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CONVEXITY, PLURISUBHARMONICITY AND THE STRONG MAXIMUM MODULUS PRINCIPLE IN BANACH SPACES

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Abstract. In this article, we first try to make the known analogy between convexity and plurisubharmonicity more precise. Then we introduce a notion of strict plurisubharmonicity analogous to strict convexity, and we show how this notion can be used to study the strong maximum modulus principle in Banach spaces. As an application, we define a notion of L^p direct integral of a family of Banach spaces, which includes at once Bochner L^p spaces, ℓ^p direct sums and Hilbert direct integrals, and we show that under suitable hypotheses, when $p < \infty$, an L^p direct integral satisfies the strong maximum modulus principle if and only if almost all members of the family do. This statement can be considered as a rewording of several known results, but the notion of strict plurisubharmonicity yields a new proof of it, which has the advantage of being short, enlightening and unified.

1. Introduction

1.1. **Convexity and plurisubharmonicity.** Plurisubharmonic functions (PSH, for short) were introduced independently by Oka [24, p. 40] and Lelong [20, p. 306, déf. 1]. Since the origins of the theory, it has been observed that there is a certain analogy between convex functions and PSH functions; in fact, Oka called the latter *pseudoconvex functions*. To make this analogy apparent, Bremermann [4, pp. 34-38] collected a list of properties satisfied by convex functions, and showed that for each of them, there is a corresponding property satisfied by PSH functions.

Here are Bremermann's main ideas, slightly reformulated. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex, or *sublinear*, if for all compact intervals $I \subset \mathbb{R}$ and for all affine functions $\alpha : I \to \mathbb{R}$, the inequality $f \leqslant \alpha$ holds on I as soon as it holds on the boundary of I. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex, or *plurisublinear*, if for all affine maps $\gamma : \mathbb{R} \to \mathbb{R}^n$, the composition $f \circ \gamma$ is convex.

In the same way, an upper semicontinuous function $f : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is said to be subharmonic if for all connected, smoothly bounded compact sets $K \subset \mathbb{C}$ and for all functions $\alpha : K \to \mathbb{R}$ continuous and harmonic in the interior of K, the inequality $f \leqslant \alpha$ holds on K as soon as it holds on the boundary of K. An upper semicontinuous function $f : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be plurisubharmonic if for all affine maps $\gamma : \mathbb{C} \to \mathbb{C}^n$, the composition $f \circ \gamma$ is subharmonic.

As Bremermann remarks, convexity and plurisubharmonicity can be defined more quickly in the following way. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for all affine maps $\gamma : \mathbb{R} \to \mathbb{R}^n$,

$$f(\gamma(0)) \leqslant \frac{f(\gamma(-1)) + f(\gamma(1))}{2}. \tag{1.1}$$

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This amounts to asking that the value of $f \circ \gamma$ at the centre of the unit ball of $\mathbb{R}$ be less than its mean on the sphere. Similarly, an upper semicontinuous function $f : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is PSH if and only if for all affine maps $\gamma : \mathbb{C} \to \mathbb{C}^n$,

$$f(\gamma(0)) \leqslant \frac{1}{2\pi} \int_0^{2\pi} f\left(\gamma(e^{it})\right) \, \mathrm{d}t, \tag{1.2}$$

which amounts to asking that the value of $f \circ \gamma$ at the centre of the unit ball of $\mathbb{C}$ be less than its mean on the sphere.

From the above discussion, it is tempting to conclude, as does Bremermann, that the analogy between convex functions and PSH functions is obtained by replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, real affine maps with complex affine maps, and sublinearity conditions with subharmonicity conditions. These ideas permeate much of the literature on the subject; see for instance [16, p. 225]. Yet we will show that this dictionary misses an essential point and therefore is unsatisfactory.

Indeed, a fundamental result, due to Lelong [20, p. 325, no. 17], states that PSH functions are stable under composition with a holomorphic map, from which one can define the notion of PSH function on a holomorphic manifold. This result does not appear in Bremermann's list, which is understandable, since from his point of view, there is no analogous result for convex functions.

In fact, the natural domain of a convex function is a real affine space, while the natural domain of a PSH function is a holomorphic manifold, or even a complex analytic space[1]. Therefore, real affine maps do not correspond to complex affine maps, but rather to holomorphic maps.

Thus it is preferable to define the notion of PSH function in the following way: if X is a holomorphic manifold, an upper semicontinuous function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be PSH if the inequality (1.2) holds for all *holomorphic* maps $\gamma : \overline{\mathbb{D}} \to X$, where $\overline{\mathbb{D}} \subset \mathbb{C}$ is the closed unit disc, with the understanding that a map defined on $\overline{\mathbb{D}}$ is holomorphic if it extends to a holomorphic map on a neighbourhood of $\overline{\mathbb{D}}$.

In the case $X = \mathbb{C}^n$, if f satisfies this inequality for all *affine* maps $\gamma : \overline{\mathbb{D}} \to \mathbb{C}^n$, then f is PSH: this is the contents of Lelong's result. But this fact is best seen, not as a definition, but rather as a characterisation, valid in a special case, and which is far from obvious. We will not use it in this article.

Beyond the aesthetic aspect, a good understanding of the analogy between convexity and plurisubharmonicity enables one to obtain certain non-trivial results about PSH functions by adapting the proofs of the corresponding, usually easier, statements concerning convex functions. We hope that the results presented in this article will serve to illustrate this phenomenon.

1.2. **Strict plurisubharmonicity.** A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if the inequality (1.1) holds strictly for all non-constant affine maps $\gamma : \mathbb{R} \to \mathbb{R}^n$. According to Bremermann's dictionary, this suggests the following definition:

[1]The approach taken in this article would allow one to define the notion of convex function more generally on any topological space X, as soon as one chooses a class of continuous maps $\gamma : [-1; 1] \to X$ playing the role of affine maps. An important case is when X is a Riemannian manifold and the maps γ are geodesic segments. In the same way, one could define the notion of PSH function on any topological space X equipped with a class of maps $\gamma : \overline{\mathbb{D}} \to X$ playing the role of holomorphic maps.

an upper semicontinuous function $f : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is strictly PSH if the inequality (1.2) holds strictly for all non-constant affine maps $\gamma : \mathbb{C} \to \mathbb{C}^n$. This is, with a different formulation, Carmignani's definition [5, pp. 285-286, Defs. 1.1 and 1.2][2]. However, this definition is very unsatisfactory: indeed, we will see an example showing that the class of functions thus obtained is not stable under composition with a biholomorphism.

It is therefore preferable, according to the principles explained above, to say that an upper semicontinuous function $f : X \to \mathbb{R} \cup \{-\infty\}$ on a holomorphic manifold X is strictly PSH if the inequality (1.2) holds strictly for all non-constant *holomorphic* maps $\gamma : \overline{\mathbb{D}} \to X$. We will see several results showing that strict plurisubharmonicity, thus defined, is a natural notion, analogous to strict convexity.

In the real case, there exists a stronger notion than strict convexity: a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly convex if it can be written locally as the sum of a convex function and a $\mathcal{C}^2$ function ε such that $d^2\varepsilon$ is a positive definite symmetric bilinear form at every point.

The analogous notion in the complex case is the following: a function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be strongly PSH if it can be written locally as the sum of a PSH function and a $\mathcal{C}^2$ function ε such that $\partial\overline{\partial}\varepsilon$ is a positive definite hermitian form at every point.

Unfortunately, it is a common practice in the literature to call strongly PSH functions *strictly PSH*. This situation is unhappy, because strong plurisubharmonicity is analogous to strong convexity, and not to strict convexity.

1.3. **The strong maximum modulus principle in Banach spaces.** An $\mathbb{R}$-Banach space E is said to be strictly convex if every affine map $\gamma : [-1; 1] \to E$ whose image is contained in the unit sphere is constant.

By analogy, it might be tempting to say that a $\mathbb{C}$-Banach space E is strictly convex in the complex sense if every affine map $\gamma : \overline{\mathbb{D}} \to E$ whose image is contained in the unit sphere is constant. This definition was strongly suggested by Thorp and Whitley [25], and explicitly given by Globevnik [14, p. 175, Def. 1].

However, the analogy turns out to be more satisfying if one asks instead that every *holomorphic* map $\gamma : \overline{\mathbb{D}} \to E$ whose image is contained in the unit sphere be constant. In this case, in order to keep the terminology consistent, we will say that E is strictly PSH.

The main result of Thorp and Whitley [25, p. 641, Th. 3.1], slightly reformulated, is that both definitions are actually equivalent, that is, that E is strictly PSH if and only if every *affine* map $\gamma : \overline{\mathbb{D}} \to E$ whose image is contained in the unit sphere is constant. But this fact is best seen, not as a definition, but rather as a non-trivial characterisation. We will not use it in this article.

Beyond the analogy with strictly convex spaces, the importance of strictly PSH spaces lies in the fact that a $\mathbb{C}$-Banach space is strictly PSH if and only if it satisfies the strong maximum modulus principle, that is, if and only if every holomorphic map from a connected manifold X to E whose norm has a local maximum is constant.

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[2] After changing Carmignani's definition 1.1 so that strictly subharmonic functions are assumed to be finite on a dense subset, and correcting definition 1.2, which erroneously omits the hypothesis $w \neq 0$.

Strict convexity of an $\mathbb{R}$-Banach space can be characterised in the following way: $(E, \|\cdot\|)$ is strictly convex if and only if for every (or for one) increasing, strictly convex map $\psi : \mathbb{R}_+ \to \mathbb{R}$, the composition $\psi \circ \|\cdot\|$ is strictly convex. The analogous statement for a $\mathbb{C}$-Banach space is the following: $(E, \|\cdot\|)$ is strictly PSH if and only if for every (or for one) strictly convex map $\psi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$, the composition $\psi \circ \log\|\cdot\|$ is strictly PSH.

These two results will give us simple characterisations of strict convexity and strict plurisubharmonicity of L^p direct integrals, that we will now present.

1.4. **Direct integrals.** The notion of direct integral used in this article is rather basic, but sufficient to include at once Bochner L^p spaces, ℓ^p direct sums and Hilbert direct integrals. One may consult [15, pp. 61-62] and [9, pp. 683-686] for a more elaborate theory.

