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NON-LOCAL APPROXIMATIONS OF THE GRADIENT

HAIM BREZIS AND PETRU MIRONESCU

Abstract. We revisit the proofs of a few basic results concerning non-local approximations of the gradient. A typical such result asserts that, if (ρ_ε) is a radial approximation to the identity in \mathbb{R}^N and u belongs to a homogeneous Sobolev space $\dot{W}^{1,p}$, then

$$V_\varepsilon(x) := N \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N,$$

converges in L^p to the distributional gradient ∇u as $\varepsilon \rightarrow 0$.

We highlight the crucial role played by the representation formula $V_\varepsilon = (\nabla u) * F_\varepsilon$, where F_ε is an approximation to the identity defined via ρ_ε . This formula allows to unify the proofs of a significant number of results in the literature, by reducing them to standard properties of the approximations to the identity.

We also highlight the effectiveness of a symmetric non-local integration by parts formula.

Relaxations of the assumptions on u and ρ_ε , allowing, e.g., heavy tails kernels or a distributional definition of V_ε , are also discussed. In particular, we show that heavy tails kernels may be treated as perturbations of approximations to the identity.

1. A REPRESENTATION FORMULA AND APPLICATIONS

Let $(\rho_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ be a family functions on \mathbb{R}^N such that:

$$\rho_\varepsilon \text{ is non-negative, integrable, radial, } \forall \varepsilon, \quad (1.1)$$

$$\int_{\mathbb{R}^N} \rho_\varepsilon = 1, \quad \forall \varepsilon, \quad (1.2)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| > \delta} \rho_\varepsilon(h) dh = 0, \quad \forall \delta > 0. \quad (1.3)$$

Following Mengesha and Spector [7] (with roots in Bourgain, Brezis, and Mirone-scu [1], Gilboa and Osher [6], Du, Gunzburger, Lehouck, and Zhou [5]; see also a detailed list of references in [7, p. 254]), we set, for any measurable function $u \in L^1_{loc}(\mathbb{R}^N)$, and assuming that the integral below exists,

$$\begin{aligned} V_\varepsilon(x) &= V_{\varepsilon,u}(x) \\ &= V_{\varepsilon,u,\rho_\varepsilon}(x) := N \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N. \end{aligned} \quad (1.4)$$

V_ε may be seen as a non-local approximation of the gradient. Indeed (see Remark 1.13), (i) when u is C^1 and bounded, we have the pointwise convergence $V_\varepsilon(x) \rightarrow \nabla u(x)$ as $\varepsilon \rightarrow 0$, $\forall x \in \mathbb{R}^N$; (ii) when u is C^1 and compactly supported, we have $V_\varepsilon \rightarrow \nabla u$ uniformly.

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In what follows, we revisit the proofs of a few results establishing the validity of the convergence

$$V_\varepsilon \rightarrow Du \text{ as } \varepsilon \rightarrow 0, \quad (1.5)$$

in various functional settings. Many of these results were originally obtained, in slightly different forms, in [7].

Before presenting the main results and methods, we make some easy observations concerning the existence of V_ε . Set

$$\begin{aligned} W_\varepsilon(x) &= W_{\varepsilon,u}(x) = W_{\varepsilon,u,\rho_\varepsilon}(x) \\ &:= \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N. \end{aligned}$$

Clearly, the following holds.

LEMMA 1.1. — *Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Then V_ε is well-defined a.e. and is measurable.*

Moreover, we have $|V_\varepsilon| \leq N W_\varepsilon$ a.e., and thus $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.

Remark 1.2. — In the above statements, the condition $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$ seems constraining. However, under the following assumption:

for every ε , there exist some $\delta_\varepsilon, R_\varepsilon > 0$ such that

$$\rho_\varepsilon(h) = 0 \text{ if } |h| < \delta_\varepsilon \text{ or if } |h| > R_\varepsilon, \quad (1.6)$$

we have $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$, $\forall u \in L^1_{loc}(\mathbb{R}^N)$.

This is especially relevant for Propositions 2.2, 2.3, 2.5, 2.6, and 5.1 below.

We next present a sufficient condition for having $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ (and thus, by Lemma 1.1, $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$). For $1 \leq p < \infty$, set

$$I_{\varepsilon,p} = I_{\varepsilon,p,u} = I_{\varepsilon,p,u,\rho_\varepsilon} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dx dh.$$

LEMMA 1.3. — *Assume (1.2). Let $1 \leq p < \infty$ and $u \in L^1_{loc}(\mathbb{R}^N)$. Then $\|W_\varepsilon\|_{L^p(\mathbb{R}^N)}^p \leq I_{\varepsilon,p}$.*

Consequently, if $I_{\varepsilon,p} < \infty$, then V_ε is well-defined a.e., measurable, and $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq N [I_{\varepsilon,p}]^{1/p}$.

