

CONFLUENTES MATHEMATICI

Johannes WITZIG

Abstract Excision and ℓ^1 -Homology

Tome 15 (2023), p. 107–136.

<https://doi.org/10.5802/cml.95>

© Les auteurs, 2023.

Certains droits réservés.



Cet article est mis à disposition selon les termes de la Licence Internationale d'Attribution Creative Commons BY-NC-ND 4.0

<http://creativecommons.org/licenses/by-nc-nd/4.0/>



*Confluentes Mathematici est membre du
Centre Mersenne pour l'édition scientifique ouverte*
<http://www.centre-mersenne.org/>
e-ISSN : 1793-7434

ABSTRACT EXCISION AND ℓ^1 -HOMOLOGY

JOHANNES WITZIG

Abstract. We use the abstract setting of excisive functors in the language of ∞ -categories to show that the best approximation to the ℓ^1 -homology functor by an excisive functor is trivial.

Then we make an effort to explain the used language on a conceptual level for those who do not feel at home with ∞ -categories, prove that the singular chain complex functor is indeed excisive in the abstract sense, and show how the latter leads to classical excision statements in the form of Mayer–Vietoris sequences.

0. INTRODUCTION

In order to efficiently compute the value of a (generalized) homology theory on a topological space, a divide and conquer approach is usually most convenient: Given an open cover $X = U \cup V$ of a space X , we want to infer the homology of the latter from the homologies of U , V and $U \cap V$. The existence of a link between these objects is guaranteed by the *Mayer–Vietoris sequence*, which in turn is a formal consequence of the Eilenberg–Steenrod axioms [40, Sec. 10.7], in particular the *Excision Axiom*. For example, singular homology fulfills the latter, which is the content of the classical *Excision Theorem* [40, Sec. 9.4], [16, Thm. 2.20].

It is known that *bounded cohomology* H_b^* (of spaces) and its sibling *ℓ^1 -homology* $H_*^{\ell^1}$ do *not* satisfy excision. For example, $H_k^{\ell^1}(S^1) \cong 0$ for $k \in \mathbb{N}_{\geq 1}$, but $H_2^{\ell^1}(S^1 \vee S^1) \not\cong 0$ [24, Ex. (2.8)]. One can, however, ask the question whether we can somehow enforce those functors to be excisive. In other words, can we find a best approximation to those functors (in some precise formal sense) that *is* excisive?

To answer this question in at least one possible way, we use the abstract and broadly applicable notion of excisive functors in the sense of Goodwillie calculus. The natural framework for this abstract setting is the language of ∞ -categories, which we will also use where appropriate. We show:

MAIN THEOREM. — *The best excisive approximation to the ℓ^1 -chain complex functor $C^{\ell^1} : \text{Top}_* \rightarrow \text{Ch}_{\mathbb{R}}$ from the right is trivial (i.e. mapping everything to 0).*

The precise statement and proof will be given below as Theorem 1.7. In order to import the 1-categorical ℓ^1 -chain complex functor to the world of ∞ -categories, we use the localization machinery from Cisinski’s book [6], see Remark 2.28.

Context: ℓ^1 -homology and bounded cohomology. A basic object in algebraic topology is the *singular chain complex* $C(X, A)$ with real coefficients of a pair of topological spaces (X, A) . Whereas, classically, we take its homology, or we dualize and take cohomology (yielding *singular (co)homology*), we can also obtain a more

2020 *Mathematics Subject Classification*: 55N35, 55U35, 18F50, 18N60.

Keywords: ℓ^1 -homology, excision, excisive approximation, ∞ -categories.

functional analytic theory: First, we equip $C_n(X)$ with the ℓ^1 -norm associated to the canonical basis, i.e., we set

$$\left\| \sum_{j=0}^N a_j \sigma_j \right\|_1 := \sum_{j=0}^N |a_j| \in \mathbb{R}_{\geq 0}$$

for reduced chains in $C_n(X)$, and $C_n(X, A)$ with the quotient norm. The boundary operator of $C(X, A)$ is bounded with respect to the induced operator norms, hence we can take the norm-completion of the whole complex and obtain the ℓ^1 -chain complex $C^{\ell^1}(X, A)$. Its homology is the ℓ^1 -homology of (X, A) , the topological dual of (either $C(X, A)$ or $C^{\ell^1}(X, A)$) is the *bounded cochain complex* $C_b(X, A)$ of (X, A) , and cohomology of the latter is the *bounded cohomology of (X, A)* . For details and (basic) properties, we refer the reader to the literature [24, Sec. 3.1], [23, Introduction and Ch. 2].

The prime example of an invariant defined in terms of $\|\bullet\|_1$ is the *simplicial volume* [22], which was introduced by Gromov [14] and measures how difficult it is to build the fundamental class of a manifold out of singular simplices with real coefficients. It is linked to Riemannian geometry [14] and – except for a few cases – very hard to compute. To make structural statements about simplicial volume, the theory of bounded cohomology and ℓ^1 -homology has been indispensable so far.

Bounded cohomology also has various connections to other mathematical areas (some of which are summarized in Monod’s nice “invitation to bounded cohomology” [31]), so in view of the (albeit imperfect) dualism between ℓ^1 -homology and bounded cohomology, it makes sense to obtain a more throughout understanding of the former as well. One should also point out, that some computations in bounded cohomology *cannot* be dualized to ℓ^1 -homology. For example, certain transfer maps can only be constructed in the cohomological setting [24, Caveats 5.8 and 5.6].

To get the appropriate ℓ^1 -chain complex functor between ∞ -categories via the universal property of the localization (2.28), it is important to recognize C^{ℓ^1} as a *relative* functor (2.27), i.e., mapping weak (homotopy) equivalences of topological spaces to quasi-isomorphisms of chain complexes. This follows from the fact that C_b is a relative functor [19, 6.4 Cor.] and the translation principle by Löh [24, Cor. 5.1]. (Note, that Ivanov’s result is (maybe for convenience) only stated for maps between *path-connected* spaces, but his proof of the theorem preceding the corollary can be taken over verbatim if one drops the path-connectedness assumptions and replaces the i.e.-part by the more general definition of “ k -equivalence” involving all base points and π_0 ; see the literature [40, Sec. 6.7], [16, Ch. 4].)

Variations of the main theorem. By taking opposite ∞ -categories appropriately, basically the same proof as for Theorem 1.7 shows that the best excisive approximation to the bounded cochain complex functor $\mathbf{Top}_*^{\text{op}} \rightarrow \mathbf{Ch}_{\mathbb{R}}$ *from the left* is also trivial. (One must be careful with the terminology, though: in the context of contravariant functors, one has to read Definition 1.2 as “pushout squares *in the original category*”, i.e. before applying op .)

Dually to our investigation, one might also ask for an excisive approximation of the ℓ^1 -chain complex functor *from the left* and of the bounded cochain complex functor *from the right*. It is shown by Raptis [34], that the latter is exactly the comparison map from the bounded to the singular cochain complex [34, Sec. 2.5],

and that the arguments also dualize to the comparison map in the case of ℓ^1 -homology [34, Sec. 3.3].

As for approximations to bounded cohomology and ℓ^1 -homology of groups, the setup of this article does not seem to be very useful: One fundamental problem is, that pushouts are well suited for decompositions of topological spaces, but not in the case of groups, where they only correspond to amalgamated free products.

Looking deeper into Goodwillie calculus reveals, that a functor being excisive is only the first of a whole sequence of (increasingly weaker) conditions on a functor. In generalization of the main theorem of this article, the author showed, that the n -excisive approximation to the ℓ^1 -chain complex functor is trivial for all $n \in \mathbb{N}$. The details, which become a lot more technical, can be found in the author’s PhD thesis [43, Sec. 6.3].

Notations and conventions. As we want to work with functor categories, it is necessary to avoid set theoretic issues. The reader, who does usually not worry about size issues, may find an explanation of the problem and strategies for its solution in Remark 2.10.

If one intends to do general treatment of ∞ -category theory itself, the best way to solve this, is to parametrize each and every statement over the meaning of “small” and “large”. This is, what happens in (the second half of) Cisinski’s book [6], and it has the benefit of greatest flexibility. However, as we mainly want to use ∞ -categories to treat explicit examples and to survey some aspects of the theory, this approach would add too much complexity and most certainly confusion among novice readers. For these reasons, we will choose a size hierarchy now, into which all objects of this article fit.

We fix an increasing chain of four Grothendieck universes, whose elements we refer to as *small*, *large*, *very large* and *huge*. We occasionally use the term *class* for a subset of the small universe; hence, each class is a large set. We then denote the category of

- small topological spaces by **Top**,
- large sets by **Set**, and
- large, but locally small 1-categories by **Cat**.

All the usual categories of algebraic theories (rings, modules, ...) live at the same stage as **Top**, and are objects of **Cat**. Rings are always unital. For a ring k , we denote the category of chain complexes of left k -modules by \mathbf{Ch}_k .

For a category A and objects x, y of A , we let $\mathbf{Mor}_A(x, y)$ denote the set of morphisms $x \rightarrow y$; more generally (and if A is locally large), \mathbf{Mor}_A denotes the corresponding functor $A^{\text{op}} \times A \rightarrow \mathbf{Set}$, where A^{op} denotes the opposite category of A .

When used as an ∞ -category, the symbols **Top** and \mathbf{Ch}_k need to be re-interpreted as the localization (2.28) of their 1-categorical counterpart with respect to weak homotopy equivalences and quasi-isomorphisms, respectively. We do *not* employ the common practice of using the nerve functor (2.12) implicitly.

Organization of this article. In Section 1 we lay out the results that we are able to obtain via the Goodwillie calculus framework. Section 2 contains a very brief introduction to ∞ -categories in the sense of Lurie [27] and tries to provide some insight on a conceptual level to a few aspects of the theory; we would like to

provide as much “glue” as needed for the “1-categorical reader” who is usually not working with ∞ -categories. In Section 3 we provide a full proof of the fact that the singular chain complex functor (on the level of ∞ -categories) is excisive in our abstract sense (1.2). Finally, in Section 4, we show how to obtain Mayer–Vietoris sequences from ∞ -categorical pullback squares, and in particular how to obtain the classical Mayer–Vietoris sequence for singular homology in this way.

How to read this article. Readers with at least some prior knowledge of ∞ -category theory can start at Section 1, might want to skip most of Section 2, and then (optionally) continue with Section 3 and Section 4. The reader, who is proficient in model category theory, can skip Section 3.1.

For the reader without ∞ -categorical background, we recommend to begin with Section 2, ignoring Section 2.4 on the first visit. Depending on the familiarity with modern homotopy theory, one might also want to read Section 3.1 to get a feeling for the nature of homotopical situations. Afterwards, the remaining parts can be consumed as desired, e.g. linearly from Section 1.

1. EXCISIVE APPROXIMATION

In this section, we investigate the excisive approximation of ℓ^1 -homology – or more precisely, of the ℓ^1 -chain complex functor. First, we introduce a concept of *excision* whose formulation applies to *any* functor between ∞ -categories (1.2), we quote a theorem by Lurie ensuring the existence of such an approximation (1.5), and we explain in what sense such an approximation is universal (1.6). We then use these notions to show that the excisive approximation of $C^{\ell^1} : \mathbf{Top}_* \rightarrow \mathbf{Ch}_{\mathbb{R}}$, i.e. the ℓ^1 -chain complex functor on pointed spaces, is trivial (1.7). An explicit formula by Lurie makes the argument short and accessible.