Let (S, Σ, μ) be a measure space, let $\mathcal{E} = (E_s)_{s \in S}$ be a *measurable* family, in a sense that we will define, of real or complex Banach spaces, and let $p \in [1; \infty]$. A section of $\mathcal{E}$ is an element of the product $\prod_{s \in S} E_s$. Given a section σ satisfying an appropriate measurability condition, let $\|\sigma\|_p$ be the p-norm of the function $s \mapsto \|\sigma(s)\|_{E_s}$; explicitly,

$$\|\sigma\|_p = \begin{cases} \left(\int_S \|\sigma(s)\|_{E_s}^p \mathrm{d}\mu(s)\right)^{\frac{1}{p}} & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{s \in S} \|\sigma(s)\|_{E_s} & \text{if } p = \infty. \end{cases} \tag{1.3}$$

Then σ is said to be p-integrable if $\|\sigma\|_p < \infty$, and the L^p direct integral of the family $\mathcal{E}$ is defined to be the space of p-integrable sections, up to equality almost everywhere, equipped with the norm $\|\cdot\|_p$. It is a Banach space, denoted by $L^p(\mathcal{E})$.

If the family $\mathcal{E}$ is constant, equal to a Banach space E, then $L^p(\mathcal{E})$ is the Bochner space $L^p(S, \Sigma, \mu; E)$. In the case where E has dimension 1, one recovers Lebesgue L^p spaces.

Suppose that the σ-algebra Σ is discrete, that μ is the counting measure and that the E_s are pairwise distinct. Then $L^p(\mathcal{E})$ is essentially the ℓ^p direct sum of the family $\mathcal{E}$, denoted by $\ell^p(\mathcal{E})$. More precisely, $L^p(\mathcal{E})$ is the closed subspace of $\ell^p(\mathcal{E})$ whose elements are the sections with countable support; this subspace coincides with $\ell^p(\mathcal{E})$ except when $p = \infty$ and the set of those $s \in S$ such that E_s is non-zero is uncountable.

Finally, Hilbert direct integrals correspond to the case where $p = 2$ and each E_s is equal to one of the spaces ℓ^2_n, for $n \in \mathbb{N}$, or to ℓ^2_∞.

Here are now the results promised. For conciseness purposes, the statements are slightly less general than what will be proved in the article.

Theorem 1.1. — *Suppose that μ is σ-finite, that $\mathcal{E}$ is a discrete measurable family of $\mathbb{R}$-Banach spaces, and that $1 < p < \infty$. The direct integral $L^p(\mathcal{E})$ is strictly convex if and only if E_s is strictly convex for almost all s.*

Theorem 1.2. — *Suppose that μ is σ-finite, that $\mathcal{E}$ is a discrete measurable family of $\mathbb{C}$-Banach spaces, and that $1 \leqslant p < \infty$. The direct integral $L^p(\mathcal{E})$ is strictly PSH if and only if E_s is strictly PSH for almost all s.*

The notion of *discrete* family which appears in these statements is, as we will see, insignificant in practice: indeed, when μ is σ-finite, the existence of a non-discrete measurable family cannot be proved in ZFC.

Even if we will prove Theorems 1.1 and 1.2 through entirely parallel methods, it is important to note that the statements themselves are not rigorously analogous. Indeed, definition (1.3) is problematic in this respect, because the true complex analogue of $\|\cdot\|_{E_s}$ is $\log\|\cdot\|_{E_s}$, and not $\|\cdot\|_{E_s}$. An examination of the proofs reveals that this discrepancy is the origin of the difference between the hypothesis $1 < p < \infty$ in the real case and the hypothesis $1 \leqslant p < \infty$ in the complex case.

Let us now give a few immediate consequences of Theorems 1.1 and 1.2.

Corollary 1.3. — *Suppose that μ is non-zero and σ-finite and that $1 < p < \infty$, and let E be an $\mathbb{R}$-Banach space. The Bochner space $L^p(S, \Sigma, \mu; E)$ is strictly convex if and only if E is. In particular, the Lebesgue space $L^p(S, \Sigma, \mu; \mathbb{R})$ is strictly convex.*

Corollary 1.4. — *Suppose that μ is non-zero and σ-finite and that $1 \leqslant p < \infty$, and let E be a $\mathbb{C}$-Banach space. The Bochner space $L^p(S, \Sigma, \mu; E)$ is strictly PSH if and only if E is. In particular, the Lebesgue space $L^p(S, \Sigma, \mu; \mathbb{C})$ is strictly PSH.*

Corollary 1.5. — *Suppose that S is countable, that $1 < p < \infty$, and that the E_s are $\mathbb{R}$-Banach spaces. Then $\ell^p(\mathcal{E})$ is strictly convex if and only if E_s is strictly convex for all s.*

Corollary 1.6. — *Suppose that S is countable, that $1 \leqslant p < \infty$, and that the E_s are $\mathbb{C}$-Banach spaces. Then $\ell^p(\mathcal{E})$ is strictly PSH if and only if E_s is strictly PSH for all s.*

It is not difficult to see that Corollaries 1.5 and 1.6 imply the same statements without the countability hypothesis on S. Taking this remark into account, it turns out that Theorem 1.1 is a consequence of Corollaries 1.3 and 1.5, and that Theorem 1.2 is a consequence of Corollaries 1.4 and 1.6: indeed, we will see that L^p direct integrals are in fact ℓ^p direct sums of Bochner L^p spaces. Thus one sees that the notion of direct integral is a means to state and prove results about ℓ^p direct sums and Bochner spaces in a unified way.

Corollary 1.5 was proved by Day [7, p. 314] [8, p. 520, Th. 6] through a simple and direct method, which can also give Corollary 1.3, as the author remarks [8, p. 521]. This method can actually be used to prove Theorems 1.1 and 1.2 when $1 < p < \infty$, but probably not in the general case, as we will see in the appendix.

Corollary 1.6 was proved by Jamison, Loomis and Rousseau [19, p. 205, Th. 3.15 and p. 208, Th. 4.2]. The case $1 < p < \infty$ is dealt with through a method due to V. and I. Istrăţescu [18, p. 424, Th. 2.4], using Thorp and Whitley's results and Lumer's theory of semi-scalar products. The method used for $p = 1$ in fact works for all values of p, but on the other hand it does not yield Corollary 1.4, because it relies on an explicit description of the dual space of $\ell^1(\mathcal{E})$ which does not hold in general for Bochner spaces [10, p. 98, Th. 1].

Finally, Corollary 1.4 was proved by Thorp and Whitley when $E = \mathbb{C}$, and by Dilworth [11, p. 499, Th. 2.5] in the general case, using Thorp and Whitley's results. It is essentially Dilworth's method that we will follow to prove Theorem 1.2, but in a conceptual framework which makes the proof more enlightening, shorter and less technical, and avoids appealing to Thorp and Whitley's work.

1.5. **Organisation of the article and generalities.** Subsections 2.1 and 2.2 give a few basic facts about convex and PSH functions, emphasising the striking parallelism between the two theories, which is apparent in definitions as well as in statements and proofs.

Subsection 2.3 proves a form of Jensen's inequality which is used in the next subsection. This result is known, at least in similar contexts.

Subsection 2.4, essentially independent from the rest of the article, contains two results which are intended to show that the notion of strict plurisubharmonicity is natural and analogous to strict convexity.

Section 3 deals with strictly convex and strictly PSH spaces, again insisting on the parallelism between both theories, and shows the connection to the strong maximum modulus principle.

Section 4 develops the notion of direct integral, and, as an application of the concepts introduced in the article, proves Theorems 1.1 and 1.2. Finally, the appendix proves the same theorems again in the case $1 < p < \infty$ using Day's method.

Recall that a map defined on the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$ is said to be holomorphic if it extends to a holomorphic map on a neighbourhood of $\overline{\mathbb{D}}$.

We will use the fact that the origin of a topological vector space on a non-discretely valued field has a fundamental system of neighbourhoods whose members are balanced, that is, stable under multiplication by scalars of absolute value less than 1 [3, I, p. 7, prop. 4].

A topological affine space is an affine space E whose direction $\overrightarrow{E}$ is a topological vector space. In this case, there is a unique topology on E such that for all $x \in E$, the bijection $\overrightarrow{E} \to E$ which maps v to $x + v$ is a homeomorphism.

We will call a topological vector space whose topology can be defined by a norm a *normable space*.

The holomorphic manifolds considered in this article are sets equipped with a holomorphic atlas whose charts have target an open set in a complete $\mathbb{C}$-normable space. Please refer to [1] or [12, pp. 7-16] about these manifolds.

For Bochner integration theory and Bochner L^p spaces, one may consult the introductory text [6, pp. 397-404] and the more detailed expositions [10, pp. 41-52] and [17, pp. 1-30]. Let us remark that Bochner theory, in full generality, enables one to integrate functions defined on a measure space with values in a complete *normable* space.

2. Functions

2.1. Convex functions.

Definition 2.1. — Let E be a topological $\mathbb{R}$-affine space and $X \subset E$ a convex subset. A continuous function $f : X \to \mathbb{R}$ is said to be convex if for all affine maps $\gamma : [-1; 1] \to E$ with values in X,

$$f(\gamma(0)) \leqslant \frac{f(\gamma(-1)) + f(\gamma(1))}{2}. \tag{2.1}$$

Moreover, f is said to be strictly convex if the inequality (2.1) is strict as soon as γ is non-constant.

PROPOSITION 2.2. — *Let E_1 and E_2 be two topological $\mathbb{R}$-affine spaces, $X_1 \subset E_1$ and $X_2 \subset E_2$ convex subsets, $\varphi : E_1 \to E_2$ a continuous affine map such that $\varphi(X_1) \subset X_2$, and $f : X_2 \to \mathbb{R}$ a convex function. The composition $f \circ \varphi$ is convex. If φ is injective and f is strictly convex, then $f \circ \varphi$ is strictly convex.*

Proof. — Immediate. □

PROPOSITION 2.3 (Maximum principle). — *Let E be a topological $\mathbb{R}$-affine space and $X \subset E$ a convex open subset. Every convex function $f : X \to \mathbb{R}$ having a global maximum is constant.*

Proof. — Indeed, let M be the maximum of f, and let $U = \{x \in X \mid f(x) = M\}$. It is a non-empty closed set; if we show that it is also open, then we will be able to conclude that $U = X$ by connectedness.