In the above and in what follows, the L^p -norms of vector fields $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are computed with respect to the Euclidean norm $|\cdot|$, i.e.,

$$\|F\|_{L^p(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |F(x)|^p dx.$$

Similarly, the mass of a measure $F \in \mathcal{M}(\mathbb{R}^N; \mathbb{R}^N)$ is computed with respect to the Euclidean norm, i.e.,

$$\|F\|_{\mathcal{M}(\mathbb{R}^N)} = \sup \left\{ \sum_{j=1}^N \int_{\mathbb{R}^N} \zeta_j dF_j; \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), |\zeta(x)| \leq 1, \forall x \right\}.$$

Proof of Lemma 1.3. — The conclusion follows by integrating in x the inequality

$$\begin{aligned} [W_\varepsilon(x)]^p &\leq \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dh \left(\int_{\mathbb{R}^N} \rho_\varepsilon(h) dh \right)^{p-1} \\ &= \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dh, \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

□

We now recall a few sufficient conditions for having $I_{\varepsilon,p} < \infty$. Set

$$\begin{aligned} \dot{W}^{1,p} &:= \{u \in \mathcal{D}'(\mathbb{R}^N); Du \in L^p(\mathbb{R}^N)\} \\ &= \{u \in L^1_{loc}(\mathbb{R}^N); Du \in L^p(\mathbb{R}^N)\}, \quad 1 \leq p < \infty, \\ \dot{B}V &:= \{u \in \mathcal{D}'(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N)\} \\ &= \{u \in L^1_{loc}(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N)\}. \end{aligned}$$

In what follows, for $u \in \dot{W}^{1,p}$, we denote the distributional gradient ∇u .

Let $K_{p,N} := \int_{\mathbb{S}^{N-1}} |h_j|^p d\sigma(h)$ (which does not depend on $j \in \llbracket 1, N \rrbracket$).

We have the following

LEMMA 1.4 ([1]). — Assume (1.1).

(1) Let $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}$. Then $I_{\varepsilon,p} \leq K_{p,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$.

In particular, $V_\varepsilon \in L^p(\mathbb{R}^N)$ and

$$\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq N [K_{p,N}]^{1/p} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

(2) Let $u \in \dot{B}V$. Then $I_{\varepsilon,1} \leq K_{1,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$.

In particular, $V_\varepsilon \in L^1(\mathbb{R}^N)$ and

$$\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq N K_{1,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}.$$

We next present a crucial identity that illuminates the validity of (1.5): (1.10), and its avatar (1.11). Although (1.10) was probably known to experts (it is implicit in [7, proof of Lemma 3.3] and related to several identities in [5]), its intimate connection to (1.5) seems to have remained relatively unnoticed.

Assume (1.1). Let $f_\varepsilon : (0, \infty) \rightarrow [0, \infty)$ be a measurable function such that $\rho_\varepsilon(x) = f_\varepsilon(|x|)$ for a.e. $x \in \mathbb{R}^N$. Set

$$F_\varepsilon(h) := N \int_{|h|}^{\infty} \frac{f_\varepsilon(t)}{t} dt, \quad \forall h \in \mathbb{R}^N \setminus \{0\}. \quad (1.7)$$

Let us note that

$$\begin{aligned} \|F_\varepsilon\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} F_\varepsilon = N |\mathbb{S}^{N-1}| \int_0^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t} dt dr \\ &= |\mathbb{S}^{N-1}| \int_0^\infty t^{N-1} f_\varepsilon(t) dt = \int_{\mathbb{R}^N} \rho_\varepsilon(h) dh = \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}, \end{aligned} \quad (1.8)$$

so that, in particular,

$$F_\varepsilon \in L^1(\mathbb{R}^N). \quad (1.9)$$

LEMMA 1.5. — Assume (1.1). Let F_ε be as in (1.7).