As this section’s setup is entirely ∞ -categorical, the word “category” will always mean ∞ -category in the sense of Lurie [27] (2.7) and we will use 1-category to refer to categories in the sense of traditional category theory. Likewise, we will deal with other concepts such as functors, limits, etc., i.e. “functor” will mean a functor of ∞ -categories, “(co)limit”/“pushout”/“pullback” will mean the ∞ -categorical notion, etc. In particular, we remind the reader of our convention that \mathbf{Top} and $\mathbf{Ch}_{\mathbb{R}}$ become ∞ -categories via localization (page 109). We highlight these choices by the following:

CONVENTION 1.1. — In this section, and this section only, the term *category* refers to ∞ -category.

1.1. **Abstract excision.** First of all, we introduce the concept of an excisive functor of ∞ -categories:

DEFINITION 1.2 (excisive functor). — Let C be a category with pushouts and let $F : C \rightarrow D$ be a functor. Then F is *excisive* if it sends pushout squares to pullback squares, i.e., if the following holds: Given a pushout square

$$\begin{array}{ccc} W & \xrightarrow{f} & U \\ g \downarrow & & \downarrow g' \\ V & \xrightarrow{f'} & X \end{array}$$

in C , the diagram

$$\begin{array}{ccc} F(W) & \xrightarrow{F(f)} & F(U) \\ F(g) \downarrow & & \downarrow F(g') \\ F(V) & \xrightarrow{F(f')} & F(X) \end{array}$$

is a pullback square in D .

Note again, that this must be read in the ∞ -categorical setting. In particular, pushout/pullback squares do *not* have to commute “on the nose” but only “up to homotopy” and the latter is part of the data(!), see also Example 2.19 and Remark 2.21.

Example 1.3. — The constant functor $\mathbf{Top} \rightarrow \mathbf{Top}$ that sends everything to a point is excisive. On the other hand, the identity functor on \mathbf{Top} is *not* excisive: for example, there is a pushout square

$$\begin{array}{ccc} S^0 & \xrightarrow{\text{inclusion}} & D^1 \\ \downarrow & \text{quotient} & \downarrow \\ * & \longrightarrow & D^1/S^0 \end{array}$$

that is *not* a pullback square. We will get back to this in Example 3.7.

As it turns out, given a functor between categories with enough limits, it always has a “best approximation” by an excisive functor in a precise sense. The following definition captures the niceness assumptions that we have to impose on the codomain category:

DEFINITION 1.4. — Let C be a category. Then C is *differentiable* if it admits finite limits and sequential colimits and if the formation of the latter commutes with the former.

More explicitly, this means: every finite [26, 0130] diagram $K \rightarrow C$ admits a limit, every diagram $\mathbb{N}(\mathbb{N}) \rightarrow C$ admits a colimit in C , and the colimit functor $\text{Fun}(\mathbb{N}(\mathbb{N}), C) \rightarrow C$ commutes with finite limits. Here, \mathbb{N} is viewed as the preorder category (2.2) of the usual ordering on the natural numbers and \mathbb{N} is the nerve functor (2.12) from 1-categories to ∞ -categories.

THEOREM 1.5 ([28, Thm. 6.1.1.10 and Ex. 6.1.1.28]). — *Let C be a category with a terminal object and finite colimits and let D be a differentiable category. Let $\text{Exc}(C, D)$ be the full subcategory of the functor category $\text{Fun}(C, D)$, spanned by the excisive functors. Then the inclusion functor $\text{Exc}(C, D) \hookrightarrow \text{Fun}(C, D)$ has a left adjoint $P_1: \text{Fun}(C, D) \rightarrow \text{Exc}(C, D)$.*

Furthermore, if $F: C \rightarrow D$ is reduced, which means that F maps every terminal object of C to a terminal object of D , and if D has a zero object, the following holds:

$$P_1 F \simeq \text{colim}_{n \in \mathbb{N}} \Omega^n \circ F \circ \Sigma^n$$

where Ω and Σ are the loop and suspension functor on D and C , respectively.

It should be noted, that the functor categories that appear in the theorem are usually not locally small anymore. Though the existence of an adjoint as well as

the statement that two given functors are adjoint to each other do *not* depend on the ambient universe [6, Rem. 6.1.12 and Thm. 6.1.23(v)], the use of mapping space functors (“Hom space” in Cisinski’s book [6, Sec. 5.8]) will be convenient in the following remark. In order to meet the smallness assumptions for this, it is necessary to view the functor categories from our very large universe. For more discussion of mapping spaces, see Section 4.2 (with the small caveat, that our current shift of universes implies that the targets of mapping space functors in Remark 1.6 are not \mathbf{Top} , but the category of *large* topological spaces).

For any functor $F: C \rightarrow D$, the functor $P_1F: C \rightarrow D$ can be seen as a best approximation of F by an excisive functor from the right, as we shall now explain:

Remark 1.6 (best approximation through adjunction). — One way of making the term *best approximation* precise in a categorical setting is by so-called *universal arrows* [29, Sec. III.1], i.e. by a certain mapping property:

Let C' be a full subcategory of some category C and let X be an object of C that we want to approximate by an object of C' . For example, C could be a functor category and C' could be its full subcategory of excisive functors. Then a *best approximation of X by an object of C' from the right* consists of an object Y of C' and a morphism $f: X \rightarrow Y$ in C with the following property: for every other morphism $f': X \rightarrow Y'$ to an object of C' , there exists an (up to homotopy) unique morphism $g: Y \rightarrow Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow f' & \downarrow g \\ & & Y' \end{array}$$

commutes. Intuitively, we search for the “biggest quotient” of X that has the desired properties. Of course, the whole paragraph could be dualized, by which we would obtain approximations *from the left*, i.e. “biggest subobjects”.

It is a classical 1-categorical fact [29, Thm. IV.1.1] that an adjunction gives rise to universal arrows. We will sketch the argument for our situation:

Assume that we have the inclusion functor $i: \mathbf{Exc} \hookrightarrow \mathbf{Fun}$ of a full subcategory and that it has a left adjoint $P: \mathbf{Fun} \rightarrow \mathbf{Exc}$. Using mapping space functors, one can define adjunctions in ∞ -categories analogously to the 1-categorical case [6, Def. 6.1.3], i.e. we have an invertible natural transformation

$$\mathrm{Map}_{\mathbf{Fun}}(\bullet, i(\bullet)) \xrightarrow{c} \mathrm{Map}_{\mathbf{Exc}}(P(\bullet), \bullet).$$

Here, invertibility means, that each component of c is an invertible morphism [6, Def. 1.6.10 and 1.5.1], or equivalently, that c is an invertible morphism in the corresponding functor category [6, Cor. 3.5.12].

Now let F be an object of \mathbf{Fun} , choose an inverse of c and let $\eta := c_{F, PF}^{-1}(\mathrm{id}_{PF}) \in \mathrm{Map}_{\mathbf{Fun}}(F, iPF)$. In order to show that this morphism

$$F \xrightarrow{\eta} PF$$

is universal in the above sense, let P' be another object of Exc , let $(\eta': F \rightarrow P') \in \text{Map}_{\text{Fun}}(F, iP')$ be a morphism and set $\theta := c_{F, P'}(\eta')$. Then the triangle

$$\begin{array}{ccc} F & \xrightarrow{\eta} & PF \\ & \searrow \eta' & \downarrow \theta \\ & & P' \end{array}$$

commutes, which can be seen by chasing η through the following diagram:

$$\begin{array}{ccc} \text{Map}_{\text{Fun}}(F, PF) & \xrightarrow{\text{Map}_{\text{Fun}}(\text{id}_F, \theta)} & \text{Map}_{\text{Fun}}(F, P') \\ \downarrow c_{F, PF} & & \downarrow c_{F, P'} \\ \text{Map}_{\text{Exc}}(PF, PF) & \xrightarrow{\text{Map}_{\text{Exc}}(\text{id}_{PF}, \theta)} & \text{Map}_{\text{Exc}}(PF, P'). \end{array}$$

1.2. Excisive approximation of the ℓ^1 -chain complex functor. We can now apply Theorem 1.5 to ℓ^1 -homology, but we have to be a little bit careful: After a bit of thought, it is quite apparent that the classical *excision axiom* for a homology theory in the sense of Eilenberg and Steenrod is *not* a property of a single functor H_n for a fixed n , but of a whole sequence of functors: after all, we know [39, cf. Ch. 7, esp. 7.34 and 7.35] that excision can equivalently be recast in the form of a Mayer–Vietoris sequence running through all “dimensions” of a homology theory. (In the usual form of the Eilenberg–Steenrod axioms the excision axiom seems to depend only on a single functor at a time, but this is deceptive because it uses *relative* homology which is tightly linked to its absolute version in different(!) dimensions by the long exact sequence of a pair.)

So the upshot of this discussion is that we should not try to approach each of the functors $H_n^{\ell^1}$ individually, but instead we have to look at the corresponding functor $\text{Top} \rightarrow \text{Ch}_{\mathbb{R}}$ to chain complexes. We will see later (3) that indeed the usual singular chain complex functor *is* excisive in our sense. In order to apply the formula of Theorem 1.5, we need a reduced functor, but $C^{\ell^1}(\ast)$ is *not* a contractible chain complex. This is why we consider pointed spaces and the functor $C^{\ell^1}: \text{Top}_{\ast} \rightarrow \text{Ch}_{\mathbb{R}}$. We then obtain:

THEOREM 1.7 (excisive approximation of ℓ^1 -homology). — *There exists a best excisive approximation of $C^{\ell^1}: \text{Top}_{\ast} \rightarrow \text{Ch}_{\mathbb{R}}$ from the right, i.e. an excisive functor $P: \text{Top}_{\ast} \rightarrow \text{Ch}_{\mathbb{R}}$ and a natural transformation $\eta: C^{\ell^1} \rightarrow P$, such that for every other excisive functor $P': \text{Top}_{\ast} \rightarrow \text{Ch}_{\mathbb{R}}$ and natural transformation $\eta': C^{\ell^1} \rightarrow P'$ there exists a natural transformation $\theta: P \rightarrow P'$ (unique up to homotopy) that makes the triangle*

$$\begin{array}{ccc} C^{\ell^1} & \xrightarrow{\eta} & P \\ & \searrow \eta' & \downarrow \theta \\ & & P' \end{array}$$

commutative.

This best approximation is trivial, i.e. $P(X) \simeq 0$ for all pointed spaces X .

Proof. — The category Top_{\ast} has all small limits and colimits. Furthermore, $\text{Ch}_{\mathbb{R}}$ is stable [28, Prop. 1.3.5.9 (with Prop. 1.3.5.15)] and has all small colimits, so it is in particular differentiable [28, Ex. 6.1.1.7]. Hence, the prerequisites

of Theorem 1.5 are fulfilled and the functors $P_1: \text{Fun}(\text{Top}_*, \text{Ch}_{\mathbb{R}}) \rightarrow \text{Exc}(\text{Top}_*, \text{Ch}_{\mathbb{R}})$ and $P_1(C^{\ell^1}): \text{Top}_* \rightarrow \text{Ch}_{\mathbb{R}}$ exist. Since P_1 is left adjoint to the inclusion functor $i: \text{Exc}(\text{Top}_*, \text{Ch}_{\mathbb{R}}) \hookrightarrow \text{Fun}(\text{Top}_*, \text{Ch}_{\mathbb{R}})$, we also know (see Remark 1.6) that the unit morphism $F = \text{Id}(F) \rightarrow (i \circ P_1)(F)$ of the adjunction provides a universal arrow in the sense described in the claim.