Thus, let $x \in U$, and let V be a balanced neighbourhood of the origin in $\vec{E}$ such that $x + V$ is contained in X. Let $y \in x + V$, and let $\gamma : [-1; 1] \to E$ be given by $\gamma(t) = x + t(y - x)$; it is an affine map with values in X such that $\gamma(0) = x$ and $\gamma(1) = y$. Since f is convex,

$$f(\gamma(0)) \leqslant \frac{f(\gamma(-1)) + f(\gamma(1))}{2}.$$

The left side of this inequality is M, and the right side is at most M, since $f(\gamma(-1))$ and $f(\gamma(1))$ are at most M. Hence we have equality; therefore, $f(\gamma(-1)) = f(\gamma(1)) = M$, so $f(y) = M$, which means that $y \in U$, and finally $x + V \subset U$. □

2.2. **Plurisubharmonic functions.**

DEFINITION 2.4. — Let X be a holomorphic manifold. An upper semicontinuous function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be PSH if for all holomorphic maps $\gamma : \overline{\mathbb{D}} \to X$,

$$f(\gamma(0)) \leqslant \frac{1}{2\pi} \int_0^{2\pi} f\left(\gamma(e^{it})\right) \mathrm{d}t. \tag{2.2}$$

Moreover, f is said to be strictly PSH if the inequality (2.2) is strict as soon as γ is non-constant.

Remark 2.5. — The integral is a well-defined element of $\mathbb{R} \cup \{-\infty\}$, because f is bounded above on every compact set by semicontinuity.

Remark 2.6. — In the case where X is an open subset of a complete $\mathbb{C}$-normable space E, if f satisfies the inequality (2.2) for all *affine* maps $\gamma : \overline{\mathbb{D}} \to E$ with values in X, then f is PSH: see [20, p. 325, no. 17] when E has finite dimension, and [21, p. 172, th. 4.3] in the general case. On the other hand, it can happen that the inequality (2.2) is strict for all non-constant affine maps without f being strictly PSH, as the following example shows.

Example 2.7. — Let $f : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ be a strictly PSH function, and let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be the projection on the second factor. The composition $f \circ \pi$ is PSH by Proposition 2.8, but not strictly PSH, because its composition with the affine map $\gamma : z \mapsto (z, 0)$ is constant. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ be the biholomorphism given by $\varphi(z_1, z_2) = (z_1, z_1^2 + z_2)$. The preceding discussion shows that the composition $f \circ \pi \circ \varphi$ is PSH, but not strictly PSH. However, if $\gamma : z \mapsto (az+b, cz+d)$ is an affine map such that

$$(f \circ \pi \circ \varphi)(\gamma(0)) = \frac{1}{2\pi} \int_0^{2\pi} (f \circ \pi \circ \varphi)\left(\gamma(e^{it})\right) \mathrm{d}t,$$

then $\pi \circ \varphi \circ \gamma$ is constant, since f is strictly PSH. Thus the map $z \mapsto (az+b)^2 + cz + d$ is constant. This implies that $a = c = 0$, so γ is constant.

Proposition 2.8. — *Let X_1 and X_2 be two holomorphic manifolds, $\varphi : X_1 \to X_2$ a holomorphic map and $f : X_2 \to \mathbb{R} \cup \{-\infty\}$ a PSH function. The composition $f \circ \varphi$ is PSH. If the differential of φ is injective outside a discrete subset of X_1 and f is strictly PSH, then $f \circ \varphi$ is strictly PSH.*

Proof. — The first claim is immediate. Thus, suppose that $\mathrm{d}\varphi$ is injective outside a discrete set $D \subset X_1$ and that f is strictly PSH, and let $\gamma : \overline{\mathbb{D}} \to X_1$ be a holomorphic map such that

$$(f \circ \varphi)(\gamma(0)) = \frac{1}{2\pi} \int_0^{2\pi} (f \circ \varphi)\left(\gamma(e^{it})\right) \mathrm{d}t.$$

Since f is strictly PSH, $\varphi \circ \gamma$ is constant, so for all $z \in \mathbb{D}$, $(\varphi \circ \gamma)'(z) = \mathrm{d}\varphi(z)(\gamma'(z)) = 0$. Thus the restriction of γ to the open set $U = \{z \in \mathbb{D} \mid \gamma'(z) \neq 0\}$ has values in D, which is discrete; therefore, γ is constant on each connected component of U, hence its derivative is zero, so $U = \varnothing$ and γ is constant on $\mathbb{D}$, and thus on $\overline{\mathbb{D}}$ by continuity. □

Proposition 2.9 (Maximum principle). — *Let X be a connected holomorphic manifold. Every PSH function $f : X \to \mathbb{R} \cup \{-\infty\}$ having a global maximum is constant.*

Proof. — Indeed, let M be the maximum of f, and let $U = \{x \in X \mid f(x) = M\}$. It is a non-empty set, which is closed since f is upper semicontinuous and the condition $f(x) = M$ is equivalent to $f(x) \geqslant M$. We will see that U is open, from which we will be able to conclude that $U = X$ by connectedness.

Thus, let $x \in U$, and let V be a balanced open neighbourhood of the origin in a complete $\mathbb{C}$-normable space and φ a biholomorphism between V and an open neighbourhood of x, satisfying $\varphi(0) = x$. Let $y \in \varphi(V)$, and let $\gamma : \overline{\mathbb{D}} \to X$ be given by $\gamma(z) = \varphi(z \cdot \varphi^{-1}(y))$; it is a holomorphic map such that $\gamma(0) = x$ and $\gamma(1) = y$. Since f is PSH,

$$f(\gamma(0)) \leqslant \frac{1}{2\pi} \int_0^{2\pi} f\left(\gamma(e^{it})\right) \mathrm{d}t.$$

The left side of this inequality is M, and the right side is at most M, since all the values $f(\gamma(e^{it}))$ are at most M. Hence we have equality; therefore, for almost all $t \in [0; 2\pi]$, $f(\gamma(e^{it})) = M$. By semicontinuity, this relation actually holds for all t, so in particular $f(y) = M$, which means that $y \in U$, and finally $\varphi(V) \subset U$. □

2.3. Jensen's inequality.

Lemma 2.10. — *Let E be a topological $\mathbb{R}$-affine space and $X \subset E$ a convex open subset, and let $f : X \to \mathbb{R}$ be a convex function and $x_0 \in X$. There exists* *a continuous affine map $\alpha : E \to \mathbb{R}$ such that $\alpha \leqslant f$ and $\alpha(x_0) = f(x_0)$. If f is strictly convex, then α and f only coincide at x_0.*

Proof. — Define

$$C = \{(x,t) \in E \times \mathbb{R} \mid x \in X \text{ and } f(x) < t\}.$$

It is a convex open subset of $E \times \mathbb{R}$, which does not contain the point $(x_0, f(x_0))$. According to the Hahn–Banach theorem [3, II, p. 39, th. 1], there exists a closed hyperplane containing $(x_0, f(x_0))$ and not intersecting C. This amounts to saying that there exists a continuous affine map $\alpha : E \to \mathbb{R}$ and a real number λ such that $\alpha(x_0) = \lambda f(x_0)$ and for all $(x,t) \in C$, $\alpha(x) < \lambda t$.

We observe that $\lambda \neq 0$: indeed, there exists $t \in \mathbb{R}$ such that $(x_0, t) \in C$, so if λ were zero, we would have both $\alpha(x_0) = 0$ and $\alpha(x_0) < 0$. Thus, by replacing α by $\lambda^{-1}\alpha$, one reduces to $\lambda = 1$, in which case the two desired properties $\alpha \leqslant f$ and $\alpha(x_0) = f(x_0)$ are satisfied.

Let $x \in X$ such that $\alpha(x) = f(x)$, and let $\gamma : [-1;1] \to E$ be the affine map such that $\gamma(-1) = x_0$ and $\gamma(1) = x$. Its image is contained in X. Since α is affine and f is convex,

$$\alpha(\gamma(0)) = \frac{\alpha(\gamma(-1)) + \alpha(\gamma(1))}{2}, \qquad f(\gamma(0)) \leqslant \frac{f(\gamma(-1)) + f(\gamma(1))}{2}. \tag{2.3}$$

The inequality $\alpha(\gamma(0)) \leqslant f(\gamma(0))$ and the equalities $\alpha(x_0) = f(x_0)$ and $\alpha(x) = f(x)$ show that the inequality (2.3) is an equality. Thus if f is strictly convex, then γ is constant, so $x = x_0$. ☐

Lemma 2.11 (Jensen's inequality). — *Let E be a complete $\mathbb{R}$-normable space, $X \subset E$ a convex open subset, and $f : X \to \mathbb{R}$ a convex function. Let also (S, Σ, μ) be a measure space of total mass 1 and $\eta : S \to E$ an integrable map with values in X. The integral $\int_S \eta \,\mathrm{d}\mu$ belongs to X, and*

$$f\left(\int_S \eta \,\mathrm{d}\mu\right) \leqslant \int_S (f \circ \eta) \,\mathrm{d}\mu.$$

If equality holds and f is strictly convex, then η is essentially constant.

Proof. — Let $m = \int_S \eta \,\mathrm{d}\mu$. That m belongs to X can be seen by writing X as an intersection of open half-spaces thanks to the Hahn–Banach theorem [3, II, p. 39, th. 1]. According to Lemma 2.10, there exists a continuous affine map $\alpha : E \to \mathbb{R}$ such that $\alpha \leqslant f$ and $\alpha(m) = f(m)$. We have

$$\begin{aligned} f\left(\int_S \eta \,\mathrm{d}\mu\right) &= \alpha\left(\int_S \eta \,\mathrm{d}\mu\right) \\ &= \int_S (\alpha \circ \eta) \,\mathrm{d}\mu \\ &\leqslant \int_S (f \circ \eta) \,\mathrm{d}\mu. \end{aligned}$$

If equality holds, then $\alpha \circ \eta$ and $f \circ \eta$ coincide almost everywhere. In the case where f is strictly convex, Lemma 2.10 then shows that η equals m almost everywhere. ☐

2.4. **Relations between convexity and plurisubharmonicity.**

Theorem 2.12. — *Let E be a complete $\mathbb{C}$-normable space and $X \subset E$ a convex open subset, and let $f : X \to \mathbb{R}$ be a continuous function. If f is convex, then f is* PSH*, and if f is strictly convex, then f is strictly* PSH.