(1) Let $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}$. Then

$$V_\varepsilon = (\nabla u) * F_\varepsilon \text{ a.e.} \quad (1.10)$$

(2) Let $u \in \dot{B}V$. Then

$$V_\varepsilon = (Du) * F_\varepsilon \text{ a.e.} \quad (1.11)$$

Proof. —

Step 1. Proof of (1.10) when $u \in C^\infty(\mathbb{R}^N)$ and $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$. — In this case, we actually prove that

$$V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x), \quad \forall x \in \mathbb{R}^N.$$

For this purpose, we note that

$$F_\varepsilon \text{ is compactly supported,} \quad (1.12)$$

$$\nabla F_\varepsilon(h) = -N \frac{h}{|h|^2} \rho_\varepsilon(h), \quad \forall h \in \mathbb{R}^N \setminus \{0\}, \quad (1.13)$$

$$F_\varepsilon(h) = O(|\ln |h||) \text{ as } h \rightarrow 0. \quad (1.14)$$

Using (1.12)–(1.14), we find, via an integration by parts, that

$$\begin{aligned} V_\varepsilon(x) &= - \int_{\mathbb{R}^N} [u(x+h) - u(x)] \nabla F_\varepsilon(h) dh \\ &= - \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N \setminus \overline{B}(0,\delta)} [u(x+h) - u(x)] \nabla F_\varepsilon(h) dh \\ &= \int_{\mathbb{R}^N} \nabla u(x+h) F_\varepsilon(h) dh = \int_{\mathbb{R}^N} \nabla u(x-h) F_\varepsilon(h) dh \\ &= [(\nabla u) * F_\varepsilon](x), \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

Step 2. Proof of (1.10) and (1.11) when $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. — Let η be a radial non-increasing normalized bump function. By Step 1, we have

$$V_{\varepsilon, u * \eta_\delta}(x) = (\nabla(u * \eta_\delta)) * F_\varepsilon(x) = (Du) * (F_\varepsilon * \eta_\delta)(x), \quad \forall x \in \mathbb{R}^N. \quad (1.15)$$

On the one hand, as $\delta \rightarrow 0$, the right-hand side of (1.15) converges (possibly along a subsequence) a.e. to $(Du) * F_\varepsilon(x)$. (This follows by combining the Young inequality with the fact that $F_\varepsilon * \eta_\delta \rightarrow F_\varepsilon$ in L^1 .)

In order to obtain (1.10), respectively (1.11), it suffices to prove that

$$V_{\varepsilon, u * \eta_\delta, \rho_\varepsilon}(x) \rightarrow V_{\varepsilon, u, \rho_\varepsilon}(x) \text{ as } \delta \rightarrow 0 \text{ for a.e. } x \in \mathbb{R}^N. \quad (1.16)$$

Property (1.16) is obtained via dominated convergence, using the standard inequality

$$|u * \eta_\delta(y)| \leq \mathcal{M}_1 u(y), \quad \forall 0 < \delta < 1, \quad \forall y \in \mathbb{R}^N, \quad (1.17)$$

(see, e.g., [13, eq (17), p. 57]), where $\mathcal{M}_1 u$ is the centered truncated maximal function of u ,

$$\mathcal{M}_1 u(x) := \sup \left\{ \int_{B_r(x)} |u|; 0 < r \leq 1 \right\}.$$

(Here, we use the fact that η is radial, non-increasing, and supported in the unit ball.) Using (1.17) and the extra assumptions on ρ_ε , we obtain the domination

$$\frac{|u * \eta_\delta(x+h) - u * \eta_\delta(x)|}{|h|} \rho_\varepsilon(h) \leq [\mathcal{M}_1 u(x+h) + \mathcal{M}_1 u(x)] g(h), \forall 0 < \delta < 1, \quad (1.18)$$

with $g(h) := \rho_\varepsilon(h)/|h|$ bounded and compactly supported. The right-hand side of (1.18) is in $L^1(\mathbb{R}^N)$ since $\mathcal{M}_1 u \in L^1_{loc}(\mathbb{R}^N)$ (and thus, in particular, $\mathcal{M}_1 u$ is finite a.e.). The latter property follows by combining the Sobolev embeddings $\dot{W}^{1,p}, BV \hookrightarrow L^{N/(N-1)}_{loc}(\mathbb{R}^N)$ with the fact that, by the maximal function theorem, we have $\mathcal{M}_1 u \in L^r_{loc}(\mathbb{R}^N)$ when $u \in L^r_{loc}(\mathbb{R}^N)$ for some $r > 1$. We obtain that the convergence in (1.16) holds on the full measure set

$$\{x \in \mathbb{R}^N; \mathcal{M}_1 u(x) < \infty \text{ and } x \text{ is a Lebesgue point of } u\}.$$

