and so $K(\alpha)$ is too, contradiction. Thus $K(\alpha)$ is a dimension p extension of K.

Now let f be the minimum polynomial of α over K. Put

$$\Lambda{:=}Diag(K) \cup \Sigma_{PAC} \cup \{\exists\, x\ (f(x) = 0)\} \\ \cup \{\forall\, x\ (g(x) \neq 0), g \in \Theta\} \cup \Delta(K)$$

were Σ_{PAC} denotes the set of sentences expressing the condition of being pseudo-algebraically closed, Θ is the set of polynomials in one variable over k which are unsolvable in K, and $\Delta(K)$ is the Galois diagram of K.

Claim 4.4. — Λ *is consistent.*

Proof. — We do a compactness argument. Consider a finite subset Λ_0 of Λ. It involves a finite set $c_0, \ldots, c_m$ from K including coefficients of f and a finite part of $Diag(K)$, finitely many $g_1, \ldots, g_l$ from Θ, a t with $f(t) = 0$, and a finite part of the Galois diagram $\Delta(K)$. Given a finite part S of the Galois diagram $\Delta(K)$, S contains finitely many statements describing the isomorphism types of fields generated by finitely many finite subsets $S_1, \ldots, S_k$ of K, and translations to the language of rings of finitely many coformulas. The translations of the coformulas involve Galois groups of finitely many finite extensions of K. The compositum of these is a finite Galois extension $K(T)$ of K, for a finite set T.

Note that $tr.deg.(k(\alpha, c_0, \ldots, c_m, S_0, \ldots, S_k, T)) \geqslant 1$, so by Theorem 4.2,

$$k' := k\,(\alpha, c_0, \ldots, c_m, S_0, \ldots, S_k, T)$$

is Hilbertian. f is irreducible of dimension p over k'. Now if we adjoin to k' a root α of f, then none of $g_1, \ldots, g_l$ get a root. For if one does, that root is either in k' which is impossible, or has dimension congruent to zero modulo p over k', and then $\alpha \in K(\beta)$, for some β which is algebraic over k.

We apply Theorem 4.3 to k' and deduce that the set of all $\sigma \in G(k')$ such that $Fix(\sigma)$ is pseudofinite has measure 1. Note that given a polynomial $g(x)$ over k, the set

$$G_g := \{\sigma \in G(k') : Fix(\sigma) \text{ does not contain a root of } g\}$$

is open in $G(k')$ since $U := Gal(k'^{alg}/F)$ is a basic open set containing the identity in $G(k')$ where F is the splitting field of $g(x)$, and $\sigma U \in G_g$ for any $\sigma \in G_g$. Thus the set of all $\sigma \in G(k')$ such that $g_1, \ldots, g_l$ do not have a root in $Fix(\sigma)$ and $Fix(\sigma)$ is pseudofinite has non-zero measure.

Note that for any such σ, the fixed field $Fix(\sigma)$ contains the given finite part of $Diag(K)$, contains a root of f (namely α), and contains T. Thus $Fix(\sigma)$ must satisfy the finitely many given statements from the Galois diagram $\Delta(K)$ and the diagram $Diag(K)$ as these can be witnessed by finitely many elements from

$$S_1 \cup \cdots \cup S_k \cup T$$

(by adding constants symbols). We deduce that $Fix(\sigma)$ is a model of Λ_0. Thus Λ has a model L. □

We need to show that $\Delta(K) = \Delta(L)$. It is obvious that $\Delta(K) \subseteq \Delta(L)$. We show that $\Delta(L) \subseteq \Delta(K)$. Suppose that $\psi \in \Delta(L)$ and $\psi \notin \Delta(K)$. Then ψ involves statements on isomorphism type and translations of coformulas corresponding to a finite subset of $Abs(L)$ that does not hold for K. But $Abs(K) = Abs(L)$, so $\neg\psi$ holds for the finitely many elements of $Abs(K)$, hence $\neg\psi \in \Delta(L)$ by adding constants for the distinguished elements of $Abs(K) = Abs(L)$, which is a contradiction.

Applying Theorem 4.1, we deduce that K and L are elementarily equivalent. Clearly K is not an elementary submodel of L. This completes the proof. □

Acknowledgements

Angus Macintyre was supported by a Leverhulme Emeritus Fellowship. We are very grateful to the Referee for several valuable comments and remarks. The first author would like to thank Sylvy Anscombe and Jochen Koenigsmann for many interesting discussions.

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Manuscript received 5th May 2022,
revised 21st June 2023,
accepted 11th September 2023.

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