Furthermore, since we consider C^{ℓ^1} relative to the base point, the second part of Theorem 1.5 is applicable. For a pointed space X this gives us

$$P_1(C^{\ell^1})(X) \simeq \text{colim}_n \Omega^n \left(C^{\ell^1}(\Sigma^n X) \right),$$

where $\Sigma: \text{Top}_* \rightarrow \text{Top}_*$ is the (unreduced) suspension functor and $\Omega: \text{Ch}_{\mathbb{R}} \rightarrow \text{Ch}_{\mathbb{R}}$ is the loop functor on chain complexes. But since $\Sigma^n X$ is simply-connected for $n \in \mathbb{N}_{\geq 2}$ and we know that C^{ℓ^1} vanishes on path-connected spaces with amenable (so in particular trivial) fundamental group [19, 8.4 Thm.], [24, Cor. 5.1], we get $P_1(C^{\ell^1})(X) \simeq \text{colim}_{n \in \mathbb{N}_{\geq 2}} \Omega^n(0)$. Consider the fact that the loop functor Ω on chain complexes can be realized by just shifting a complex down by one (i.e. $(\Omega C)_n = C_{n+1}$). Using this, it follows that we have

$$P_1(C^{\ell^1})(X) \simeq \text{colim}_{n \in \mathbb{N}_{\geq 2}} 0 \simeq 0. \quad \square$$

2. A BRIEF GLIMPSE OF INFINITY

We now take the time to discuss some aspects of ∞ -category theory for readers who mostly or exclusively work in a 1-categorical setting. In this survey section, we will not include any proofs, but we give lots of pointers to the literature for further reading. We hope that the reader who is not familiar with ∞ -category theory will find enough insights as to get a comfortable feeling about the statements and arguments of the other sections.

In Sections 2.1 and 2.2, we give a rigorous definition of the term *∞ -category* and explain how it relates to the concept of higher morphisms. Section 2.3 is dedicated to the question how interesting ∞ -categories can be constructed from 1-categorical data, and in Section 2.4 we note some caveats regarding the nomenclature around the term “ ∞ -category”.

We presume some familiarity with basic category theory, e.g. as in the first two chapters of Mac Lane’s book [29, Sec. I.1–4, I.8, II.1–4]. Furthermore, we note:

CONVENTION 2.1. — Starting with this section, *category* means *1-category*, and we will explicitly use *∞ -category* as in Definition 2.7.

2.1. From 1 to ∞ in a nutshell. Our first goal is to give a precise definition of ∞ -categories in terms of Lurie and to observe that there’s a formal way to turn 1-categories and functors into ∞ -categorical ones. We refer the reader to the literature [6, 15, 26, 27, 35] for a more comprehensive treatment of the subject.

The underlying foundational object for this theory is the *simplicial set*, a generalization of (ordered) simplicial complexes, for which we give a rigorous definition in the following but almost no illustration or background. We recommend the beautiful survey article by Friedman [12] as an introduction to the subject and the textbook by Goerss and Jardine [13] for further reading.

DEFINITION 2.2 (simplicial set). —

- (1) Let (X, \leq) be a preorder, i.e. \leq is a transitive and reflexive relation on X . We define the *preorder category* $\text{Pre}(X, \leq)$ to have X as objects and a unique morphism from x to x' exactly if $x \leq x'$ (and no further morphisms).
- (2) For $n \in \mathbb{N}$ let $[n] := \text{Pre}(\{0, \dots, n\}, \leq)$. Let Δ be the full subcategory of Cat on the objects $\{[n] \mid n \in \mathbb{N}\}$.
- (3) A *simplicial set* is a functor $\Delta^{\text{op}} \rightarrow \text{Set}$. The *category sSet of simplicial sets* is the functor category $\text{Fun}(\Delta^{\text{op}}, \text{Set})$. Hence, a morphism of simplicial sets is a natural transformation of functors. For a simplicial set X and $n \in \mathbb{N}$ one often uses the notation $X_n := X([n])$.
- (4) Let X be a simplicial set. Then a *simplicial subset* of X is a simplicial set U such that for all $n \in \mathbb{N}$ we have $U_n \subseteq X_n$ and for all morphisms $f: [m] \rightarrow [n]$ in Δ we have $U(f) = X(f)|_{U_n}$.

Remark 2.3 (smaller simplicial sets). — As a variant of Definition 2.2(3), we could replace Set , the category of large sets, by the category of small sets. The resulting simplicial sets would then be suitable for modelling objects of Top as in simplicial homotopy theory [13], but would be too small for our purposes.

Remark 2.4 (sets as simplicial sets). — By an extension of a set U to a simplicial set, we mean a simplicial set X with $X_0 = U$. Such an extension always exists by virtue of the constant functor at U .

Remark 2.5 (operations with simplicial sets). — Many operations one normally does with plain sets can be lifted to simplicial sets by just applying them “dimensionwise”. For instance, for two simplicial subsets U, V of the same simplicial set, one can define $U \cap V$ by $(U \cap V)_n := U_n \cap V_n$ for all $n \in \mathbb{N}$ and restriction on morphisms. In particular the expression “smallest simplicial subset with some property” can be made precise by taking intersections as usual.

The formal reason why this works, is the fact that limits and colimits in a functor category are just computed “pointwise” if they exist in the target category. Taking intersection is a pullback of sets, for example.

With Remark 2.5 in mind, we can now introduce some fundamental simplicial sets:

Examples 2.6 (standard simplex, horn). —

- (1) Let $n \in \mathbb{N}$. The simplicial set $\Delta^n := \text{Mor}_\Delta(\bullet, [n])$ is the *standard n -simplex*. For $n \geq 1$ and $i \in \{0, \dots, n\}$ the unique functor $d_i^{n-1} \in \text{Mor}_\Delta([n-1], [n]) = (\Delta^n)_{n-1}$ with $d_i^{n-1}(\{0, \dots, n-1\}) = \{0, \dots, n\} \setminus \{i\}$ is the i^{th} *coface map*.
- (2) For $n \in \mathbb{N}$, the *simplicial n -sphere* is the smallest simplicial subset of Δ^{n+1} that contains $\{d_0^n, \dots, d_{n+1}^n\} \subseteq (\Delta^{n+1})_n$.
- (3) Let $n \in \mathbb{N}_{\geq 1}$ and $i \in \{0, \dots, n\}$. The *(n, i) -horn* Λ_i^n is the smallest simplicial subset of the standard n -simplex that contains $\{d_0^{n-1}, \dots, d_n^{n-1}\} \setminus \{d_i^{n-1}\} \subseteq (\Delta^n)_{n-1}$.

Informally speaking, the simplicial $(n-1)$ -sphere is obtained from Δ^n by removing the “interior” of the n -simplex; and by further removing the $(n-1)$ -dimensional face opposite to the i^{th} vertex we get the horn Λ_i^n . The latter now plays the main role in the definition of an ∞ -category:

DEFINITION 2.7 (∞ -category). — An ∞ -category is a simplicial set C with the following property: For all $n \in \mathbb{N}_{\geq 1}$ and all $i \in \{1, \dots, n-1\}$ every morphism $\Lambda_i^n \rightarrow C$ factors over the inclusion $\Lambda_i^n \hookrightarrow \Delta^n$.

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & C \\ \downarrow & \nearrow \exists & \\ \Delta^n & & \end{array}$$

A usual formulation of this property is the following: In an ∞ -category all so-called *inner horns* ($0 < i < n$) have a *not* necessarily unique(!) “filler”. Those fillers are witnesses of (higher dimensional) compositions in an ∞ -category and the existence statement then ensures that a composition can always be found. See Remark 2.16 for the case of Λ_1^2 ; for examples using higher dimensional horns and the existence of their fillers, we refer to the literature [27, Sec. 1.2.3], [6, proof of Lem. 1.6.2], [26, 003U], [26, 0041]. For nomenclature linking the different pieces of an ∞ -category to classical categorical terms, see Definition 2.15.

Example 2.8. — For all $n \in \mathbb{N}$, the standard n -simplex is an ∞ -category. On the other hand, for $n \in \mathbb{N}_{\geq 1}$, the simplicial n -sphere is *not* an ∞ -category.

Remark 2.9 (functor ∞ -categories). — For ∞ -categories C and D , a functor $C \rightarrow D$ is simply taken to be a morphism of simplicial sets.

A certain construction [26, 0060], [13, Sec. I.5] turns the category of simplicial sets into a *cartesian closed* category [29, Sec. IV.6]: it provides a simplicial set Y^X that extends the set of morphisms between two simplicial sets X and Y , i.e. with $(Y^X)_0 = \text{Mor}_{\text{Set}}(X, Y)$. This is often called an *internal Hom*, because it is again an object of the ambient category, or in this specific case the *function complex from X to Y* . It can be shown [6, Cor. 3.2.10] that Y^X is again an ∞ -category whenever Y is an ∞ -category.

In particular, this shows how functors $C \rightarrow D$ for fixed ∞ -categories C and D give rise to a functor ∞ -category $\text{Fun}(C, D)$.

Remark 2.10 (size issues). — As the reader has probably already noticed, the operation of taking “all functors $C \rightarrow D$ ” might or might not yield a well-defined object, depending on the set theory that one chooses as a mathematical foundation. To be more specific: If one considers functors between categories “of the same size” (in terms of the “number” of objects and morphisms) one often obtains something that is “bigger” than that. On the other hand, if one imposes the correct size constraints on C and D , the result can also be kept “small enough”.

For example, suppose that we work within NBG (von Neumann, Bernays, Gödel) set theory. Then a category is usually defined to allow for a proper class of objects, but most often only morphism *sets* between any two objects are permitted. If we consider such categories, it is well-known that one is usually not permitted to form the functor category between two categories whose classes of objects are both proper classes. Though, if we consider a *small* category C , i.e. its class of objects is a set, and another category D , the functor category $\text{Fun}(C, D)$ is again a well-defined category in the sense of this paragraph.

Analogously, such considerations can be made for ∞ -categories when appropriate size restrictions are in place [6, Cor. 5.7.7].

Even then, however, the “two layers” of NBG set theory are not enough for the arguments in Section 1.2: there we want to look at the functor category $\text{Fun}(\text{Top}_*, \text{Ch}_{\mathbb{R}})$ where the size conditions are *not* met. A usual way to circumvent this problem is the usage of so-called *Grothendieck universes*. Roughly speaking, this means the following: A set U is called a *universe* if it satisfies certain closure properties like the *power set axiom*:

$$\forall x \in U, \quad \{y \mid y \subseteq x\} \in U.$$

Then an additional axiom, the *universe axiom*, is imposed:

$$\forall x \quad \exists U \text{ universe, } x \in U.$$

One can then just choose a universe that is big enough to support all the operations one wants to make, i.e. such that “enough layers exist”. As one may convince oneself, for the formulation of the statements in this article, it is actually enough to consider just “one more layer” on top of NBG, giving it a name like “superclasses” or “conglomerates” if one wants to maintain the hierarchy that is already in place.

For further information, we refer the reader to the literature about size issues and possible solutions: A concise introduction to Grothendieck universes can be found in a paper by Low [25]; a gentle explanation of the “conglomerate solution” can be found in the category theory book by Adámek, Herrlich and Strecker [1, 2 Foundations]; an expository article by Shulman compares set theoretical foundations for category theory from a modern perspective [38].

Remark 2.11 (size levels of our objects). — We say a category is of a specific size, if its set of objects and set of morphisms are both of that size. Under the explicit choices that we made in the introduction (page 109), we have, among others:

- The sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and all underlying sets of objects of Top are small, and Δ is a small category.
- The categories Top, Ch_k and the category of small sets are large categories. However, they are locally small, i.e. the morphism set between every two objects is small.
- The categories Set, Cat and sSet are very large, but locally large.
- The category of very large categories is huge.

Now that we have defined what an ∞ -category is, we would certainly want to “import” all the 1-categories that we already know of into this setting. Put differently, we really want that ∞ -categories provide a generalization of 1-categories. This can be done via the *nerve* functor:

DEFINITION 2.12 (nerve of a 1-category). — Let $\iota: \Delta \hookrightarrow \text{Cat}$ denote the inclusion functor. Then the *nerve functor* $N: \text{Cat} \rightarrow \text{sSet}$ arises from the functor

$$\text{Mor}_{\text{Cat}}(\iota^{\text{op}} \bullet, \bullet): \Delta^{\text{op}} \times \text{Cat} \rightarrow \text{Set}$$

by *currying*, i.e. $N(X) = \text{Mor}_{\text{Cat}}(\iota^{\text{op}} \bullet, X)$.