Step 3. Proof of (1.10) and (1.11) in the general case. — For fixed ε , we approximate ρ_ε in L^1 with a sequence $(\rho_{\varepsilon,j})_j$ of kernels $\rho_{\varepsilon,j} \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying (1.1). By Step 2, the corresponding associated kernels $F_{\varepsilon,j}$ satisfy

$$V_{\varepsilon,u,\rho_{\varepsilon,j}} = (Du) * F_{\varepsilon,j} \text{ a.e.} \quad (1.19)$$

Let us note that $F_{\varepsilon,j} \rightarrow F_\varepsilon$ in $L^1(\mathbb{R}^N)$ as $j \rightarrow \infty$. (This follows from a straightforward variant of (1.8).)

We obtain (1.10), respectively (1.11), by letting $j \rightarrow \infty$ in (1.19). Passing to the limits is justified, on the left-hand side, by Lemma 1.4, and, on the right-hand side, by the Young inequality combined with the fact that $F_{\varepsilon,j} \rightarrow F_\varepsilon$ in $L^1(\mathbb{R}^N)$. \square

Using Lemma 1.5, (1.8), and the Young inequality, we obtain the following

LEMMA 1.6. — Assume (1.1)–(1.2).

- (1) Let $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}$. Then $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)}$.
- (2) Let $u \in BV$. Then $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$.

This is an improvement of Lemma 1.4, since $N[K_{p,N}]^{1/p} > 1$ when $N \geq 2$. Indeed, the Jensen inequality yields

$$\begin{aligned} K_{p,N} &\geq \left(\int_{\mathbb{S}^{N-1}} |h_j| d\sigma(h) \right)^p \\ &= \left(\frac{1}{N} \int_{\mathbb{S}^{N-1}} \sum_{k=1}^N |h_k| d\sigma(h) \right)^p > \left(\frac{1}{N} \int_{\mathbb{S}^{N-1}} d\sigma(h) \right)^p = \frac{1}{N^p}. \end{aligned}$$

We next present two direct consequences of Lemma 1.5, originally obtained, with different arguments, in [7].

PROPOSITION 1.7 ([7, Theorem 1.1 (b)]). — Assume (1.1)–(1.3). Let $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. Then

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

PROPOSITION 1.8 ([7, Theorem 1.2]). — Assume (1.1)–(1.3). Let $u \in \dot{B}V$. Then

$$V_\varepsilon \rightharpoonup Du \text{ }^*\text{-weakly in } \mathcal{M}(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)}.$$

Proof of Propositions 1.7 and 1.8. — By Lemma 1.9 below, (F_ε) is an approximation to the identity. We conclude by combining this fact with Lemma 1.5. \square

LEMMA 1.9. — Under the assumptions (1.1)–(1.3), (F_ε) is an approximation to the identity.

Proof. — If $\delta > 0$ is fixed, then (1.7) and (1.3) yield

$$\begin{aligned} \int_{|h| > \delta} F_\varepsilon(h) dh &= N |\mathbb{S}^{N-1}| \int_{\delta}^{\infty} r^{N-1} \int_r^{\infty} \frac{f_\varepsilon(t)}{t} dt dr \\ &= N |\mathbb{S}^{N-1}| \int_{\delta}^{\infty} \int_{\delta}^t r^{N-1} dr \frac{f_\varepsilon(t)}{t} dt \\ &\leq |\mathbb{S}^{N-1}| \int_{\delta}^{\infty} t^{N-1} f_\varepsilon(t) dt \\ &= \int_{|h| > \delta} \rho_\varepsilon(h) dh \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned} \tag{1.20}$$

The conclusion of the lemma follows from (1.8) and (1.20). \square

We next present two a.e. versions of the above results.

PROPOSITION 1.10. — Assume (1.1)–(1.3). Let $u \in \dot{W}^{1,p}$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla u(x)$ as $\varepsilon \rightarrow 0$.

PROPOSITION 1.11. — Assume (1.1)–(1.3). Let $u \in \dot{B}V$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla^{ac} u(x)$ as $\varepsilon \rightarrow 0$.

These results clearly follow from Lemmas 1.5 and 1.9 and the following well-known measure-theoretical result (see, e.g., [12, Chapter III, Section 2.2, Theorem 2] for the first item, and the discussion in [12, Chapter III, Section 4.1] for the second one).