Proof. — Suppose that f is convex and let $\gamma : \overline{\mathbb{D}} \to X$ be a holomorphic map. According to Cauchy's formula and Jensen's inequality,

$$\begin{aligned} f(\gamma(0)) &= f\left(\frac{1}{2\pi}\int_0^{2\pi} \gamma(e^{it})\,\mathrm{d}t\right) \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} f\left(\gamma(e^{it})\right)\,\mathrm{d}t. \end{aligned}$$

This shows that f is PSH. If equality holds, in the case where f is strictly convex, γ is essentially constant on the circle, and thus constant on $\overline{\mathbb{D}}$ by analytic continuation, so f is strictly PSH. □

Remark 2.13. — Theorem 2.12 shows in particular that every continuous real linear form $f : E \to \mathbb{R}$ is PSH. Of course, this special case is a direct consequence of Cauchy's formula, which does not require Jensen's inequality.

Theorem 2.14. — *Let E be a complete $\mathbb{C}$-normable space. Let σ be a continuous anti-linear involution of E, and let $E_{\mathbb{R}}$ be the real subspace of E fixed by σ, so that $E = E_{\mathbb{R}} \oplus iE_{\mathbb{R}}$. Denote by $\pi_{\mathbb{R}} : E \to E_{\mathbb{R}}$ the projection along $iE_{\mathbb{R}}$ onto $E_{\mathbb{R}}$. Let $f_{\mathbb{R}} : E_{\mathbb{R}} \to \mathbb{R}$ be a continuous function, and let $f = f_{\mathbb{R}} \circ \pi_{\mathbb{R}}$. Then $f_{\mathbb{R}}$ is convex if and only if f is* PSH*, and $f_{\mathbb{R}}$ is strictly convex if and only if f is strictly* PSH.

Proof. — Suppose that $f_{\mathbb{R}}$ is convex and let $\gamma : \overline{\mathbb{D}} \to E$ be a holomorphic map. According to Cauchy's formula and Jensen's inequality,

$$\begin{aligned} f(\gamma(0)) &= f\left(\frac{1}{2\pi}\int_0^{2\pi} \gamma(e^{it})\,\mathrm{d}t\right) \\ &= f_{\mathbb{R}}\left(\frac{1}{2\pi}\int_0^{2\pi} \pi_{\mathbb{R}}\left(\gamma(e^{it})\right)\,\mathrm{d}t\right) \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} f\left(\gamma(e^{it})\right)\,\mathrm{d}t. \end{aligned}$$

This shows that f is PSH. If equality holds, in the case where $f_{\mathbb{R}}$ is strictly convex, $\pi_{\mathbb{R}} \circ \gamma$ is essentially constant on the circle, and thus constant by continuity, and this constant is $(\pi_{\mathbb{R}} \circ \gamma)(0)$. If $\lambda : E_{\mathbb{R}} \to \mathbb{R}$ is a continuous linear form, the same facts are true for $\lambda \circ \pi_{\mathbb{R}} \circ \gamma$. So this function, which is PSH on $\mathbb{D}$, has a maximum point in $\mathbb{D}$; thus it is constant. We deduce that $\pi_{\mathbb{R}} \circ \gamma$ is itself constant on $\mathbb{D}$. Therefore, at every point of $\mathbb{D}$, the image of the differential of γ is a complex subspace of E contained in $iE_{\mathbb{R}}$; but such a subspace is necessarily zero, thus γ is constant, so f is strictly PSH.

Conversely, suppose that f is PSH and let $\gamma : [-1;1] \to E_{\mathbb{R}}$ be an affine map. Let $\lambda : [-1;1] \to \mathbb{R}$ be the linear map such that

$$\lambda(1) = \frac{f_{\mathbb{R}}(\gamma(-1)) - f_{\mathbb{R}}(\gamma(1))}{2},$$

and let $g = f_{\mathbb{R}} \circ \gamma + \lambda$, so that

$$g(0) = f_{\mathbb{R}}(\gamma(0)), \qquad g(-1) = g(1) = \frac{f_{\mathbb{R}}(\gamma(-1)) + f_{\mathbb{R}}(\gamma(1))}{2}. \tag{2.4}$$

Let $U \subset \mathbb{C}$ be the open set defined by the condition $-1 < \mathfrak{Re}(z) < 1$, and let $\overline{U}$ be its closure. There exists a unique complex affine map $\widetilde{\gamma} : \overline{U} \to E$ extending γ; moreover, $\pi_{\mathbb{R}} \circ \widetilde{\gamma} = \gamma \circ \mathfrak{Re}$, so $f \circ \widetilde{\gamma} = f_{\mathbb{R}} \circ \gamma \circ \mathfrak{Re}$. Thus the function $g \circ \mathfrak{Re} = f \circ \widetilde{\gamma} + \lambda \circ \mathfrak{Re}$ is PSH on U. According to the maximum principle, thanks to the equalities (2.4),

$$f_{\mathbb{R}}(\gamma(0)) \leqslant \frac{f_{\mathbb{R}}(\gamma(-1)) + f_{\mathbb{R}}(\gamma(1))}{2}.$$

This shows that $f_{\mathbb{R}}$ is convex. If equality holds, then $g \circ \mathfrak{Re}$ is constant on U, so $f \circ \widetilde{\gamma}$ cannot be strictly PSH; in the case where f is strictly PSH, we deduce that $\widetilde{\gamma}$ is constant, thus γ is also constant, so $f_{\mathbb{R}}$ is strictly convex. □

COROLLARY 2.15. — *A continuous function $f : \mathbb{C} \to \mathbb{R}$ which is constant on every vertical line is PSH (resp. strictly PSH) if and only if its restriction to $\mathbb{R}$ is convex (resp. strictly convex).*

3. Spaces

3.1. Strictly convex spaces.

PROPOSITION 3.1. — *Let $(E, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space. The function $\|\cdot\| : E \to \mathbb{R}$ is convex.*

Proof. — This is an immediate consequence of the triangle inequality. □

THEOREM 3.2. — *Let $(E, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space. The following conditions are equivalent:*

1. *Every affine map $\gamma : [-1; 1] \to E$ whose image is contained in the unit sphere is constant.*
2. *For all increasing, strictly convex maps $\psi : \mathbb{R}_+ \to \mathbb{R}$, the composition $\psi \circ \|\cdot\|$ is strictly convex.*
3. *There exists an increasing, strictly convex map $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that the composition $\psi \circ \|\cdot\|$ is strictly convex.*

Proof. — Suppose condition (1), and let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, strictly convex map and $\gamma : [-1; 1] \to E$ an affine map. By successively using the convexity of $\|\cdot\|$, the monotonicity of ψ and the convexity of ψ, one obtains

$$\begin{aligned}\psi(\|\gamma(0)\|) &\leqslant \psi\left(\frac{\|\gamma(-1)\| + \|\gamma(1)\|}{2}\right) \\ &\leqslant \frac{\psi(\|\gamma(-1)\|) + \psi(\|\gamma(1)\|)}{2}.\end{aligned}$$

If equality holds, by using the strict monotonicity and the strict convexity of ψ, one obtains $\|\gamma(0)\| = \|\gamma(-1)\| = \|\gamma(1)\|$. Therefore, the function $\|\cdot\| \circ \gamma$, which is convex, has a maximum point in the interior of $[-1; 1]$; thus it is constant. So the image of γ is contained in a sphere, so γ is constant. This proves condition (2).

Condition (2) trivially implies condition (3); suppose that the latter is satisfied and let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, strictly convex map such that $\psi \circ \|\cdot\|$ is strictly convex, and $\gamma : [-1; 1] \to E$ whose image is contained in the unit sphere of

$\|\cdot\|$. The composition $\psi \circ \|\cdot\| \circ \gamma$ is constant, so γ is constant by the strict convexity of $\psi \circ \|\cdot\|$, whence condition (1). □

Definition 3.3. — If the conditions of Theorem 3.2 are satisfied, then $(E, \|\cdot\|)$ is said to be strictly convex.

3.2. **Strictly plurisubharmonic spaces.**

Proposition 3.4. — *Let $(E, \|\cdot\|)$ be a $\mathbb{C}$-Banach space. The function $\log\|\cdot\| : E \to \mathbb{R} \cup \{-\infty\}$ is PSH.*

Proof. — Let $\gamma : \overline{\mathbb{D}} \to \mathbb{C}$ be a holomorphic map, and for $r \in [0;1]$, let $N(r)$ be the number of zeros of γ whose modulus is less than r. According to Jensen's formula,

$$\log|\gamma(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|\gamma(e^{it})\right| \mathrm{d}t - \int_0^1 \frac{N(r)}{r} \mathrm{d}r.$$

This proves the result when $E = \mathbb{C}$ and $\|\cdot\| = |\cdot|$; now our aim is to reduce to this case. Let $\gamma : \overline{\mathbb{D}} \to E$ be a holomorphic map. According to the Hahn–Banach theorem [3, II, p. 67, cor. 1], there exists a continuous linear form $\lambda : E \to \mathbb{C}$ of norm at most 1 such that $\lambda(\gamma(0)) = \|\gamma(0)\|$. We have

$$\begin{aligned}\log\|\gamma(0)\| &= \log|\lambda(\gamma(0))| \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \log\left|\lambda\left(\gamma(e^{it})\right)\right| \mathrm{d}t \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \log\left\|\gamma(e^{it})\right\| \mathrm{d}t.\end{aligned}$$

□

Definition 3.5. — A map $\psi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ is said to be convex if it is continuous and its restriction to $\mathbb{R}$ is finite and convex. If in addition $\psi_{|\mathbb{R}}$ is strictly convex, then ψ is said to be strictly convex.

Remark 3.6. — Every convex (resp. strictly convex) map $\psi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ is increasing (resp. strictly increasing).

The next result is a variation of Jensen's inequality for functions with values in $\mathbb{R} \cup \{-\infty\}$.

Lemma 3.7. — *Let (S, Σ, μ) be a measure space of total mass 1 and $f : S \to \mathbb{R} \cup \{-\infty\}$ a measurable function bounded from above, and let $\psi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ be a convex map. We have*

$$\psi\left(\int_S f \mathrm{d}\mu\right) \leqslant \int_S (\psi \circ f) \, \mathrm{d}\mu.$$

Moreover, if ψ is strictly convex and both sides are finite and equal, then f is essentially constant.