More formally, this currying construction can be seen as an application of the exponential law in the category of very large categories, i.e. $\text{Fun}(C \times C', D) \cong \text{Fun}(C', \text{Fun}(C, D))$, analogously to the exponential law for functions [29, Exerc. 2 of Sec. II.5].

Example 2.13 (standard simplex via nerve). — For $n \in \mathbb{N}$ we have $\Delta^n = N([n])$.

PROPOSITION 2.14 (nerve yields ∞ -category [6, Prop. 1.4.11, Ex. 1.5.3], [27, Prop. 1.1.2.2], [26, 002Z]). — For every category C , the nerve $N(C)$ is an ∞ -category. Furthermore, the nerve functor N is fully faithful, i.e. for all categories C and D the map $\text{Mor}_{\text{Cat}}(C, D) \ni F \mapsto N(F) \in \text{Mor}_{\text{sSet}}(N(C), N(D))$ is bijective.

This means that whenever one embeds two 1-categories into the ∞ -world via the nerve, there will be exactly “the same” functors between them as there were before. So in this sense, ∞ -categories generalize 1-categories.

However, while it is nice that we have 1-categories as ∞ -categories now, it’s not an enhancement of the original 1-categories. Often, one builds ∞ -categories by other means and then they contain some homotopy theoretic information of the original 1-category. In Section 2.3 we will go into a few more details.

2.2. Working in ∞ -categories. Now that we know how to model ∞ -categories by simplicial sets, we can set up some nomenclature to make *working* in an ∞ -category more similar to working in a 1-category.

DEFINITION 2.15 (nomenclature for ∞ -categories). — Let C be an ∞ -category and let us use the notations of Definition 2.2(3) and Examples 2.6(1). Then

- *objects of C* are the elements of C_0 ,
- *morphisms of C* are the elements of C_1 ,
- *2-morphisms of C* are the elements of C_2 ,
- and generally for $n \in \mathbb{N}$ the *n -morphisms of C* are the elements of C_n .

For two objects x and y in C , a morphism $x \rightarrow y$ is a morphism f of C such that $C(d_1^0)(f) = x$ and $C(d_0^0)(f) = y$. For an object x of C the morphism $C([1] \rightarrow [0])(x)$ of C is the *identity morphism of x* , where we simply write $[1] \rightarrow [0]$ for the unique such functor.

Note that “morphism” and “1-morphism” mean exactly the same thing, as do “object” and “0-morphism”, though the latter is not used by all authors. With this notation at hand, one can, at least superficially, start to use an ∞ -category just as a 1-category. However, there is one big catch:

Remark 2.16 (lack of composition map). — In an ∞ -category C , there is usually no “composition map $\text{Mor}(y, z) \times \text{Mor}(x, y) \rightarrow \text{Mor}(x, z)$ ” like in 1-categories. Instead, given morphisms $f: x \rightarrow y$ and $g: y \rightarrow z$ there could be many possible compositions of f and g – and this is indeed one of the key notions in the generalization from 1-categories to ∞ -categories. More precisely, f and g give rise to a map $\Lambda_1^2 \rightarrow C$ of simplicial sets, and any extension $\sigma: \Delta^2 \rightarrow C$ to the standard simplex



determines a composition h of f and g , namely:

$$h = C(d_1^1)(\sigma_{[2]}(\text{Id}_{[2]})) \in C_1.$$

The existence of such an extension is guaranteed by the fact that C is an ∞ -category, but there can (and usually will) be many! In the situation above, we will say that σ witnesses that h is a composition of f and g .

A useful analogy is the following: when defining the concatenation of two compatible paths with common domain interval in a topological space, one has to subdivide the interval, which is usually done by splitting it in the middle. This might look most symmetric at first, but as soon as we consider three paths, in fact *no* such subdivision will make the concatenation map associative. One way to enforce this, is passing to the homotopy category, but as mentioned below (2.26), this might be a bad idea. Instead, we could just declare that *any* concatenation (regardless of the chosen interval subdivision) is a valid composition of the paths, i.e. we allow many different compositions of two paths.

With this in mind, the next natural question is: How does the notion of (*commutative*) *diagrams* [26, 005H] enter ∞ -category theory? Indeed, since we already know what a functor of ∞ -categories is, we could say, analogously to the usual definition for 1-categories, that a diagram in an ∞ -category C is a functor $C' \rightarrow C$ from some ∞ -category C' (the “shape of the diagram”) to C . Though this would suffice on a purely technical level [6, Thm. 7.3.22][27, Prop. 4.2.3.14] it is more convenient to allow “shapes” that are not ∞ -categories themselves:

DEFINITION 2.17 (diagram in an ∞ -category). — Let C be an ∞ -category and let K be a simplicial set. A *diagram in C indexed by K* is a morphism of simplicial sets $K \rightarrow C$.

Example 2.18 (diagrams indexed by simplices). — Let $n \in \mathbb{N}$ and let C be an ∞ -category. A diagram $\Delta^n \rightarrow C$ is equivalently an n -morphism of C ; more precisely: the map

$$\text{Mor}_{\text{Set}}(\Delta^n, C) \rightarrow C_n, \quad \sigma \mapsto \sigma_{[n]}(\text{Id}_{[n]})$$

is a bijection. (This is an instance of the Yoneda lemma [29, Sec. III.2].) It is common to identify n -morphisms of C with diagrams $\Delta^n \rightarrow C$ via this bijection whenever it is convenient.

Example 2.19 (commutative squares). — Let $K := N([1] \times [1])$ be the nerve of the product category $[1] \times [1]$ (taken in Cat). More explicitly, the latter can be depicted as

$$\begin{array}{ccc} (0, 0) & \xrightarrow{f} & (1, 0) \\ f' \downarrow & \searrow h & \downarrow g \\ (0, 1) & \xrightarrow{g'} & (1, 1) \end{array}$$

and its nerve has the obvious additional 2-simplices σ and τ “filling the two triangles”. Then a diagram $p: K \rightarrow C$ amounts to the data of

- four objects $p((0, 0))$, $p((1, 0))$, $p((0, 1))$ and $p((1, 1))$,
- five morphisms $p(f)$, $p(g)$, $p(f')$, $p(g')$ and $p(h)$, and
- two 2-morphisms $p(\sigma)$ and $p(\tau)$

of C , such that

- the morphisms have the correct source and target and
- the 2-morphisms $p(\sigma)$ and $p(\tau)$ witness that $p(h)$ is a composition of $p(f)$ and $p(g)$ as well as of $p(f')$ and $p(g')$. (Here we apply Example 2.18.)

As an illustration, we consider the case that C is an ∞ -category of topological spaces, e.g. as in Remark 2.25 via the homotopy coherent nerve or as the localization of \mathbf{Top} at weak homotopy equivalences in the sense of Remark 2.28. Then in a diagram as above, $p(\sigma)$ and $p(\tau)$ are homotopies $p(h) \simeq p(g) \circ p(f)$ and $p(h) \simeq p(g') \circ p(f')$. So in particular, every such diagram in C gives a 1-categorical commutative diagram in the homotopy category of topological spaces. But it is crucial to understand that the latter notion is just a *property* of a 1-categorical diagram while an actual diagram $K \rightarrow C$ carries the 2-morphisms $p(\sigma)$ and $p(\tau)$ as additional *data*.

Remark 2.20 (homotopy coherence). — The last example already shows that commutative diagrams in ∞ -categories are quite a bit more delicate to handle than in 1-categories. Of course, more complex diagrams than commutative squares require even more data to be specified by a diagram. In particular, diagrams indexed by “infinite shapes”, e.g. $\mathbf{N}(\mathbf{Pre}(\mathbb{N}, \leq))$, may require chosen n -morphisms in the target ∞ -category for *all* $n \in \mathbb{N}$ in a compatible way – one also speaks of a *coherent* choice of higher morphisms. For further reading about homotopy coherence, we recommend notes by Riehl [36], which also contain an example [36, Ex. I.3.4] of a homotopy commutative diagram that cannot be made homotopy coherent.

Remark 2.21 (sloppy notation). — As we have seen in Example 2.19 (we will reuse its notation here), a commutative square in an ∞ -category needs to specify a “diagonal” morphism. Indeed, if one defines a commutative square in a 1-category C' analogously as a functor $F: [1] \times [1] \rightarrow C'$, one a priori has to define $F(h)$. But since compositions in 1-categories are unique, one easily sees that keeping this piece of data is *redundant*, i.e. inferable from $(F(f), F(g))$. So for such a 1-categorical commutative square, the restriction of F to the “outer” four morphisms uniquely determines F and this is why we usually don’t mention the “diagonal” at all.

However, this is not true anymore for diagrams in ∞ -categories! So when speaking of a commutative square

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

in an ∞ -category C , one really means a diagram $p: K \rightarrow C$ where $p((0, 0)) = a \rightarrow d = p((1, 1))$ as well as $p(\sigma)$ and $p(\tau)$ are not visible, although they are implicit data associated to the square.

Outlook 2.22 ((co)limits). — Having said what a diagram is, one would like to speak about limits and colimits of such. Unfortunately, we cannot fully define this notion here, because it requires a more technical insight into the theory of ∞ -categories and simplicial sets. To give an idea of why this is the case, we may try to transplant the usual picture of how one thinks about limits in 1-categories to the ∞ -categorical world: traditionally, one starts with the notion of a *cone* on a diagram. While this is easy for 1-categorical, i.e. “1-dimensional” diagrams, one already has to reflect on the issue, how to define a cone on an ∞ -categorical, i.e. “higher dimensional” simplicial set shaped diagram. However, it is indeed not

difficult to find explicit formulae that define the cone of a simplicial set [26, 0172], [26, 0177], which in turn may be used to formulate what a cone on an ∞ -diagram is.

Next, we would like to single out *universal* cones. In 1-category theory, those are then called limits or limit cones of the original diagram, and the universal property roughly says that every cone factors through such a limit cone. The latter property, though, cannot immediately be transferred to ∞ -cones for several reasons: first, remember that we do not have unique compositions of morphisms (2.16), and second, one would have to take higher simplices into account.

To solve this problem, one arranges for the ∞ -cones on a given diagram to be the objects of a certain ∞ -category. The latter then includes all the information about the higher morphisms and certain objects of this ∞ -category will then be called universal ∞ -cones.

All of this applies to cones, yielding *limits*, as well as dually to cocones, yielding *colimits*. In the first case, the cone point is the initial vertex of each simplex, in the second case the final vertex. For an expository account of the matter, skipping most technicalities yet providing all necessary intermediate steps, see Groth’s notes [15, Sec. 2]. All details can be found in the literature [6, Sec. 6.2], [27, Sec. 1.2.13].

Example 2.23 (pullbacks and pushouts). — As an informal example of the previous outlook, we say a bit more about pullback and pushout squares. For this, we consider diagrams indexed by the horns Λ_2^2 and Λ_0^2 . The cone on Λ_2^2 and the cocone on Λ_0^2 are both isomorphic to $N([1] \times [1])$; a fact that is intuitively clear by the following pictures:



i.e. both of them may index commutative squares (2.19) in an ∞ -category. Given a commutative square $p: N([1] \times [1]) \rightarrow C$, we say that p is

- a *pullback square* if the cone on Λ_2^2 that it determines is universal and it is
- a *pushout square* if the cocone on Λ_0^2 that it determines is universal,

both in the sense of Outlook 2.22.