LEMMA 1.12. — Let (F_ε) be a an approximation to the identity, with F_ε radial and non-increasing. Then

- (1) For every $1 \leq p < \infty$ and $G \in L^p(\mathbb{R}^N)$, we have $G * F_\varepsilon(x) \rightarrow G(x)$ at each Lebesgue point of G .

- (2) For every finite Borel measure ν singular with respect to the Lebesgue measure in \mathbb{R}^N , we have $\nu * F_\varepsilon \rightarrow 0$ a.e.

Proof of Lemma 1.12. — The assumptions on F_ε imply that there exist (unique) non-negative Borel measures μ_ε on $(0, \infty)$ such that

$$F_\varepsilon(x) = \mu_\varepsilon(|x|, \infty), \text{ for a.e. } x \in \mathbb{R}^N, \quad (1.21)$$

$$\frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N d\mu_\varepsilon(t) = 1, \quad \forall \varepsilon, \quad (1.22)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{|\mathbb{S}^{N-1}|}{N} \int_\delta^\infty t^N d\mu_\varepsilon(t) = 1, \quad \forall \delta > 0. \quad (1.23)$$

Proof of item (1). — We have

$$\begin{aligned} & |G * F_\varepsilon(x) - G(x)| \\ &= \left| \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} [G(x - r\omega) - G(x)] ds(\omega) \mu_\varepsilon((r, \infty)) dr \right| \\ &\leq \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) \mu_\varepsilon((r, \infty)) dr \\ &= \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) \int_r^\infty d\mu_\varepsilon(t) dr \\ &= \int_0^\infty \int_0^t r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) dr d\mu_\varepsilon(t) \quad (1.24) \\ &= \int_0^\infty \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t) \\ &= \int_0^\infty |B(x,t)| \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t) \\ &= \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t). \end{aligned}$$

We complete the proof by combining (1.22)–(1.24) with the fact that

$$\lim_{t \rightarrow 0} \int_{B(x,t)} |G(y) - G(x)| dy = 0 \text{ at each Lebesgue point } x \text{ of } G$$

and the straightforward inequality

$$\begin{aligned} \int_{B(x,t)} |G(y) - G(x)| dy &\leq \frac{1}{|B(x,t)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)| \\ &\leq \frac{1}{|B(x,\delta)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)|, \quad \forall t \geq \delta. \end{aligned}$$

Proof of item (2). — We have

$$\begin{aligned} \nu * F_\varepsilon(x) &= \int_{\mathbb{R}^N} \mu_\varepsilon(|x-y|, \infty) d\nu(y) = \int_{\mathbb{R}^N} \int_{|x-y|}^{\infty} d\mu_\varepsilon(t) d\nu(y) \\ &= \int_0^\infty \int_{|x-y| < t} d\nu(y) d\mu_\varepsilon(t) = \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \frac{\nu(B(x,t))}{|B(x,t)|} d\mu_\varepsilon(t). \end{aligned} \tag{1.25}$$

We complete the proof combining (1.22), (1.23), and (1.25) with the fact that (by the Lebesgue–Besicovitch differentiation theorem) we have

$$\lim_{t \rightarrow 0} \frac{\nu(B(x,t))}{|B(x,t)|} = 0 \text{ for a.e. } x \in \mathbb{R}^N,$$

and the upper bound

$$\frac{\nu(B(x,t))}{|B(x,t)|} \leq \frac{\nu(\mathbb{R}^N)}{|B(x,\delta)|}, \quad \forall t \geq \delta. \quad \square$$

For other results in the spirit of Propositions 1.10 and 1.11, see Spector [11, Theorem 1.2] and Brezis and Nguyen [2, Theorems 1 and 2].

Remark 1.13. — Here are two additional quick consequences of the fact that, under the assumptions (1.1)–(1.3), (F_ε) is an approximation to the identity. It is straightforward that the pointwise equality $V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x)$, $\forall x \in \mathbb{R}^N$, holds if $u \in C_c^1(\mathbb{R}^N)$, and therefore for such u we have $V_\varepsilon \rightarrow \nabla u$ uniformly in \mathbb{R}^N as $\varepsilon \rightarrow 0$. Similarly, this equality holds when $u \in (C^1 \cap L^\infty)(\mathbb{R}^N)$, and in this case we have $V_\varepsilon(x) \rightarrow \nabla u(x)$, as $\varepsilon \rightarrow 0$, $\forall x \in \mathbb{R}^N$.