Proof. — Let $m = \int_S f \mathrm{d}\mu \in \mathbb{R} \cup \{-\infty\}$. A slight extension of Lemma 2.10 shows that there exists a continuous map $\alpha : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ whose restriction to $\mathbb{R}$ is affine, such that $\alpha \leqslant \psi$ and $\alpha(m) = \psi(m)$. We have

$$\begin{aligned}\psi\left(\int_S f \mathrm{d}\mu\right) &= \alpha\left(\int_S f \mathrm{d}\mu\right)\\ &= \int_S (\alpha\circ f)\,\mathrm{d}\mu\\ &\leqslant \int_S (\psi\circ f)\,\mathrm{d}\mu.\end{aligned}$$

If both sides are finite and equal, then $\alpha\circ f$ and $\psi\circ f$ coincide almost everywhere. Suppose that ψ is strictly convex. If m is finite, then α and ψ only coincide at m and possibly at $-\infty$; but in this case f is finite almost everywhere, thus f equals m almost everywhere. If $m=-\infty$, then $\psi(-\infty)\in\mathbb{R}$ and α is constant, equal to $\psi(-\infty)$, so α and ψ only coincide at $-\infty$, thus f equals $-\infty$ almost everywhere. □

Theorem 3.8. — *Let $(E,\|\cdot\|)$ be a $\mathbb{C}$-Banach space. The following conditions are equivalent:*

(1) *Every holomorphic map $\gamma:\overline{\mathbb{D}}\to E$ whose image is contained in the unit sphere is constant.*
(2) *For all strictly convex maps $\psi:\mathbb{R}\cup\{-\infty\}\to\mathbb{R}\cup\{-\infty\}$, the composition $\psi\circ\log\|\cdot\|$ is strictly* PSH.
(3) *There exists a strictly convex map $\psi:\mathbb{R}\cup\{-\infty\}\to\mathbb{R}\cup\{-\infty\}$ such that the composition $\psi\circ\log\|\cdot\|$ is strictly* PSH.

Proof. — Suppose condition (1), and let $\psi:\mathbb{R}\cup\{-\infty\}\to\mathbb{R}\cup\{-\infty\}$ be a strictly convex map and $\gamma:\overline{\mathbb{D}}\to E$ a holomorphic map. By successively using the fact that $\log\|\cdot\|$ is PSH, the monotonicity of ψ and Lemma 3.7, one obtains

$$\begin{aligned}\psi(\log\|\gamma(0)\|) &\leqslant \psi\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left\|\gamma(e^{it})\right\|\,\mathrm{d}t\right)\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\psi\left(\log\|\gamma(e^{it})\|\right)\,\mathrm{d}t.\end{aligned}$$

If equality holds, according to Lemma 3.7, the function $\log\|\cdot\|\circ\gamma$ is essentially constant on the circle, thus constant by continuity; moreover, by using the strict monotonicity of ψ, we see that this constant is $\log\|\gamma(0)\|$. Therefore, the function $\log\|\cdot\|\circ\gamma$, which is PSH on $\mathbb{D}$, has a maximum point in $\mathbb{D}$; thus it is constant. So the image of γ is contained in a sphere, so γ is constant. This proves condition (2).

Condition (2) trivially implies condition (3); suppose that the latter is satisfied and let $\psi:\mathbb{R}\cup\{-\infty\}\to\mathbb{R}\cup\{-\infty\}$ be a strictly convex map such that $\psi\circ\log\|\cdot\|$ is strictly PSH, and $\gamma:\overline{\mathbb{D}}\to E$ whose image is contained in the unit sphere of $\|\cdot\|$. The composition $\psi\circ\log\|\cdot\|\circ\gamma$ is constant, so γ is constant since $\psi\circ\log\|\cdot\|$ is strictly PSH, whence condition (1). □

Definition 3.9. — If the conditions of Theorem 3.8 are satisfied, then $(E,\|\cdot\|)$ is said to be strictly PSH.

Remark 3.10. — Thorp and Whitley [25, p. 641, Th. 3.1] proved that if condition (1) of Theorem 3.8 is satisfied for all *affine* maps $\gamma:\overline{\mathbb{D}}\to E$, then $(E,\|\cdot\|)$ is strictly PSH.

Proposition 3.11. — *A $\mathbb{C}$-Banach space $(E, \|\cdot\|)$ which is strictly convex as an $\mathbb{R}$-Banach space is strictly* PSH.

Proof. — Indeed, if there exists an increasing, strictly convex map $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that $\psi \circ \|\cdot\|$ is strictly convex, then $\psi \circ \|\cdot\| = \psi \circ \exp \circ \log \|\cdot\|$ is strictly PSH according to Theorem 2.12; so we can conclude by using Theorem 3.8, since $\psi \circ \exp : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ is strictly convex. ☐

3.3. **Plurisubharmonicity and strong maximum modulus principle.**

Definition 3.12. — A $\mathbb{C}$-Banach space $(E, \|\cdot\|)$ satisfies the strong maximum modulus principle if every holomorphic map η from a connected holomorphic manifold X to E such that $\|\eta\|$ has a local maximum is constant.

Proposition 3.13. — *A $\mathbb{C}$-Banach space $(E, \|\cdot\|)$ satisfies the strong maximum modulus principle if and only if it is strictly* PSH.

Proof. — If $(E, \|\cdot\|)$ satisfies the strong maximum modulus principle, then clearly every holomorphic map $\gamma : \overline{\mathbb{D}} \to E$ whose image is contained in the unit sphere is constant, so $(E, \|\cdot\|)$ is strictly PSH.

Conversely, suppose that $(E, \|\cdot\|)$ is strictly PSH, let X be a connected holomorphic manifold and $\eta : X \to E$ a holomorphic map such that $\|\eta\|$ has a local maximum at $x \in X$, and let us show that η is constant.

By analytic continuation, it is enough to show that η is constant near x. Thus we can assume that $\|\eta\|$ has a *global* maximum at x, in which case this function is constant, since it is PSH; moreover, we can assume that there exists a biholomorphism φ between a balanced open neighbourhood of the origin in a complete $\mathbb{C}$-normable space and X, satisfying $\varphi(0) = x$. Now let $y \in X$. The holomorphic map $\gamma : \overline{\mathbb{D}} \to E$ given by $\gamma(z) = \eta(\varphi(z \cdot \varphi^{-1}(y)))$ is constant, since its image is contained in a sphere; in particular, $\gamma(0) = \gamma(1)$, that is, $\eta(x) = \eta(y)$, which proves that η is constant. ☐

4. Direct integrals

In this section, we let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$.

4.1. **Measurability.**

Let (S, Σ, μ) be a measure space and $\mathcal{E} = (E_s)_{s \in S}$ a family of $\mathbb{K}$-Banach spaces, and define

$$\Phi_{\mathcal{E}} = \{E_s \mid s \in S\}, \qquad \Pi_{\mathcal{E}} = \prod_{s \in S} E_s, \qquad \Omega_{\mathcal{E}} = \coprod_{E \in \Phi_{\mathcal{E}}} E.$$

Equip $\Phi_{\mathcal{E}}$ with the discrete σ-algebra, $\Pi_{\mathcal{E}}$ with the product topology, and $\Omega_{\mathcal{E}}$ with the disjoint union topology and with the corresponding Borel σ-algebra.

Remarks 4.1. — $\Pi_{\mathcal{E}}$ is canonically identified with a closed subset of $(\Omega_{\mathcal{E}})^S$. Also, $\Omega_{\mathcal{E}}$ is metrisable, as a disjoint union of metrisable spaces [2, IX, p. 16].

Definition 4.2. — If $\mathcal{E}$, seen as a map from S to $\Phi_{\mathcal{E}}$, is measurable, then it is said to be a measurable family of $\mathbb{K}$-Banach spaces.

Definition 4.3. — A section of $\mathcal{E}$ is an element of $\Pi_{\mathcal{E}}$. A measurable section of $\mathcal{E}$ is a section which, seen as a map from S to $\Omega_{\mathcal{E}}$, is measurable.

Definition 4.4. — Let σ be a measurable section of $\mathcal{E}$, and let $\sigma(S)^* \subset \Omega_{\mathcal{E}}$ be the set of its non-zero values.

- If $\sigma(S)^*$ is finite, then σ is said to be simple.
- If $\sigma(S)^*$ is separable, then σ is said to be strongly measurable.

Proposition 4.5. — *If the family $\mathcal{E}$ has a measurable section, then it is measurable.*

Proof. — Indeed, if σ is a measurable section of $\mathcal{E}$, then for all subsets $X \subset \Phi_{\mathcal{E}}$, the inverse image of the open set $\coprod_{E \in X} E \subset \Omega_{\mathcal{E}}$ under σ is $\mathcal{E}^{-1}(X)$; thus this set is measurable, so $\mathcal{E}$ is measurable. ☐

Proposition 4.6. — *For each $E \in \Phi_{\mathcal{E}}$, let v_E be an element of E. For all measurable sections σ of $\mathcal{E}$, the map $S \to \mathbb{R}_+$ given by $s \mapsto \|\sigma(s) - v_{E_s}\|_{E_s}$ is measurable.*

Proof. — For all $E \in \Phi_{\mathcal{E}}$, the map $\|\cdot - v_E\|_E$ is continuous on E. The maps $\|\cdot - v_E\|_E$ induce a continuous map $\Omega_{\mathcal{E}} \to \mathbb{R}_+$, which is thus measurable. By composing this map with a measurable section σ of $\mathcal{E}$, one obtains the map defined in the statement, which is therefore measurable. ☐

Corollary 4.7. — *For all measurable sections σ of $\mathcal{E}$, the map $s \mapsto \|\sigma(s)\|_{E_s}$ is measurable.*

Proposition 4.8. — *The set of measurable sections and the set of strongly measurable sections of $\mathcal{E}$ are sequentially closed in $\Pi_{\mathcal{E}}$.*

Proof. — Let $(\sigma_n)_{n \in \mathbb{N}}$ be a convergent sequence of measurable sections. Its limit σ is measurable, as a pointwise limit of measurable functions with values in a metrisable space [6, p. 245, Prop. 8.1.10].