Outlook 2.24 (stable ∞ -category). — The property of being a stable ∞ -category [28, Def. 1.1.1.9], which is especially important in the context of functor calculus [28, Ch. 6], can be defined in terms of pullbacks and pushouts [28, Prop. 1.1.3.4]:

An ∞ -category is *stable* if

- it has a zero object, i.e. an object that is both initial and terminal,
- it has all finite limits and colimits, and
- a square in it is a pushout if and only if it is a pullback.

This concept originates from stable homotopy theory, which Lurie recalls and discusses in depth in the ∞ -categorical setting [28, Sec. 1.4].

It should also be noted, that a precursor is the notion of a *stable model category* [18, Ch. 7], which can similarly be defined as a model category that has a zero object and in which homotopy pushouts and homotopy pullbacks coincide [18, Rem. 7.1.12]. (For a primer on homotopy pushouts in \mathbf{Top} , see Section 3.1.)

2.3. How to produce ∞ -categories. The following method is, historically and because of its role in Lurie’s book [27], one of the most important constructions that convert 1-categorical input into a corresponding ∞ -category, taking the homotopical structure into account:

Remark 2.25 (homotopy coherent nerve). — Let C be a *simplicially enriched category*: instead of just morphism sets, each $\text{Mor}_C(X, Y)$ is a simplicial set and the usual requirements regarding composition are applied mutatis mutandis [26, 00JQ]. For such a category C , there is a construction analogous to the nerve of a 1-category (2.12), called the *homotopy coherent nerve* or *simplicial nerve*, that builds a simplicial set $N^{\text{hc}}(C)$ out of C [27, Def. 1.1.5.5], [8, Sec. 2.4]. Under suitable conditions, $N^{\text{hc}}(C)$ is an ∞ -category [27, Prop. 1.1.5.10], which is in some sense a “homotopical thickening” of the nerve $N(C)$ of C .

To illustrate the last comment a bit more, we consider the example $C = \text{Top}$: For spaces X and Y , the set of continuous maps $X \rightarrow Y$ can be extended to a simplicial set that satisfies

$$\text{Mor}_{\text{Top}}(X, Y)_n = \{\text{continuous maps } \Delta_{\text{top}}^n \times X \rightarrow Y\}$$

for all $n \in \mathbb{N}$ [26, 00JV]; here, Δ_{top}^n denotes the standard topological n -simplex. By easy inspection, one sees that the 2-simplices $N(\text{Top})_2$ of the usual nerve correspond to commuting triangles in Top , i.e. to triples (f, g, h) of morphisms in Top such that $g \circ f = h$. By contrast, $N^{\text{hc}}(\text{Top})_2$ will correspond to triangles commuting up to homotopy together with such a homotopy [26, 00KX], i.e. to quadruples (f, g, h, H) where f, g, h are morphisms in Top and H is a homotopy from $g \circ f$ to h in the sense of simplicially enriched categories [26, 00JW]. In our example, the latter notion happens to coincide with the usual notion of homotopies in Top [26, 00JX]. The upshot is, that we still have the strictly commuting triangles, but also a lot of other ones, which allows for more flexibility.

In certain situations, the homotopy coherent nerve can provide ∞ -categorical versions of homotopy categories:

Remark 2.26 (homotopy categories). — Let C be a *simplicial model category*, i.e. a simplicially enriched category that is also a *model category* in a compatible way. The structure of a model category [6, Def. 2.2.1] that is subject to some conditions, of course, is the following: There is a distinguished class of morphisms W of C , called *weak equivalences*, that one is interested in from a homotopical point of view. Furthermore there are two distinguished classes of auxiliary morphisms, called *fibrations* and *cofibrations*.

There is a well-known construction that assigns to a model category C with weak equivalences W its *homotopy category*, which in turn presents itself as a model for the localization $C[W^{-1}]$, i.e. [6, Def. 2.2.8] the universal category obtained from C by making all morphisms in W isomorphisms. For example, looking at the category Top of topological spaces with W the class of homotopy equivalences, the corresponding homotopy category can be obtained from Top by modding out the “is homotopic to” equivalence relation on all morphism sets.

While 1-categorical homotopy categories are certainly useful, working with them tends to lose information in the sense that the homotopies that make diagrams commutative are not part of the data. They are merely required to exist for any

choice of two parallel morphisms, but not necessarily in a coherent way, see also Remark 2.20.

In the case of the simplicial model category C , this is remedied by an ∞ -categorical enhancement of $C[W^{-1}]$: One takes a certain full subcategory C° of C (which inherits a simplicial enrichment from C) and forms $N^{\text{hc}}(C^\circ)$. This gives an ∞ -category that is tightly linked to the homotopy category of C . (More specifically: one can also define what the homotopy category of an ∞ -category is and if one applies this notion to $N^{\text{hc}}(C^\circ)$ the result will be a 1-category that is equivalent to $C[W^{-1}]$. This can be seen by combining several facts about homotopy categories [26, 00M4], [17, Prop. 9.5.24(2)], [17, Sec. 7.5.6].)

However, while these constructions are certainly useful and applicable in some important cases, they require a lot of structure – which, in applications, is often not naturally available. For example, one might want to consider (model) categories where no extension to a *simplicial* model category exists or is known. And even if there is, the functors one would like to consider can most likely not easily be lifted to respect the simplicial enrichment. Thus, it is useful to have alternative approaches, specifically ones that allow to import a given 1-categorical situation, including functors, more directly. The following notion captures the minimal setting that is necessary to talk about homotopical situations:

DEFINITION 2.27 (relative category). — A *relative category* (C, W) consists of

- a category C , together with
- a class of morphisms W of C , called *weak equivalences*.

A *relative functor*, also called *homotopical functor*, from a relative category (C_1, W_1) to a relative category (C_2, W_2) is a functor $F: C_1 \rightarrow C_2$ that maps weak equivalences to weak equivalences, i.e. with $F(W_1) \subseteq W_2$.

Barwick and Kan have shown [2] that relative categories can be used to model the homotopy theory of ∞ -categories (see also Remark 2.30). Also note, that every model category yields an example of a relative category by simply forgetting the fibrations and cofibrations.

Similarly to Remark 2.26, one is usually not only interested in a 1-categorical localization $C[W^{-1}]$ of a relative category (C, W) , but rather in a better behaved ∞ -categorical version thereof. Classically, this was done by a process called *simplicial localization*, which was introduced and studied in a series of papers by Dwyer and Kan[11, 9, 10]. From this, one gets a simplicially enriched category – but not of the form to which Remark 2.25 applies directly. Instead of trying to rectify this, one can go straight to localizations of ∞ -categories [6, Sec. 7.1]:

Remark 2.28 (localization). — Let (C, W') be a relative category (2.27). Using the terminology of Definition 2.15, we can view W' as a subset of the morphisms of either C itself or of its nerve $N(C)$. Let W denote the smallest simplicial subset of $N(C)$ that contains W' . One can then form the localization $W^{-1}N(C)$, which is an ∞ -category right away [6, Prop. 7.1.3]. (It will have [6, Rem. 7.1.6] the same link to $C[W^{-1}]$ as described at the end of Remark 2.26.)

2.4. Other names and other models. As the final act of this section, we should emphasize, that we merely presented one facet of the theory of ∞ -categories. So as not to confuse the novice reader, we purposely refrained from mentioning this

before; but since its complete concealment might lead to frustration when consulting the literature, we deem it necessary to inform the reader of the seemingly bizarre terminology landscape:

Remark 2.29 (other words for ∞ -category). — In the literature, there are three terms for the *same* object, listed in reverse chronological order of appearance:

- ∞ -category (2.7), as used by Lurie [27],
- *quasi-category*, Joyal’s terminology [20, 21], and
- *weak Kan complex*, introduced by Boardman and Vogt [5, 41].

All of them are still commonly used, although it is customary to annotate the choice of “ ∞ -category” with the phrase “in terms of Lurie” or similar, in order to avoid misunderstandings (see Remark 2.30).

Remark 2.30 (other objects that are called ∞ -category). — On the other hand, there are *different* objects that are also termed “ ∞ -categories” in the literature:

- topologically enriched categories,
- simplicially enriched categories,
- Segal categories,
- complete Segal spaces,
- ...

This stems mainly from the fact, that during the evolution of higher category theory, the term “ ∞ -category” was used to denote the somewhat vague concept of “category with infinitely many levels of morphisms”. To make this notion precise, several people developed different approaches, which resulted (among others) in the objects listed above. Nowadays, it is common to denote the conceptual idea, which they are all a model of, by the term “ $(\infty, 1)$ -category”. While a little longer than “ ∞ -category”, this prevents ambiguities.

For a more comprehensive account of the history and comparison between the different conceptual aspects and models, we refer to the literature [4, 3], [21, Introduction and Sec. 2–5], [27, Ch. 1] and the *nLab* [33], starting at the articles “higher category theory”, “ (n, r) -category”, and “quasi-category”.

3. EXCISION FOR SINGULAR HOMOLOGY

Let us consider singular homology (with arbitrary constant coefficients) on \mathbf{Top} . It satisfies classical excision, e.g. in the sense of having Mayer–Vietoris sequences. As discussed before (1.2), this should rather be seen as a property of the underlying chain complex functor C , and indeed formal arguments then can be used to derive Mayer–Vietoris sequences from this fact, see Section 4. In this section we will show that C really is excisive in the sense of Definition 1.2, i.e. that the latter notion is a generalization of the classical property.

For the entire section, we fix a ring k and a left k -module M . Since neither k nor M play any role in the following, we simply denote by $C := C(\bullet; M)$ the singular chain complex functor with M -coefficients, and by $\mathbf{Ch} := \mathbf{Ch}_k$ the category of chain complexes over k .

THEOREM 3.1 (C is excisive). — *The ∞ -categorical singular chain complex functor $C: \mathbf{Top} \rightarrow \mathbf{Ch}$ is excisive.*

As this is often used as an example or motivation for the definition of an excisive functor, it seems to be an easy fact for homotopy theorists, but an explicit proof seems to be hard to find in the literature, so we will include one here.

3.1. Model category theoretical bits. To prove the main theorem of this section, we will use the formalism of model categories and we will show the following more explicit (1-categorical) version:

PROPOSITION 3.2. — *Homotopy pushout squares in \mathbf{Top} are sent to homotopy pullback squares in \mathbf{Ch} by the singular chain complex functor.*

Here, we consider the *classical* or *Quillen model structure* on \mathbf{Top} [18, Sec. 2.4] and the *projective model structure* on \mathbf{Ch} [18, Sec. 2.3]. Weak equivalences in the former are the weak homotopy equivalences, i.e. continuous maps that induce isomorphism on all homotopy groups [40, Sec. 6.7], and weak equivalence in the latter are the quasi-isomorphisms, i.e. chain maps that induce an isomorphism on homology. No further familiarity with the model structures is required to follow the rest of this and the overall logic of the subsequent sections. However, in the proofs of the latter, we cannot fully avoid to mention (co)fibrations, for example when we need to splice together results from different references. The desperate reader may find an introduction to (the nomenclature of) model categories in Cisinski’s book [6, Sec. 2.2].

We will now give an ad hoc definition of homotopy pushouts in \mathbf{Top} , which can be thought of as “thick pushouts”, as witnessed by the *double mapping cylinder* construction in the below definition. As we shall see shortly, we will actually concern ourselves only with homotopy pushouts of a certain form, but using that as our definition would hide the essence of this homotopy theoretic notion entirely. Thus we have [17, Def. 15.5.8], [32, Def. 3.7.1]:

DEFINITION 3.3 (homotopy pushout (square) in \mathbf{Top}). — A commutative square

$$\begin{array}{ccc} W & \xrightarrow{f} & U \\ g \downarrow & & \downarrow p \\ V & \xrightarrow{q} & X \end{array}$$

in \mathbf{Top} is a *homotopy pushout (square)* if the composition

$$(U \sqcup (W \times [0, 1]) \sqcup V) / \sim \xrightarrow[\substack{\text{“}w \mapsto u, v \mapsto v\text{”} \\ \text{“}(w,t) \mapsto f(w)\text{”}}]{\text{“}w \mapsto u, v \mapsto v\text{”}} U \sqcup_{f,g} V \xrightarrow[p \sqcup_{f,g} q]{} X$$

is a weak equivalence; here

- the first space is the *double mapping cylinder* of f and g , where \sim is the equivalence relation generated by $f(w) \sim (w, 0)$ and $(w, 1) \sim g(w)$ for $w \in W$.
- $U \sqcup_{f,g} V$ is the pushout $(U \sqcup V) / \approx$ where \approx is generated by $f(w) \approx g(w)$ for $w \in W$.
- $p \sqcup_{f,g} q$ is the map induced by p and q by the universal property of the pushout.