2. AN INTEGRATION BY PARTS FORMULA AND APPLICATIONS

The representation formula (1.10) naturally leads to the following formal calculation, with $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $\check{f}(x) := f(-x)$, $\forall x \in \mathbb{R}^N$, and $\{e_j\}_{1 \leq j \leq N}$ the canonical basis of \mathbb{R}^N :

$$\begin{aligned}
\int_{\mathbb{R}^N} V_{\varepsilon,u} \cdot \zeta &= \sum_j \int_{\mathbb{R}^N} [V_{\varepsilon,u} \cdot e_j] \zeta_j = \sum_j [V_{\varepsilon,u} \cdot e_j] * \check{\zeta}_j(0) \\
&= \sum_j [(\partial_j u) * F_\varepsilon] * \check{\zeta}_j(0) \\
&= \sum_j u * [F_\varepsilon * \partial_j \check{\zeta}_j] (0) = \sum_j u * V_{\varepsilon, \check{\zeta}_j} \cdot e_j(0) \\
&= \int_{\mathbb{R}^N} u(x) \sum_j [V_{\varepsilon, \check{\zeta}_j} \cdot e_j] (-x).
\end{aligned} \tag{2.1}$$

Combining (2.1) with the (formal) identity $V_{\varepsilon, \check{f}}(-x) = -V_{\varepsilon, f}(x)$, we obtain the formal identity

$$\int_{\mathbb{R}^N} V_{\varepsilon,u} \cdot \zeta = - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \tag{2.2}$$

and its more symmetric avatar

$$\begin{aligned}
&\int_{\mathbb{R}^N} [V_{\varepsilon,u} \cdot e_j] \psi \\
&= - \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \psi}(x) \cdot e_j] dx, \forall j, \forall \psi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}).
\end{aligned} \tag{2.3}$$

Similar “non-local integration by parts” identities were known in the literature (see, e.g., [5], [7, Theorem 1.4], and, in a slightly different setting, Šilhavý [10, Section 6]). As we will see below, (2.2) holds under mild assumptions on u (this is to be compared with the more restrictive assumptions in Lemma 1.5). The importance of such identities is that they provide a first direction for generalizing the results in Section 1, consisting of weakening the *assumption* $u \in \dot{W}^{1,p}$ (respectively $u \in \dot{B}V$), widely used in Section 1, to a reasonable one allowing V_ε to be well-defined a.e. and to obtain the *property* $u \in \dot{W}^{1,p}$ (respectively $u \in \dot{B}V$) as a *conclusion*.

We first formalize the validity of (2.2).

LEMMA 2.1. — *Let $\varepsilon > 0$ be fixed. Assume (1.1). Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.*

(1) *If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} V_{\varepsilon,u} \cdot \zeta = - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N). \tag{2.4}$$

(2) *If ρ_ε is compactly supported, then (2.4) holds.*

Proof. — We first note the following equalities, valid (thanks to the Fubini theorem applied to the first line) for every $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$:

$$\begin{aligned}
\int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h \cdot \zeta(x)}{|h|} \rho_\varepsilon(h) dh \right) dx \\
&= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h \cdot \zeta(x)}{|h|} \rho_\varepsilon(h) dx \right) dh \\
&= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u(x) \frac{h \cdot [\zeta(x-h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\
&= -N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dx \right) dh.
\end{aligned} \tag{2.5}$$

We next claim that we may apply the Fubini theorem to the last integral in (2.5). By linearity, in item (1) we may assume that either $u \in L^1(\mathbb{R}^N)$ or $u \in L^\infty(\mathbb{R}^N)$.

Proof of the claim when $u \in L^1(\mathbb{R}^N)$. — In this case, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\
\leq \|\nabla \zeta\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.
\end{aligned}$$

Proof of the claim when $u \in L^\infty(\mathbb{R}^N)$. — By Lemma 1.4(1), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\
\leq K(1, N) \|u\|_{L^\infty(\mathbb{R}^N)} \sum_{j=1}^N \|\nabla \zeta_j\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.
\end{aligned}$$

Proof of the claim when ρ_ε is compactly supported. — Let $r, R > 0$ be such that $\text{supp } \zeta \subset B(0, r)$ and $\text{supp } \rho_\varepsilon \subset B(0, R)$. Set $v := u \chi_{B(0, r+R)} \in L^1(\mathbb{R}^N)$. Then

$$\begin{aligned}
u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) \\
= v(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h), \quad \forall x, h \in \mathbb{R}^N,
\end{aligned}$$

and we then argue as in the case where $u \in L^1(\mathbb{R}^N)$.