If in addition the σ_n are strongly measurable, then for all $n \in \mathbb{N}$, $\sigma_n(S)^*$ is separable; thus the union $\bigcup_{n \in \mathbb{N}} \sigma_n(S)^*$ is also separable, as well as its closure; but the latter contains $\sigma(S)^*$, which is therefore separable. So σ is strongly measurable. ☐

Notation. — Given $E \in \Phi_{\mathcal{E}}$, $A \in \Sigma$ such that $\mathcal{E}$ is identically E on A, and $v \in E$, we will denote by $\sigma_{A,v}$ the section of $\mathcal{E}$ which has value v on A and zero elsewhere.

Theorem 4.9. — *Suppose that $\mathcal{E}$ measurable.*

(1) *The simple sections of $\mathcal{E}$ are the finite sums of sections of the form $\sigma_{A,v}$. In particular, the set of simple sections is a vector subspace of $\Pi_{\mathcal{E}}$.*
(2) *Every strongly measurable section σ of $\mathcal{E}$ is the limit of a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of simple sections such that for all $n \in \mathbb{N}$ and for all $s \in S$, $\|\sigma_n(s)\|_{E_s} \leqslant \|\sigma(s)\|_{E_s}$.*

   In particular, the set of strongly measurable sections is the sequential closure of the set of simple sections; thus it is a vector subspace of $\Pi_{\mathcal{E}}$.

Proof of assertion (1). — For all simple sections σ,

$$\sigma = \sum_{v \in \sigma(S)^*} \sigma_{A_v, v},$$

where $A_v = \sigma^{-1}(\{v\})$. Conversely, consider a section of the form

$$\sigma = \sum_{i \in I} \sigma_{A_i, v_i},$$

where I is a finite set. After rearranging, we can assume that the A_i are non-empty and pairwise disjoint and that the v_i are non-zero and pairwise distinct. Denote by A the union of the A_i. Given any subset $Z \subset \Omega_{\mathcal{E}}$, we have

$$\sigma^{-1}(Z) = \left(\bigcup_{v_i \in Z} A_i \right) \cup \left(\mathcal{E}^{-1}(\{E \in \Phi_{\mathcal{E}} \mid 0_E \in Z\}) \setminus A \right),$$

which shows that σ is measurable. Moreover, $\sigma(S)^* = \{v_i \mid i \in I\}$ is finite, so σ is a simple section. □

Proof of assertion (2). — Let σ be a strongly measurable section of $\mathcal{E}$, let $D \subset \sigma(S)^*$ be a dense countable subset, and let $(v_n)_{n \in \mathbb{N}}$ be an enumeration of the set $\mathbb{Q}^\times D$ of non-zero rational multiples of elements of D. For $n \in \mathbb{N}$ and $s \in S$, define

$$R_n(s) = \{k \in [\![0; n]\!] \mid v_k \in E_s \text{ and } \|v_k\|_{E_s} \leqslant \|\sigma(s)\|_{E_s}\}.$$

If $R_n(s) = \varnothing$, define $\sigma_n(s) = 0_{E_s}$; otherwise, let k be the smallest element of $R_n(s)$ minimising the quantity $\|\sigma(s) - v_k\|_{E_s}$, and define $\sigma_n(s) = v_k$. Then $\|\sigma_n(s)\|_{E_s} \leqslant \|\sigma(s)\|_{E_s}$, and the sequence $(\sigma_n)_{n \in \mathbb{N}}$ converges to σ.

Moreover, for all $n \in \mathbb{N}$ and for all $k \in [\![0; n]\!]$, the set $A_{n,k} = \sigma_n^{-1}(\{v_k\})$ is defined by the condition $v_k \in E_s$ and by inequalities involving the quantities $\|\sigma(s)\|_{E_s}$ and $\|\sigma(s) - v_l\|_{E_s}$, for $l \in [\![0; n]\!]$. Thus, thanks to Proposition 4.6, it is measurable, so $\sigma_n = \sum_{k=0}^{n} \sigma_{A_{n,k}, v_k}$ is a simple section. □

4.2. **Integrability.** Henceforth, we assume that $\mathcal{E}$ is measurable, and we fix $p \in [1; \infty]$.

Definition 4.10. — Let σ be a measurable section of $\mathcal{E}$. We denote by $\|\sigma\|_p$ the p-norm of the function $s \mapsto \|\sigma(s)\|_{E_s}$. We say that σ is p-integrable if $\|\sigma\|_p < \infty$.

Proposition 4.11. — *The set $\mathcal{L}^p(\mathcal{E})$ of strongly measurable p-integrable sections of $\mathcal{E}$ is a vector subspace of $\Pi_{\mathcal{E}}$, and $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(\mathcal{E})$, whose kernel is the space $\mathcal{N}(\mathcal{E})$ of strongly measurable sections equal to zero almost everywhere.*

Proof. — The fact that the function $\|\cdot\|_p$ on the space of strongly measurable sections is positive, homogeneous and satisfies the triangle inequality comes from the corresponding properties for the norms $\|\cdot\|_{E_s}$ and for the p-norm of real measurable functions. This shows that $\mathcal{L}^p(\mathcal{E})$ is a vector subspace of $\Pi_{\mathcal{E}}$ and that $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(\mathcal{E})$. The fact that the kernel of $\|\cdot\|_p$ is $\mathcal{N}(\mathcal{E})$ again comes from the corresponding property for the p-norm of real measurable functions. □

Definition 4.12. — The L^p direct integral of the family $\mathcal{E}$, denoted by $L^p(\mathcal{E})$, is the quotient $\mathcal{L}^p(\mathcal{E})/\mathcal{N}(\mathcal{E})$, equipped with the norm $\|\cdot\|_p$.

Example 4.13. — If the family $\mathcal{E}$ is constant, equal to a Banach space E, then $L^p(\mathcal{E})$ is the Bochner space $L^p(S, \Sigma, \mu; E)$. In particular, if $E = \mathbb{K}$, then $L^p(\mathcal{E})$ is the Lebesgue space $L^p(S, \Sigma, \mu; \mathbb{K})$.

Example 4.14. — Suppose that Σ is discrete, that μ is the counting measure, and that the E_s are pairwise distinct. Then $L^p(\mathcal{E})$ is the closed subspace $\ell^p_{\mathrm{cnt}}(\mathcal{E})$ of $\ell^p(\mathcal{E})$ whose elements are the sections with countable support. Note that $\ell^p_{\mathrm{cnt}}(\mathcal{E}) = \ell^p(\mathcal{E})$ except when $p = \infty$ and the set of those $s \in S$ such that E_s is non-zero is uncountable.

Example 4.15. — When all the E_s are Hilbert spaces, $L^2(\mathcal{E})$ is a Hilbert space. In particular, one recovers the notion of direct integral of Hilbert spaces, as it is defined in [23, pp. 22-27]. Indeed, given a field $\mathcal{H}$ of Hilbert spaces along with a coherence α, according to the terminology of that book, one obtains canonically a family $\mathcal{E}$ whose terms are equal to one of the spaces ℓ^2_n, for $n \in \mathbb{N}$, or ℓ^2_∞. If $\mathcal{H}$ is Borel, then $\mathcal{E}$ is measurable, and $L^2(\mu; \mathcal{H}, \alpha)$ is identified with $L^2(\mathcal{E})$.

The next result generalises the first two examples above.

THEOREM 4.16. — *For $E \in \Phi_{\mathcal{E}}$, let $S_E = \mathcal{E}^{-1}(\{E\})$, and denote by Σ_E and μ_E the restrictions of Σ and μ to S_E. There exists a canonical isometric isomorphism*

$$L^p(\mathcal{E}) \simeq \ell^p_{\mathrm{cnt}}\big((L^p(S_E, \Sigma_E, \mu_E; E))_{E \,\in\, \Phi_{\mathcal{E}}} \big).$$

Proof. — First, for $\sigma \in \Pi_{\mathcal{E}}$ and $E \in \Phi_{\mathcal{E}}$, denote by $\sigma_E : S_E \to E$ the restriction of σ to S_E. The map $\sigma \mapsto (\sigma_E)_{E \,\in\, \Phi_{\mathcal{E}}}$ is an isomorphism

$$\Pi_{\mathcal{E}} \simeq \prod_{E \,\in\, \Phi_{\mathcal{E}}} E^{S_E}.$$

Next, the image of the space of strongly measurable sections under this isomorphism is the space of families with countable support of strongly measurable sections.

Indeed, if σ is strongly measurable, then $(\sigma_E)_{E \,\in\, \Phi_{\mathcal{E}}}$ is a family of strongly measurable sections, which has countable support since $\sigma(S)^*$ is separable. Conversely, if $(\sigma_E)_{E \,\in\, \Phi_{\mathcal{E}}}$ is a family with countable support of strongly measurable sections, the measurability of $\mathcal{E}$ implies that for all $E \in \Phi_{\mathcal{E}}$, the section of $\mathcal{E}$ obtained by extending σ_E by zero is strongly measurable. Therefore, σ is strongly measurable, being the sum of a countable family of strongly measurable sections.

Finally, if σ is strongly measurable, then

$$\|\sigma\|_p = \big\| \big(\|\sigma_E\|_p \big)_{E \in \Phi_{\mathcal{E}}} \big\|_p.$$

Thus one obtains an isometric isomorphism

$$\mathcal{L}^p(\mathcal{E}) \simeq \ell^p_{\mathrm{cnt}}\big((\mathcal{L}^p(S_E, \Sigma_E, \mu_E; E))_{E \,\in\, \Phi_{\mathcal{E}}} \big).$$

To conclude, it remains only to take the quotient in both sides by the kernel of the corresponding seminorm. □

DEFINITION 4.17. — The family $\mathcal{E}$ is said to be discrete if the image measure of μ under $\mathcal{E}$ is concentrated on the set of singletons of positive measure, that is, if $\mu(\mathcal{E}^{-1}(X)) = 0$, where $X = \{E \in \Phi_{\mathcal{E}} \mid \mu(S_E) = 0\}$.

Remark 4.18. — Suppose that μ is σ-finite and that $\mathcal{E}$ is not discrete. Then there exists a measurable subset of $\mathcal{E}^{-1}(X)$ of finite non-zero measure. The image measure under $\mathcal{E}$ of the restriction of μ to this subset is a finite non-zero measure on $\Phi_{\mathcal{E}}$, for which every singleton has measure zero. This implies that the cardinal of $\Phi_{\mathcal{E}}$ is greater than or equal to some inaccessible cardinal [13, pp. 58-59]. Therefore, when μ is σ-finite, the existence of a measurable, non-discrete family $\mathcal{E}$ cannot be proved in ZFC. Thus, in practice, all measurable families are discrete in this situation.