Similarly, homotopy pullback squares can be defined in \mathbf{Ch} . For the following, however, we will not need this explicitly.

Example 3.4. — There are homotopy pushouts

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & U \\
 \downarrow & & \downarrow \\
 V & \longrightarrow & U \sqcup V
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 X & \longrightarrow & X \times [0, 1] \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & X
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 X & \longrightarrow & \text{Cone}(X) \\
 \downarrow & & \downarrow \\
 \text{Cone}(X) & \longrightarrow & \Sigma X
 \end{array}$$

for all topological spaces U, V, X (where $\text{Cone}(X)$ is the *cone on X* and ΣX is the *suspension of X*).

Remark 3.5 (philosophy behind homotopy (co)limits). — Generally speaking, homotopy (co)limits make up for the fact that ordinary (co)limits do *not* preserve the notion of weak equivalence. As a concrete example in \mathbf{Top} , we consider the commutative diagram

$$\begin{array}{ccccc}
 * & \longleftarrow & S^0 & \xrightarrow{\text{inclusion}} & D^1 \\
 \downarrow & & \text{id} \downarrow & & \downarrow \\
 * & \longleftarrow & S^0 & \longrightarrow & *
 \end{array}$$

and note, that even though all vertical maps are (weak) homotopy equivalences, the map $D^1/S^0 \rightarrow *$ that they induce on the ordinary pushouts of the rows is *not* a (weak) homotopy equivalence.

Moreover, homotopy (co)limits in a model category are strongly connected to ∞ -categorical (co)limits in the associated ∞ -category, i.e. the localization (2.28) at weak equivalences. For example: the latter compute the former [6, Rem. 7.9.10], (certain) homotopy pushout squares are sent to pushout squares in the ∞ -category by the localization functor [6, dual of Cor. 7.5.20], and in the setting of simplicially enriched categories (see Remark 2.25) a theorem by Lurie compares both notions [27, Thm. 4.2.4.1].

Remark 3.6 (existence and uniqueness of homotopy pushouts). — Given the solid part of the diagram

$$\begin{array}{ccc}
 W & \xrightarrow{f} & U \\
 g \downarrow & & \downarrow \\
 V & \dashrightarrow & ?
 \end{array}$$

one can always extend it to an ordinary pushout square, e.g. with $? = U \sqcup_{f,g} V$ (in the notation of Definition 3.3). However, it may happen, that there is *no* extension to a commutative homotopy pushout square! For example, assume that there is a homotopy pushout

$$\begin{array}{ccc}
 S^0 & \longrightarrow & * \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & X
 \end{array}$$

and denote the corresponding double mapping cylinder by Z . Then the weak equivalence $Z \rightarrow X$ factors through $*$ by definition, but this is impossible as Z is homeomorphic to S^1 .

On the other hand, if there are two homotopy pushouts

$$\begin{array}{ccc} W & \xrightarrow{f} & U \\ g \downarrow & & \downarrow p \\ V & \xrightarrow{q} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \xrightarrow{f} & U \\ g \downarrow & & \downarrow p' \\ V & \xrightarrow{q'} & X' \end{array}$$

with the same W, U, V, f, g , the spaces X and X' must be in the same equivalence class that is generated by weak equivalences, because the double mapping cylinder of f and g is weakly equivalent to both of them. One may thus call (the weak equivalence class of) the double mapping cylinder the *homotopy pushout of the diagram*

$$V \xleftarrow{g} W \xrightarrow{f} U,$$

and this is well-defined, even if there is no extension to a homotopy pushout square.

Example 3.7. — In Example 1.3 we claimed that there is an ∞ -categorical pushout

$$\begin{array}{ccc} S^0 & \xrightarrow{i} & D^1 \\ \downarrow & & \downarrow q \\ * & \longrightarrow & D^1/S^0 \end{array} \quad (\square)$$

in \mathbf{Top} , where i is the inclusion and q the quotient map, that is *not* a pullback. Using the language of homotopy pushouts/pullbacks and the references given in Remark 3.5, we can now give a proof of this:

The given square Equation (\square) is a homotopy pushout and thus sent to an ∞ -pushout by the localization functor. However, it cannot be an ∞ -pullback: Dually to Remark 3.6, there is a well-defined weak equivalence class of spaces that one can call *homotopy pullback of the diagram*

$$* \longrightarrow D^1/S^0 \xleftarrow{q} D^1.$$

Now if we assume, for a contradiction, that Equation (\square) is an ∞ -pullback, the assertion that ∞ -pullbacks compute homotopy pullbacks just means, that S^0 is in this weak equivalence class. We will now show that also the loop space ΩS^1 , at an arbitrary base point $e \in S^1$, is in this equivalence class, which cannot both be true, as we would get

$$2 = \#\pi_0(S^0) = \#\pi_0(\Omega S^1) = \#\pi_1(S^1) = \#\mathbb{Z}$$

as cardinal numbers. To that end, consider the following diagram:

$$\begin{array}{ccccc} * & \longrightarrow & D^1/S^0 & \xleftarrow{q} & D^1 \\ \uparrow & & \uparrow & & \uparrow \\ * & \xrightarrow{e} & S^1 & \xleftarrow{e} & * \end{array}$$

We can certainly find vertical maps such that they are all weak equivalences and such that both squares commute. Now because the homotopy pullback of diagrams *does* respect such a “weak equivalence of diagrams” [17, Prop. 13.3.4] (compare Remark 3.5 for ordinary limits), the homotopy pullbacks of the rows yield the same weak equivalence class. Using an explicit definition of homotopy pullback [32,

Def. 3.2.4] it is then easy to see, that the homotopy pullback of the diagram in the bottom row is given by ΩS^1 [32, Ex. 3.2.10].

In the rest of this section, we will give a proof of Proposition 3.2 and then show how this implies Theorem 3.1.

3.2. Proof of the model categorical statement. First of all, we may reduce our attention to homotopy pushouts of a specific type by virtue of the following three facts:

PROPOSITION 3.8 ([32, Prop. 3.7.4 and Ex. 3.7.5]). — *Let S' be a homotopy pushout square in \mathbf{Top} . Then there exists a homotopy pushout square*

$$S = \begin{pmatrix} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{pmatrix}$$

and a map of squares $\eta: S \rightarrow S'$, such that

- U and V are open subsets of X and $W = U \cap V$,
- all maps in S are inclusions, and
- η is component-wise a weak equivalence.

Furthermore, any square S as above is a homotopy pushout square.

PROPOSITION 3.9 ([16, Prop. 4.21]). — *The singular chain complex functor is homotopical, i.e. it sends weak equivalences in \mathbf{Top} to weak equivalences in \mathbf{Ch} .*

PROPOSITION 3.10. — *Being a homotopy pullback square in \mathbf{Ch} is invariant under weak equivalences of squares in the following sense: Given two squares S and S' in \mathbf{Ch} and a map of diagrams $S \rightarrow S'$ that is component-wise a weak equivalence, then S is a homotopy pullback if and only if S' is.*

Proof. — This is Proposition 13.3.13 in Hirschhorn's book [17]. Its prerequisites are fulfilled because every object in \mathbf{Ch} is fibrant [18, Thm. 2.3.11 and subsequent paragraph] and thus \mathbf{Ch} is right proper [17, Cor. 13.1.3]. \square

With this, we are left to show:

PROPOSITION 3.11. — *Let S be as in Proposition 3.8. Then the induced square*

$$\begin{array}{ccc} C(W) & \longrightarrow & C(U) \\ \downarrow & & \downarrow \\ C(V) & \longrightarrow & C(X) \end{array}$$

is a homotopy pullback in \mathbf{Ch} .

Proof. — Let $C(S)$ denote the square from the claim and let $\iota: C^{U,V}(X) \hookrightarrow C(X)$ be the inclusion of the subcomplex generated by the singular simplices that are contained in U or V . Then it is well-known [16, Prop. 2.21] that ι is a weak equivalence (this is the core of the proof of classical excision statements). Now we consider the square

$$\left(\begin{array}{ccc} C(W) & \xrightarrow{i_{WU}} & C(U) \\ \downarrow i_{WV} & & i_{UX} \downarrow \\ C(V) & \xrightarrow{i_{VX}} & C^{U,V}(X) \end{array} \right) =: S^{U,V}$$

where all morphisms denote the canonical injections and the component-wise inclusion $S^{U,V} \rightarrow C(S)$, which is component-wise a weak equivalence. Hence, Proposition 3.10 applies and we can reduce to the problem of showing that $S^{U,V}$ is a homotopy pullback.

It is known, that \mathbf{Ch} is a *stable* model category [18, Sec. 7.1], so we may equivalently [18, Rem. 7.1.12] show that $S^{U,V}$ is a homotopy *pushout*. But in the model structure on \mathbf{Ch} , all objects in $S^{U,V}$ are cofibrant [18, Lem. 2.3.6] and the morphism i_{WU} is a cofibration by Lemma 3.12 (see below), hence it suffices [37, Ex. 8.8] to show that $S^{U,V}$ is a pushout. It is easily seen that this is the case if (and only if)

$$C(W) \xrightarrow{(i_{WU}, i_{WV})} C(U) \oplus C(V) \xrightarrow{i_{UX} - i_{VX}} C^{U,V}(X) \longrightarrow 0$$

is an exact sequence in \mathbf{Ch} . But this is true, basically by construction of $C^{U,V}(X)$, so we are done. \square

This finishes the proof of Proposition 3.2. In the last section we promote this model categorical result to the ∞ -categorical setting.

In the proof of the previous proposition, we made use of the following lemma, for which we include a proof for the sake of completeness. The reader who does not want to dive into the inner workings of the model category \mathbf{Ch} may safely skip the proof and accept it as a technical fact.

LEMMA 3.12 (*C* of injection). — *Let $i: A \rightarrow X$ be an injective map in \mathbf{Top} . Then $C(i): C(A) \rightarrow C(X)$ is a cofibration in \mathbf{Ch} .*

Proof. — By definition of the singular chain complex functor, $C(A)$ and $C(X)$ are composed of free modules, the chain map $C(i)$ is dimensionwise injective and maps a basis of $C(A)$ to a subset of a basis of $C(X)$. Thus, $C(i)$ is dimensionwise a split injection with free cokernel, hence a cofibration in \mathbf{Ch} [18, Prop. 2.3.9 and Lem. 2.3.6]. \square

3.3. Proof of the ∞ -categorical statement.

Proof that Proposition 3.2 implies Theorem 3.1. — To make this proof more transparent, let us fix the following notation: \mathbf{Top} denotes the 1-category, \mathbf{Top}_∞ the corresponding ∞ -category, i.e. the localization (2.28) at weak homotopy equivalences, similarly, \mathbf{Ch} denotes the 1-category, \mathbf{Ch}_∞ the ∞ -category, i.e. the localization at quasi-isomorphisms, and $C: \mathbf{Top} \rightarrow \mathbf{Ch}$ is the 1-categorical functor, while $C_\infty: \mathbf{Top}_\infty \rightarrow \mathbf{Ch}_\infty$ is the induced functor on ∞ -categories (which is well-defined since C is homotopical, Proposition 3.9).