Applying the Fubini theorem in (2.5), we find that

$$\begin{aligned}
 \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= -N \int_{\mathbb{R}^N} u(x) \left(\int_{\mathbb{R}^N} \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dh \right) dx \\
 &= -N \int_{\mathbb{R}^N} u(x) \left(\int_{\mathbb{R}^N} \sum_{j=1}^N \frac{\zeta_j(x+h) - \zeta_j(x)}{|h|} \frac{h_j}{|h|} \rho_\varepsilon(h) dh \right) dx \\
 &= - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx,
 \end{aligned}$$

so that (2.4) holds. \square

Here are two quick consequences of (2.4), in the spirit of [7, Theorems 1.5 and 1.6].

PROPOSITION 2.2. — Assume (1.1)–(1.3). Let $1 < p < \infty$. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, $\forall \varepsilon$.

(1) If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \left(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p} \right). \quad (2.6)$$

(2) If there exists some $R < \infty$ such that

$$\text{supp } \rho_\varepsilon \subset B(0, R), \quad \forall \quad 0 < \varepsilon < \varepsilon_0, \quad (2.7)$$

then (2.6) holds.

PROPOSITION 2.3. — Assume (1.1)–(1.3). Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, $\forall \varepsilon$.

(1) If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \quad \left(\text{with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV \right). \quad (2.8)$$

(2) If (2.7) holds, then (2.8) holds.

OPEN PROBLEM 2.4. — Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Is it true that (2.6), respectively (2.8), hold, without assuming the support assumption (2.7)?

Proof of Propositions 2.2 and 2.3. — In view of Lemma 1.4(2) and Propositions 1.7 and 1.8, it suffices to prove the following. If $\ell := \liminf_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} < \infty$, then $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in BV$ if $p = 1$. Clearly, this holds provided

$$\int_{\mathbb{R}^N} u \operatorname{div} \zeta \leq \ell \|\zeta\|_{L^q(\mathbb{R}^N)}, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \quad (2.9)$$

where q is the conjugate exponent of p . In turn, (2.9) holds provided

$$-\int_{\mathbb{R}^N} u \operatorname{div} \zeta = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N). \quad (2.10)$$

In order to complete the proof, it suffices to establish (2.10) under the assumptions of Proposition 2.2, respectively 2.3 (with no boundedness assumption on $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)}$).

In view of (2.4), in order to obtain (2.10) it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u V_{\varepsilon, \zeta_j} \cdot e_j = \int_{\mathbb{R}^N} u \partial_j \zeta_j, \quad 1 \leq j \leq N. \quad (2.11)$$

When $u \in L^\infty(\mathbb{R}^N)$, this follows from Proposition 1.7 applied to ζ_j with $p = 1$.

When $u \in L^1(\mathbb{R}^N)$, we note the domination

$$|u V_{\varepsilon, \zeta_j} \cdot e_j| \leq \|\nabla \zeta_j\|_{L^\infty(\mathbb{R}^N)} |u| \in L^1(\mathbb{R}^N).$$

We conclude by dominated convergence, using the fact that $V_{\varepsilon, \zeta_j} \cdot e_j$ converges to $\partial_j \zeta_j$ pointwise as $\varepsilon \rightarrow 0$ (see Remark 1.13).

The argument for $u \in L^1_{loc}(\mathbb{R}^N)$ under the support condition (2.7) is similar.

The proof of Propositions 2.2 and 2.3 is complete. \square

One may consider versions of Propositions 2.2 and 2.3 for families of functions instead of a fixed function. Here are, for example, two versions of [7, Theorem 3.7].

PROPOSITION 2.5. — Assume (1.1)–(1.3). Let $1 < p < \infty$. Let, for every ε , $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N). \quad (2.12)$$

- (1) If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in \dot{W}^{1,p}$ such that, up to a subsequence $\varepsilon_k \rightarrow 0$,

$$u_\varepsilon \rightharpoonup u \text{ * -weakly in } \mathcal{M}_{loc}(\mathbb{R}^N), \quad (2.13)$$

$$\|\nabla u\|_{L^p(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^p(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \quad (2.14)$$

- (2) If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (2.7) holds, then (2.13)–(2.14) hold.

PROPOSITION 2.6. — Assume (1.1)–(1.3). Let, for every ε , $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^1(\mathbb{R}^N). \quad (2.15)$$

- (1) If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in BV$ such that, up to a subsequence $\varepsilon_k \rightarrow 0$,

$$u_\varepsilon \rightharpoonup u \text{ * -weakly in } \mathcal{M}_{loc}(\mathbb{R}^N), \quad (2.16)$$

$$\|Du\|_{\mathcal{M}(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^1(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \quad (2.17)$$

- (2) If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (2.7) holds, then (2.16)–(2.17) hold.