Proposition 4.19. — *Suppose that μ is σ-finite and that $\mathcal{E}$ is discrete. Then for almost all $s \in S$, there exists an isometric embedding $E_s \to L^p(\mathcal{E})$.*

Proof. — One only has to show that every $E \in \Phi_{\mathcal{E}}$ such that $\mu(S_E) > 0$ can be imbedded into $L^p(\mathcal{E})$. Now, if this condition is satisfied, since μ is σ-finite, there exists a measurable subset $A \subset S_E$ of finite non-zero measure, and the map which associates to each $v \in E$ the class of $\mu(A)^{-\frac{1}{p}}\sigma_{A,v}$ in $L^p(\mathcal{E})$ is an isometric embedding. □

4.3. **Completeness of direct integrals.**

Definition 4.20. — Let X be a separable topological space, and let $\sigma : X \to \mathcal{L}^p(\mathcal{E})$ be a map such that $\sigma(\cdot)(s) : X \to E_s$ is continuous for all $s \in S$. We call σ a parametric section, and we denote by $\|\sigma\|_p$ the p-norm of the function $s \mapsto \sup_{x\in X}\|\sigma(x)(s)\|_{E_s}$.

Remark 4.21. — This function is measurable, because if $(x_n)_{n\in\mathbb{N}}$ is a dense sequence of points of X, then $\sup_{x\in X}\|\sigma(x)(s)\|_{E_s} = \sup_{n\in\mathbb{N}}\|\sigma(x_n)(s)\|_{E_s}$.

Theorem 4.22. — *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of parametric sections such that $\sum_{n=0}^{\infty}\|\sigma_n\|_p < \infty$. There exists a parametric section $\sigma : X \to \mathcal{L}^p(\mathcal{E})$ such that*

$$\lim_{N\to\infty}\left\|\sigma - \sum_{n=0}^{N}\sigma_n\right\|_p = 0.$$

Moreover, for almost all $s \in S$, the series of functions $\sum_{n=0}^{\infty}\sigma_n(\cdot)(s) : X \to E_s$ converges normally to $\sigma(\cdot)(s)$.

Proof. — Let $f : S \to [0;\infty]$ be the measurable function defined by

$$f(s) = \sum_{n=0}^{\infty}\sup_{x\in X}\|\sigma_n(x)(s)\|_{E_s}.$$

We have $\|f\|_p \leqslant \sum_{n=0}^{\infty}\|\sigma_n\|_p < \infty$, so f is finite almost everywhere. For $x \in X$ and $s \in S$, define

$$\sigma(x)(s) = \begin{cases}\displaystyle\sum_{n=0}^{\infty}\sigma_n(x)(s) & \text{if } f(s) < \infty,\\ 0 & \text{otherwise.}\end{cases}$$

For all $s \in S$ such that $f(s) < \infty$, we have

$$\sup_{x\in X}\left\|\sigma(x)(s) - \sum_{n=0}^{N}\sigma_n(x)(s)\right\|_{E_s} \leqslant \sum_{n=N+1}^{\infty}\sup_{x\in X}\|\sigma_n(x)(s)\|_{E_s}.$$

Therefore,

$$\left\|\sigma - \sum_{n=0}^{N}\sigma_n\right\|_p \leqslant \sum_{n=N+1}^{\infty}\|\sigma_n\|_p \xrightarrow[N\to\infty]{} 0.$$

Finally, σ is a parametric section having all the properties claimed. □

Corollary 4.23. — *$L^p(\mathcal{E})$ is a $\mathbb{K}$-Banach space.*

Proof. — Indeed, Theorem 4.22 applied to a one-point space X shows that every absolutely convergent series of elements of $L^p(\mathcal{E})$ is convergent, from which the result follows according to [2, IX, p. 37, cor. 2]. □

4.4. **Strict convexity of direct integrals.** In this subsection, we assume that $\mathbb{K} = \mathbb{R}$.

Lemma 4.24. — *For every affine map $\gamma : [-1;1] \to L^p(\mathcal{E})$, there exists a parametric section $\widetilde{\gamma} : [-1;1] \to \mathcal{L}^p(\mathcal{E})$ lifting γ such that for all $s \in S$, the map $\widetilde{\gamma}(\cdot)(s) : [-1;1] \to E_s$ is affine.*

Proof. — There exist elements γ_0 and γ_1 of $L^p(\mathcal{E})$ such that for all $t \in [-1;1]$,

$$\gamma(t) = \gamma_0 + t\gamma_1.$$

Let $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ be elements of $\mathcal{L}^p(\mathcal{E})$ lifting γ_0 and γ_1, and for $t \in [-1;1]$, define

$$\widetilde{\gamma}(t) = \widetilde{\gamma}_0 + t\widetilde{\gamma}_1.$$

Then $\widetilde{\gamma}$ is a lifting of γ satisfying the required conditions. □

Theorem 4.25. — *Suppose that $1 < p < \infty$.*

(1) *If E_s is strictly convex for almost all $s \in S$, then $L^p(\mathcal{E})$ is strictly convex.*
(2) *Suppose that μ is σ-finite and that $\mathcal{E}$ is discrete. If $L^p(\mathcal{E})$ is strictly convex, then E_s is strictly convex for almost all $s \in S$.*

Proof. — Suppose that E_s is strictly convex for almost all s, and let us show that $L^p(\mathcal{E})$ is strictly convex. According to Theorem 3.2, this amounts to assuming that $\|\cdot\|_{E_s}^p$ is strictly convex for almost all s, and we have to show that $\|\cdot\|_p^p$ is strictly convex.

Let $\gamma : [-1;1] \to L^p(\mathcal{E})$ be an affine map, and let $\widetilde{\gamma}$ be a lifting of γ satisfying the statement of Lemma 4.24. We have

$$\begin{aligned}\|\gamma(0)\|_p^p &= \int_S \|\widetilde{\gamma}(0)(s)\|_{E_s}^p \mathrm{d}\mu(s) \\ &\leqslant \int_S \frac{\|\widetilde{\gamma}(-1)(s)\|_{E_s}^p + \|\widetilde{\gamma}(1)(s)\|_{E_s}^p}{2} \mathrm{d}\mu(s) \\ &= \frac{\|\gamma(-1)\|_p^p + \|\gamma(1)\|_p^p}{2}.\end{aligned}$$

If equality holds, then for almost all s,

$$\|\widetilde{\gamma}(0)(s)\|_{E_s}^p = \frac{\|\widetilde{\gamma}(-1)(s)\|_{E_s}^p + \|\widetilde{\gamma}(1)(s)\|_{E_s}^p}{2},$$

thus $\widetilde{\gamma}(\cdot)(s)$ is constant for almost all s, so γ is constant. This proves assertion (1). Assertion (2) is a consequence of Proposition 4.19. □

4.5. **Strict plurisubharmonicity of direct integrals.** In this subsection, we assume that $\mathbb{K} = \mathbb{C}$.

Lemma 4.26. — *For every holomorphic map $\gamma : \overline{\mathbb{D}} \to L^p(\mathcal{E})$, there exists a parametric section $\widetilde{\gamma} : \overline{\mathbb{D}} \to \mathcal{L}^p(\mathcal{E})$ lifting γ such that for almost all $s \in S$, the map $\widetilde{\gamma}(\cdot)(s) : \overline{\mathbb{D}} \to E_s$ is holomorphic.*

Proof. — There exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of $L^p(\mathcal{E})$ and a real number $r > 1$ such that $\sum_{n=0}^{\infty} r^n \|\gamma_n\|_p < \infty$ and for all $z \in \overline{\mathbb{D}}$,

$$\gamma(z) = \sum_{n=0}^{\infty} z^n \gamma_n.$$

Let $(\widetilde{\gamma}_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{L}^p(\mathcal{E})$ lifting $(\gamma_n)_{n\in\mathbb{N}}$. According to Theorem 4.22, applied to the topological space $r\overline{\mathbb{D}}$ and to the parametric sections $z \mapsto z^n\widetilde{\gamma}_n$, there exists a parametric section $\widetilde{\gamma} : \overline{\mathbb{D}} \to \mathcal{L}^p(\mathcal{E})$ lifting γ such that for almost all $s \in S$, $\sum_{n=0}^{\infty} r^n \|\widetilde{\gamma}_n(s)\|_{E_s} < \infty$, and for all $z \in \overline{\mathbb{D}}$,

$$\widetilde{\gamma}(z)(s) = \sum_{n=0}^{\infty} z^n \widetilde{\gamma}_n(s).$$

Then $\widetilde{\gamma}$ is a lifting of γ satisfying the required conditions. □

Lemma 4.27 ([22, p. 122, Lemma 9.2]). — *Let K be a compact metric space and Y a metrisable topological space, equipped with their Borel σ-algebras $\mathcal{B}(K)$ and $\mathcal{B}(Y)$, and let $f : K \times S \to Y$. Suppose that $f(x, \cdot) : S \to Y$ is measurable for all $x \in K$ and that $f(\cdot, s) : K \to Y$ is continuous for all $s \in S$. Then f is measurable with respect to the product σ-algebra $\mathcal{B}(K) \times \Sigma$.*

Proof. — For $n \geqslant 1$, let $K = \coprod_{i\in I_n} K_{n,i}$ be a finite partition of K into Borel subsets of diameter less than $\frac{1}{n}$, and for $i \in I_n$, let $x_{n,i} \in K_{n,i}$. Let $f_n : K \times S \to Y$ be defined by $f_n(x, s) = f(x_{n,i}, s)$ for $(x, s) \in K_{n,i} \times S$. Then f_n is measurable with respect to $\mathcal{B}(K) \times \Sigma$, and the sequence $(f_n)_{n\geqslant 1}$ converges pointwise to f, which implies the desired result according to [6, p. 245, Prop. 8.1.10]. □

Theorem 4.28. — *Suppose that μ is σ-finite and that $1 \leqslant p < \infty$.*

1. *If E_s is strictly* psh *for almost all $s \in S$, then $L^p(\mathcal{E})$ is strictly* psh.
2. *Suppose that $\mathcal{E}$ is discrete. If $L^p(\mathcal{E})$ is strictly* psh*, then E_s is strictly* psh *for almost all $s \in S$.*

Proof. — Suppose that E_s is strictly psh for almost all s, and let us show that $L^p(\mathcal{E})$ is strictly psh. According to Theorem 3.8, this amounts to assuming that $\|\cdot\|_{E_s}^p$ is strictly psh for almost all s, and we have to show that $\|\cdot\|_p^p$ is strictly psh.