Both, \mathbf{Ch} and \mathbf{Ch}_∞ are *stable*, \mathbf{Ch} as a model category [18, Sec. 7.1] and \mathbf{Ch}_∞ as ∞ -category [28, Prop. 1.3.5.9 (with Prop. 1.3.5.15)]. In particular, a square in \mathbf{Ch} is a homotopy pullback if and only if it is a homotopy pushout [18, Rem. 7.1.12], and a square in \mathbf{Ch}_∞ is a pullback if and only if it is a pushout [28, Prop. 1.1.3.4].

With this, we see that C_∞ being excisive is equivalent to C_∞ commuting with pushouts. Hence, it suffices to show that C_∞ commutes with *all* finite colimits. This follows from the dual of Proposition 7.5.28 in Cisinski’s book [6], applied to the composition $\bar{C}: \mathbf{N}(\mathbf{Top}) \rightarrow \mathbf{Ch}_\infty$ of $\mathbf{N}(C)$ and the localization functor $\mathbf{N}(\mathbf{Ch}) \rightarrow \mathbf{Ch}_\infty$; for this, \mathbf{Ch}_∞ is made an ∞ -category with weak equivalences and cofibrations in a trivial way [6, Ex. 7.5.4]. To apply the proposition, we just need to show that \bar{C} is a *right exact* [6, Def. 7.5.2] functor:

- (i) As $C(\emptyset) \cong 0$, initial objects are preserved by \bar{C} .
- (ii) All morphisms in \mathbf{Ch}_∞ are cofibrations and C is homotopical (3.9). Thus, \bar{C} preserves cofibrations and trivial cofibrations.
- (iii) Lastly, we need to check that pushout squares (in \mathbf{Top}) of the form

$$\begin{array}{ccc} A & \xrightarrow{\quad} & X \\ \downarrow & \scriptstyle i & \downarrow \\ A' & \longrightarrow & X' \end{array}$$

with i a cofibration and A and A' (and thus also X) cofibrant are sent to pushout squares by \bar{C} . But these conditions ensure [37, Ex. 8.8] that such a square is also a homotopy pushout square, which is preserved by C by combining Proposition 3.2 and the comment about stability of \mathbf{Ch} above. Furthermore, by the first item and Lemma 3.12, C also preserves the additional properties of i , A and A' , so the square induced by C is sent to a pushout square in \mathbf{Ch}_∞ [6, dual of Cor. 7.5.20].

□

4. RELATION BETWEEN ABSTRACT AND CLASSICAL EXCISION

In this section, we relate abstract excision (1.2) to classical excision in the form of Mayer–Vietoris sequences. To this end, we will explain why and to some extent how an excisive functor always admits Mayer–Vietoris sequences, coinciding with the usual one in the case of singular homology, for example.

If our excisive functor is defined on \mathbf{Top} , the first thing to note is that by the last part of Proposition 3.8 we may indeed apply the abstract excision property in the setting $X = U \cup V$ with U and V open in X .

This leaves us with the task of explaining how a pullback square in an ∞ -category leads to a long exact sequence. To this end, the first step is the following:

4.1. Obtaining a fiber sequence. Let C be an ∞ -category with finite limits and a zero object, i.e. an object $*$ of C that is both initial and terminal. Suppose we start with the following pullback in C :

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

Then from this, one can derive a fiber sequence like this:

$$\Omega X \rightarrow W \rightarrow U \times V, \tag{4.1}$$

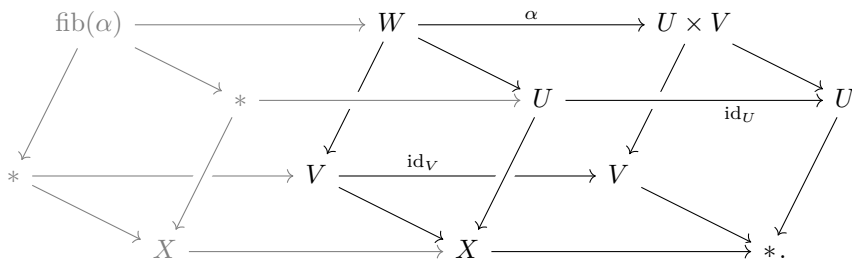
where Ω denotes the loop functor. Here, being a fiber sequence means (by definition) being a pullback square

$$\begin{array}{ccc} \Omega X & \longrightarrow & W \\ \downarrow & & \downarrow \\ * & \longrightarrow & U \times V \end{array}$$

and the loop functor $\Omega: C \rightarrow C$ assigns to an object X the pullback of $* \rightarrow X \leftarrow *$, or in other words it completes $* \rightarrow X$ to a fiber sequence $\Omega X \rightarrow * \rightarrow X$. Lurie

describes in detail how to construct loop and suspension functors [28, paragraph preceding Rem. 1.1.2.6].

For a precise treatment of the passage from pullback to fiber sequence, we would need more technicalities about limits in ∞ -categories. Informally, though, the basic ideas are as follows: We have two pullback squares and a morphism $\alpha: W \rightarrow U \times V$ induced by the universal property of the product as in the right part of this diagram:



Taking fibers (also a limit!) in the horizontal direction and using the fact that limits commute with limits [27, dual of Lem. 5.5.2.3], as is the case in 1-categories, we obtain that the left square is also a pullback. Hence, by definition of the loop functor, $\text{fib}(\alpha)$ is (equivalent to) ΩX .

4.2. Applying the mapping space functor. Let C be a 1-category. A quite important notion in 1-category theory is the functor

$$\text{Mor}_C: C^{\text{op}} \times C \rightarrow \text{Set}$$

that takes a pair (x, y) of objects of C to the set of morphisms $\text{Mor}_C(x, y)$. In the literature, this is often called ‘‘Hom-functor’’.

Now let C be an ∞ -category. One slogan of ∞ -category theory is: the role of sets in 1-category theory is taken by topological spaces in ∞ -category theory. This motivates the desire to find, for a pair of objects (x, y) of C , not only a set of morphisms $x \rightarrow y$, but instead a *space* of such morphisms – and indeed, this is always possible. The following remark indicates how, but its point is rather to illustrate the analogy between the 1- and ∞ -categorical situation than to provide the reader with a better understanding of the space itself. Thus, it can also safely be skipped.

Remark 4.1 (construction of mapping spaces). — We consider the 1-categorical situation first. Here, the set $\text{Mor}_C(x, y)$ can be found as a pullback of sets as in the following pullback square:

$$\begin{array}{ccc} \text{Mor}_C(x, y) & \longrightarrow & \text{Mor}(C) \\ \downarrow & & \downarrow (f: a \rightarrow b) \\ \{0\} & \xrightarrow{0 \rightarrow (x, y)} & \text{Ob}(C) \times \text{Ob}(C) \end{array}$$

where $\text{Mor}(C)$ and $\text{Ob}(C)$ denote the class of morphisms and objects of C , respectively.

Now if C is instead an ∞ -category, we can form the analogous diagram

$$\begin{array}{ccc} & & C^{\Delta^1} \\ & & \downarrow ((d_1^0)^*, (d_0^0)^*) \\ \Delta^0 & \xrightarrow{*(x,y)} & C^{\Delta^0} \times C^{\Delta^0} \end{array}$$

in \mathbf{sSet} and *define* the space of morphisms from x to y as (the geometric realization of) its pullback; here we use the following notation:

- C^{Δ^n} is the *internal Hom* simplicial set as in Remark 2.9,
- $(d_i^0)^*: C^{\Delta^1} \rightarrow C^{\Delta^0}$ is the morphism induced by the i^{th} coface map d_i^0 , and
- $*(x,y)$ denotes the unique morphism with

$$(\Delta^0)_0 \ni \text{Id}_{[0]} \mapsto (x, y) \in C_0 \times C_0$$

under the identification

$$C_0 \times C_0 \cong \text{Mor}_{\mathbf{sSet}}(\Delta^0, C) \times \text{Mor}_{\mathbf{sSet}}(\Delta^0, C) = (C^{\Delta^0} \times C^{\Delta^0})_0$$

from Example 2.18.

Dugger and Spivak [7] discuss multiple ways to construct mapping spaces and in particular they consider the construction above [7, Prop. 1.2].

If properly organized [6, Sec. 5.8], one can even define a functor of ∞ -categories

$$\text{Map}_C: C^{\text{op}} \times C \rightarrow \mathbf{Top},$$

which we call Map_C to distinguish it from the 1-categorical concept. (Note, however, that $\text{Map}_C(x, y)$ will in general only be weakly equivalent to the space that we constructed in Remark 4.1. Also, in Cisinski’s book the target of the mapping space functor is “ \mathcal{S} ”, so to get one to \mathbf{Top} , we have to compose with an equivalence between those two ∞ -categories [6, Rem. 7.8.10 and Rem. 7.8.11].) For this functor one can show:

PROPOSITION 4.2 ([6, Cor. 6.3.5]). — *Let C be a locally small ∞ -category and let x be an object of C . Then the functor $\text{Map}_C(x, \bullet): C \rightarrow \mathbf{Top}$ preserves limits.*

In particular, since these are defined via pullbacks, $\text{Map}_C(x, \bullet)$ maps fiber sequences to fiber sequences. This gives us a method to convert the fiber sequence Equation (4.1) from Section 4.1 to one in \mathbf{Top} : for any test object S of C we obtain

$$\text{Map}_C(S, \Omega X) \rightarrow \text{Map}_C(S, W) \rightarrow \text{Map}_C(S, U \times V), \tag{4.2}$$

a fiber sequence of spaces, and canonically also of pointed spaces by virtue of zero maps.

4.3. The Mayer–Vietoris sequence. As is well-known, a fiber sequence $F \rightarrow X \rightarrow Y$ of pointed spaces always yields a long exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(X) \rightarrow \pi_n(Y) \rightarrow \pi_{n-1}(F) \rightarrow \cdots .$$

Using this on the fiber sequence (4.2) from the last section, again the fact that $\text{Map}_C(S, \bullet)$ preserves limits (4.2), and well-known facts about the homotopy groups, we get the following exact sequence:

$$\begin{aligned} \cdots \rightarrow \pi_{n+1}\mathrm{Map}_C(S, X) &\rightarrow \pi_n\mathrm{Map}_C(S, W) \\ &\rightarrow \pi_n\mathrm{Map}_C(S, U) \times \pi_n\mathrm{Map}_C(S, V) \rightarrow \pi_n\mathrm{Map}_C(S, X) \rightarrow \cdots \end{aligned} \quad (4.3)$$

In this sense, one can always extract Mayer–Vietoris type sequences from a pullback square.

4.4. Example: singular homology. We shall now explain, how the long exact sequence Equation (4.3) from Section 4.3 about homotopy groups also yields the familiar Mayer–Vietoris sequence of *homology* groups. For this we have to use the “correct” object S plus some additional arguments. In the following, we reuse the notation from the beginning of Section 3.

Let us first recapitulate the situation: We have a space $X = U \cup V$ with U and V open in X , $W = U \cap V$, and $C: \mathbf{Top} \rightarrow \mathbf{Ch}$ the singular chain complex functor with arbitrary constant coefficients over some ring k . From the pushout formed by $U \cap V$, U , V and X , we get the according pullback of chain complexes. We now apply the previous material, where for the test complex S we choose the complex $k[0]$ that has k in degree 0 and is trivial elsewhere. Note, that $k[0]$ is just one usual notation for this and no connection to the preorder category $[0]$ from Definition 2.2 is intended. Looking at the sequence (4.3), we then get a lot of terms of the form

$$\pi_n\mathrm{Map}_{\mathbf{Ch}}(k[0], C(Z))$$

for $n \in \mathbb{N}$ and Z one of W , U , V , X , which can be computed via the following:

PROPOSITION 4.3. — *Let c be a chain complex and let $n \in \mathbb{N}$. Then:*

$$\pi_n\mathrm{Map}_{\mathbf{Ch}}(k[0], c) \cong H_n(c)$$

where the right hand side is the n^{th} homology of c .