Let $\gamma : \overline{\mathbb{D}} \to L^p(\mathcal{E})$ be a holomorphic map, and let $\widetilde{\gamma}$ be a lifting of γ satisfying the statement of Lemma 4.26. We have

$$\begin{aligned}\|\gamma(0)\|_p^p &= \int_S \|\widetilde{\gamma}(0)(s)\|_{E_s}^p \,\mathrm{d}\mu(s) \\ &\leqslant \int_S \frac{1}{2\pi}\int_0^{2\pi} \left\|\widetilde{\gamma}(e^{it})(s)\right\|_{E_s}^p \mathrm{d}t\,\mathrm{d}\mu(s) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\|\gamma(e^{it})\right\|_p^p \,\mathrm{d}t,\end{aligned}$$

thanks to Lemma 4.27, applied to the function $(t,s) \to \|\widetilde{\gamma}(e^{it})(s)\|_{E_s}^p$, and to Fubini's theorem. If equality holds, then for almost all s,

$$\|\widetilde{\gamma}(0)(s)\|_{E_s}^p = \frac{1}{2\pi}\int_0^{2\pi} \|\widetilde{\gamma}(e^{it})(s)\|_{E_s}^p \,\mathrm{d}t,$$

thus $\widetilde{\gamma}(\cdot)(s)$ is constant for almost all s, so γ is constant. This proves assertion (1). Assertion (2) is a consequence of Proposition 4.19. □

Appendix A. Day's method

The purpose of this appendix is to prove Theorems 4.25 and 4.28 again, under the hypothesis $1 < p < \infty$, by Day's method [8, p. 520, Th. 6]. In both cases, we will not come back to assertion (2), which is a consequence of Proposition 4.19.

Proof of Theorem 4.25. — Let us begin by observing that a normed $\mathbb{R}$-vector space $(E, \|\cdot\|)$ is strictly convex if and only if for all affine maps $\gamma : [-1;1] \to E$, the equalities $\|\gamma(0)\| = \|\gamma(-1)\| = \|\gamma(1)\|$ imply $\gamma(-1) = \gamma(1)$.

Now let $\gamma : [-1;1] \to L^p(\mathcal{E})$ be an affine map, let $\widetilde{\gamma}$ be a lifting of γ satisfying the statement of Lemma 4.24, let $\widetilde{\rho} : [-1;1] \to \mathcal{L}^p(S,\Sigma,\mu;\mathbb{R})$ be the map defined by $\widetilde{\rho}(t)(s) = \|\widetilde{\gamma}(t)(s)\|_{E_s}$, and let $\rho : [-1;1] \to L^p(S,\Sigma,\mu;\mathbb{R})$ be the composition of $\widetilde{\rho}$ and the quotient map. For all $s \in S$, since $\widetilde{\gamma}(\cdot)(s)$ is affine and $\|\cdot\|_{E_s}$ is convex,

$$\widetilde{\rho}(0)(s) \leqslant \frac{\widetilde{\rho}(-1)(s) + \widetilde{\rho}(1)(s)}{2}. \tag{A.1}$$

Therefore,

$$\begin{aligned}
\|\gamma(0)\|_p &= \|\rho(0)\|_p \\
&\leqslant \left\| \frac{\rho(-1) + \rho(1)}{2} \right\|_p \\
&\leqslant \frac{\|\rho(-1)\|_p + \|\rho(1)\|_p}{2} \\
&= \frac{\|\gamma(-1)\|_p + \|\gamma(1)\|_p}{2}.
\end{aligned}$$

If $\|\gamma(0)\|_p = \|\gamma(-1)\|_p = \|\gamma(1)\|_p$, then we have equality, so the inequality (A.1) is an equality for almost all s; moreover,

$$\left\| \frac{\rho(-1) + \rho(1)}{2} \right\|_p = \|\rho(-1)\|_p = \|\rho(1)\|_p. \tag{A.2}$$

Since $L^p(S,\Sigma,\mu;\mathbb{R})$ is strictly convex, the equalities (A.2) imply $\rho(-1) = \rho(1)$. Therefore, for almost all s, $\widetilde{\rho}(0)(s) = \widetilde{\rho}(-1)(s) = \widetilde{\rho}(1)(s)$, that is, $\|\widetilde{\gamma}(0)(s)\|_{E_s} = \|\widetilde{\gamma}(-1)(s)\|_{E_s} = \|\widetilde{\gamma}(1)(s)\|_{E_s}$. If E_s is strictly convex for almost all s, this shows that $\gamma(-1) = \gamma(1)$, so $L^p(\mathcal{E})$ is strictly convex. □

Lemma A.1. — *Let $(E, \|\cdot\|)$ be a strictly convex $\mathbb{R}$-Banach space, (S,Σ,μ) a measure space of total mass* 1*, and $\eta : S \to E$ an integrable map. We have*

$$\left\| \int_S \eta \,\mathrm{d}\mu \right\| \leqslant \int_S \|\eta\| \,\mathrm{d}\mu.$$

If equality holds and $\|\eta\|$ is constant, then η is essentially constant.

Proof. — The inequality is clear. Suppose that equality holds and that $\|\eta\|$ is constant, and let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, strictly convex map. We have

$$\begin{aligned}\psi\left(\left\|\int_S \eta \,\mathrm{d}\mu\right\|\right) &= \psi\left(\int_S \|\eta\| \,\mathrm{d}\mu\right)\\ &= \int_S \psi(\|\eta\|)\,\mathrm{d}\mu.\end{aligned}$$

Lemma 2.11 then shows that η is essentially constant, since $\psi \circ \|\cdot\|$ is strictly convex. □

Proof of Theorem 4.28. — Let us begin by observing that a $\mathbb{C}$-Banach space $(E, \|\cdot\|)$ is strictly PSH if and only if for all holomorphic maps $\gamma : \overline{\mathbb{D}} \to E$, the equalities $\|\gamma(0)\| = \|\gamma(z)\|$, for $|z| = 1$, imply that γ is constant on the circle.

Now let $\gamma : \overline{\mathbb{D}} \to L^p(\mathcal{E})$ be a holomorphic map, let $\widetilde{\gamma}$ be a lifting of γ satisfying the statement of Lemma 4.26, let $\widetilde{\rho} : \overline{\mathbb{D}} \to \mathcal{L}^p(S, \Sigma, \mu; \mathbb{R})$ be the map defined by $\widetilde{\rho}(z)(s) = \|\widetilde{\gamma}(z)(s)\|_{E_s}$, and let $\rho : \overline{\mathbb{D}} \to L^p(S, \Sigma, \mu; \mathbb{R})$ be the composition of $\widetilde{\rho}$ and the quotient map. For almost all $s \in S$, since $\widetilde{\gamma}(\cdot)(s)$ is holomorphic and $\|\cdot\|_{E_s}$ is PSH,

$$\widetilde{\rho}(0)(s) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\rho}(e^{it})(s)\,\mathrm{d}t. \tag{A.3}$$

Therefore, thanks to Lemma 4.27, applied to the function $(t, s) \mapsto \widetilde{\rho}(e^{it})(s)$, and to the result [17, p. 26, Prop. 1.2.25] about pointwise calculation of integrals with values in an L^p space,

$$\begin{aligned}\|\gamma(0)\|_p &= \|\rho(0)\|_p\\ &\leqslant \left\|\frac{1}{2\pi}\int_0^{2\pi} \rho(e^{it})\,\mathrm{d}t\right\|_p\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left\|\rho(e^{it})\right\|_p \,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\|\gamma(e^{it})\right\|_p \,\mathrm{d}t.\end{aligned}$$

If $\|\gamma(0)\|_p = \|\gamma(z)\|_p$ for $|z| = 1$, then we have equality, so the inequality (A.3) is an equality for almost all s; moreover, for $|z| = 1$,

$$\left\|\frac{1}{2\pi}\int_0^{2\pi} \rho(e^{it})\,\mathrm{d}t\right\|_p = \|\rho(z)\|_p. \tag{A.4}$$

Since $L^p(S, \Sigma, \mu; \mathbb{R})$ is strictly convex, the equalities (A.4) and Lemma A.1 imply that ρ is essentially constant on the circle. Thus there exists a dense sequence $(z_n)_{n \in \mathbb{N}}$ of points of the circle such that for all $n \in \mathbb{N}$, for almost all $s \in S$, $\widetilde{\rho}(z_n)(s) = \widetilde{\rho}(z_0)(s)$. We deduce that for almost all s, for all $n \in \mathbb{N}$, $\widetilde{\rho}(z_n)(s) = \widetilde{\rho}(z_0)(s)$; so, by continuity, for almost all s, for $|z| = 1$, $\widetilde{\rho}(0)(s) = \widetilde{\rho}(z)(s)$, that is, $\|\widetilde{\gamma}(0)(s)\|_{E_s} = \|\widetilde{\gamma}(z)(s)\|_{E_s}$. If E_s is strictly PSH for almost all s, this shows that γ is constant on the circle, so $L^p(\mathcal{E})$ is strictly PSH. □

Remark A.2. — In the proof of Theorem 4.25, the strict convexity of $L^p(S, \Sigma, \mu; \mathbb{R})$ and the equalities (A.2) enable one to conclude that $\rho(-1) = \rho(1)$; the reason is that there exists an affine map $[-1; 1] \to L^p(S, \Sigma, \mu; \mathbb{R})$ agreeing with ρ at -1 and 1. In the proof of Theorem 4.28, if there existed a holomorphic map $\overline{\mathbb{D}} \to L^p(S, \Sigma, \mu; \mathbb{C})$ agreeing with ρ on the circle, one could use the strict plurisubharmonicity of this space and the equalities (A.4) to conclude that ρ is constant on the circle, in which case the reasoning would remain valid for $p = 1$. Unfortunately, this is not the case a priori, which suggests that Day's method cannot be used to prove Theorem 4.28 in full generality.

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