Proof. — We split the proof as follows:

- (1) In the first part, we show the claim with \mathbf{Ch} replaced by a weaker localization, namely by the localization at chain homotopy equivalences (instead of all quasi-isomorphisms).
- (2) In the second part, we pass from this intermediate ∞ -category of chain complexes to \mathbf{Ch} . In contrast to the first part, which provides the link between mapping spaces and homology, this part can be regarded as a technicality which shows that mapping out of $k[0]$ provides the same homotopical view on other objects in both localizations.

As both steps use technology that was not previously introduced in this article, we do *not* expect the reader to understand all the details. However, we try hard to conceptually explain what happens and provide necessary literature pointers.

Ad 1. We can [26, 00PE] view the 1-category \mathbf{Ch} as a *DG-category*, which we denote by $\mathbf{Ch}^{\mathrm{dg}}$; this means [26, 00P9] that $\mathbf{Ch}^{\mathrm{dg}}$ is enriched over chain complexes of Abelian groups, i.e. $\mathrm{Mor}_{\mathbf{Ch}^{\mathrm{dg}}}(c', c)$ is an object of $\mathbf{Ch}_{\mathbb{Z}}$ for all chain complexes c', c . Then the so-called *DG-nerve* [26, 00PK], a gadget similar in spirit to the homotopy coherent nerve from Remark 2.25, provides us with an ∞ -category $\mathbf{N}^{\mathrm{dg}}(\mathbf{Ch}^{\mathrm{dg}})$, which can be seen as the localization (2.28) of \mathbf{Ch} at chain homotopy equivalences [28,

Prop. 1.3.4.5]. Mapping spaces in ∞ -categories obtained via \mathbb{N}^{dg} are particularly tractable [28, Rem. 1.3.1.12], hence

$$\text{Map}_{\mathbb{N}^{\text{dg}}(\text{Ch}^{\text{dg}})}(k[0], c) \simeq |\mathbb{K}(\tau_{\geq 0}\text{Mor}_{\text{Ch}^{\text{dg}}}(k[0], c))|,$$

where

- $|\bullet|$ denotes the geometric realization of a simplicial set (in this case: after forgetting the group structure),
- \mathbb{K} sends a non-negative chain complex to its associated simplicial Abelian group under the *Dold-Kan correspondence* [42, Sec. 8.4],
and
- $\tau_{\geq 0}$ is the good truncation functor [42, 1.2.7] that essentially forces a chain complex to be non-negative.

Thus, we see

$$\begin{aligned} \pi_n \text{Map}_{\mathbb{N}^{\text{dg}}(\text{Ch}^{\text{dg}})}(k[0], c) &\cong \pi_n \mathbb{K} \tau_{\geq 0} \text{Mor}_{\text{Ch}^{\text{dg}}}(k[0], c) \\ &\cong H_n(\text{Mor}_{\text{Ch}^{\text{dg}}}(k[0], c)) \\ &\cong H_n(c), \end{aligned}$$

because taking the homotopy group of a geometric realization is isomorphic to the simplicial homotopy group [13, p. 60] and the latter is isomorphic to the homology of the associated chain complex [42, Thm. 8.4.1]; finally, $\text{Mor}_{\text{Ch}^{\text{dg}}}(k[0], c) \cong c$ is easily seen by expanding the definitions. This shows the claim for $\mathbb{N}^{\text{dg}}(\text{Ch}^{\text{dg}})$ instead of Ch .

Ad 2. To pass from $\mathbb{N}^{\text{dg}}(\text{Ch}^{\text{dg}})$ to Ch (i.e. from localization at chain homotopy equivalences to localization at all quasi-isomorphisms), we additionally use the so-called *Hurewicz model structure* on Ch , which has chain homotopy equivalences as weak equivalences. It can be combined with the projective model structure to yield a *mixed model structure* [30, Sec. 17.3]. All three model structures are discussed in May and Ponto's book under the names "*h-q-m-model structure*" [30, Sec. 18.3, 18.4 and 18.6]. Both, projective and mixed, have the quasi-isomorphisms as weak equivalences, but the latter has the advantage of being a right Bousfield localization of the Hurewicz model structure. This can be used to see that the restriction of a mapping space functor in the sense of Cisinski [6, (dual/cofibrant version of) 7.10.7] for the Hurewicz model structure yields a mapping space functor for the mixed model structure. Together with the fact that $k[0]$ is cofibrant in all three model structures [30, Thm. 18.3.1/Prop. 18.5.2(iii)/Sec. 18.6], we get

$$\text{Map}_{\mathbb{N}^{\text{dg}}(\text{Ch}^{\text{dg}})}(k[0], c) \simeq \text{Map}_{\text{Ch}}(k[0], c). \quad \square$$

If we apply this proposition to the long exact sequence (4.3) that we get from Section 4.3 for $S = k[0]$, we indeed arrive at the familiar long exact sequence:

$$\begin{aligned} \cdots \rightarrow H_{n+1}(C(X)) \rightarrow H_n(C(U \cap V)) \\ \rightarrow H_n(C(U)) \times H_n(C(V)) \rightarrow H_n(C(X)) \rightarrow \cdots \end{aligned}$$

ACKNOWLEDGMENTS

The author was supported by the CRC 1085 *Higher Invariants* (Universität Regensburg, funded by the DFG). He is grateful to Clara Löh, Hoang Kim Nguyen, George Raptis and Georg Biedermann for helpful conversations and their comments

on earlier versions of this article. He further appreciates the detailed feedback by the anonymous referee.

REFERENCES

- [1] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories. The Joy of Cats*. John Wiley & Sons, 1990. addendum: Most recent version can be found here: <http://katmat.math.uni-bremen.de/acc/>.
- [2] C. Barwick and D. M. Kan. Relative categories: Another model for the homotopy theory of homotopy theories. *Indag. Math., New Ser.*, 23(1–2):42–68, 2012.
- [3] Julia E. Bergner. A survey of $(\infty, 1)$ -categories. In *Towards Higher Categories*, volume 152 of *IMA Volumes in Mathematics and its Applications*, pages 69–83. Springer, 2010.
- [4] Julia E. Bergner. *The Homotopy Theory of $(\infty, 1)$ -Categories*, volume 90 of *London Mathematical Society Student Texts*. Cambridge University Press, 2018.
- [5] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347 of *Lecture Notes in Mathematics*. Springer, 1973.
- [6] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2019. addendum: Last version and errata available at <http://www.mathematik.uni-regensburg.de/cisinski/publikationen.html>.
- [7] Daniel Dugger and David I. Spivak. Mapping spaces in quasi-categories. *Algebr. Geom. Topol.*, 11(1):263–325, 2011.
- [8] Daniel Dugger and David I. Spivak. Rigidification of quasi-categories. *Algebr. Geom. Topol.*, 11(1):225–261, 2011.
- [9] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980.
- [10] W. G. Dwyer and D. M. Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.
- [11] W. G. Dwyer and D. M. Kan. Simplicial localizations of categories. *J. Pure Appl. Algebra*, 17(3):267–284, 1980.
- [12] Greg Friedman. Survey article: An elementary illustrated introduction to simplicial sets. *Rocky Mt. J. Math.*, 42(2):353–423, 2012.
- [13] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Modern Birkhäuser Classics. Birkhäuser, reprint of the 1999 original edition, 2010.
- [14] Michael Gromov. Volume and bounded cohomology. *Publ. Math., Inst. Hautes Étud. Sci.*, 56:5–99, 1982.
- [15] Moritz Groth. A short course on ∞ -categories. <https://arxiv.org/abs/1007.2925>, 2015.
- [16] Allen E. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. <https://www.math.cornell.edu/~hatcher/AT/ATpage.html>.
- [17] Philip S. Hirschhorn. *Model Categories and Their Localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2003.
- [18] Mark Hovey. *Model Categories*. Mathematical Surveys and Monographs. American Mathematical Society, 1999.
- [19] Nikolai V. Ivanov. Notes on the bounded cohomology theory. <https://arxiv.org/abs/1708.05150>, 2017.
- [20] André Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1):207–222, 2002.
- [21] André Joyal. Notes on Quasi-Categories, 2008. <http://www.math.uchicago.edu/~may/IMA/Joyal.pdf>.
- [22] Clara Löh. Simplicial Volume. Manifold Atlas Project, http://www.map.mpim-bonn.mpg.de/Simplicial_volume.
- [23] Clara Löh. *ℓ^1 -Homology and Simplicial Volume*. PhD thesis, WWU Münster, 2007. addendum: available at <http://nbn-resolving.de/urn:nbn:de:hbz:6-37549578216> or (with errata) at <https://loeh.app.uni-regensburg.de/preprints/>.
- [24] Clara Löh. Isomorphisms in ℓ^1 -homology. *Münster J. Math.*, 1:237–265, 2008.
- [25] Zhen Lin Low. Universes for category theory. <https://arxiv.org/abs/1304.5227>, 2013.

- [26] Jacob Lurie. Kerodon. <https://kerodon.net>, an online resource for homotopy-coherent mathematics.
- [27] Jacob Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, 2009. addendum: Most recent version can be found here: <https://www.math.ias.edu/~lurie/>.
- [28] Jacob Lurie. Higher Algebra, 2017. <https://www.math.ias.edu/~lurie/>.
- [29] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, second edition, 1998.
- [30] J. P. May and K. Ponto. *More Concise Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 2012.
- [31] Nicolas Monod. An invitation to bounded cohomology. In *Proceedings of the International Congress of Mathematicians (ICM)*. Vol. II, pages 1183–1211. EMS Publishing House, 2006. <https://egg.epfl.ch/~nmonod/articles/icm.html>.
- [32] Brian A. Munson and Ismar Volić. *Cubical Homotopy Theory*, volume 25 of *New Mathematical Monographs*. Cambridge University Press, 2015.
- [33] nLab authors. nLab. <https://ncatlab.org/nlab/show/HomePage>, A Wiki for collaborative work on Mathematics, Physics, and Philosophy.
- [34] George Raptis. Bounded cohomology and homotopy colimits. <https://arxiv.org/abs/2103.15614>, 2021.
- [35] Emily Riehl. Homotopical categories: from model categories to $(\infty, 1)$ -categories. In *Stable categories and structured ring spectra*, volume 69 of *Mathematical Sciences Research Institute Publications*, pages 5–74. Cambridge University Press, 2022.
- [36] Emily Riehl. Homotopy coherent structures. *Expositions in Theory and Applications of Categories*, (1):1–31, 2023.
- [37] Emily Riehl and Dominic Verity. The theory and practice of Reedy categories. *Theory Appl. Categ.*, 29:256–301, 2014.
- [38] Michael A. Shulman. Set theory for category theory. <https://arxiv.org/abs/0810.1279>, 2008.
- [39] Robert M. Switzer. *Algebraic Topology – Homotopy and Homology*. Classics in Mathematics. Springer, 2002. Reprint of the 1975 original.
- [40] Tammo tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society, 2008. addendum: Errata at <https://www.uni-math.gwdg.de/tammo/algtop.html>.
- [41] Rainer M. Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973.
- [42] Charles A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1995. addendum: Errata at <https://sites.math.rutgers.edu/~weibel/Hbook-corrections.html>.
- [43] Johannes Witzig. *Universality and Examples in the Context of Functorial Semi-Norms*. PhD thesis, University of Regensburg, 2023. <https://epub.uni-regensburg.de/54396>.

Manuscript received 20th June 2022,
 revised 25th May 2023,
 accepted 9th November 2023.

Johannes WITZIG
 Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
 Johannes.Witzig@mathematik.uni-regensburg.de