Remark 2.7. — Note that, by Lemma 1.3, (2.12) (respectively, (2.15)) holds if $I_{\varepsilon, p, u_\varepsilon} \leq C < \infty$, $\forall 0 < \varepsilon < \varepsilon_0$ (respectively, $I_{\varepsilon, 1, u_\varepsilon} \leq C < \infty$, $\forall 0 < \varepsilon < \varepsilon_0$).

Proofs of Propositions 2.5 and 2.6. — We present the argument for Proposition 2.6; the proof of Proposition 2.5 is similar. Consider a (signed) Radon measure μ on \mathbb{R}^N such that, up to a subsequence, $u_{\varepsilon_k} \rightharpoonup \mu$ $*$ -weakly in $\mathcal{M}_{loc}(\mathbb{R}^N)$. Fix some $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$. We then have

$$\begin{aligned} \int_{\mathbb{R}^N} \operatorname{div} \zeta \, d\mu &= \lim_k \int_{\mathbb{R}^N} \operatorname{div} \zeta \, u_{\varepsilon_k} \\ &= \lim_k \int_{\mathbb{R}^N} \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \, u_{\varepsilon_k} \\ &\quad + \lim_k \int_{\mathbb{R}^N} \left(\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \right) u_{\varepsilon_k} \\ &:= \lim_k A_k + \lim_k B_k. \end{aligned} \tag{2.18}$$

Step 1. We have $B_k \rightarrow 0$. — We have to treat three cases: (i) (u_{ε_k}) is bounded in $L^1(\mathbb{R}^N)$; (ii) (u_{ε_k}) is bounded in $L^\infty(\mathbb{R}^N)$; (iii) (u_{ε_k}) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (2.7) holds.

Step 1.1. Proof in case (i). — We use the fact that, by Remark 1.13, $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$ uniformly in \mathbb{R}^N , together with the boundedness of (u_{ε_k}) in $L^1(\mathbb{R}^N)$.

Step 1.2. Proof in case (ii). — By Proposition 1.7, we have $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$ in $L^1(\mathbb{R}^N)$. We combine this fact with the boundedness of (u_{ε_k}) in $L^\infty(\mathbb{R}^N)$.

Step 1.3. Proof in case (iii). — By Remark 1.13, we have $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$ uniformly in \mathbb{R}^N . By the support condition (2.7), there exists some $r > 0$ such that, for each ε , $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] = 0$ in $\mathbb{R}^N \setminus B(0, r)$. We find that

$$|B_k| \leq \left\| \operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \right\|_{L^\infty(\mathbb{R}^N)} \|u_{\varepsilon_k}\|_{L^1(B(0, r))} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Step 2. Conclusion. — By Lemma 2.1, we have

$$A_k = - \int_{\mathbb{R}^N} V_{\varepsilon_k, u_{\varepsilon_k}} \cdot \zeta. \tag{2.19}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Combining (2.18), Step 1, (2.19), and the assumption (2.15), we find that, when $\zeta \in C_c^\infty(\Omega; \mathbb{R}^N)$,

$$\int_{\Omega} \operatorname{div} \zeta \, d\mu \leq \|\zeta\|_{L^\infty(\Omega)} \liminf_{\varepsilon} \|V_{\varepsilon, u_{\varepsilon}}\|_{L^1(\Omega)}.$$

It follows that $\mu \in B\dot{V}$ and that (2.17) holds. \square

Remark 2.8. — In view of [1, Theorem 4] and Ponce [8, Theorem 1.2], it is likely, but not known, that, in Propositions 2.5 and 2.6, the boundedness assumptions on u_{ε} can be removed, and that the $*$ -weak convergence in \mathcal{M}_{loc} can be improved to strong L^p_{loc} convergence. In this direction, we formulate below two open questions.

OPEN PROBLEM 2.9. — Let $1 \leq p < \infty$. Let, for every ε , $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N), \quad (2.20)$$

$$\int_{B(0,1)} u_\varepsilon = 0, \quad \forall \varepsilon. \quad (2.21)$$

- (1) Is it true that (u_ε) is bounded in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?
- (2) Is it true that (u_ε) is relatively compact in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?

Note the natural condition (2.21). Such a “normalization” condition is needed since V_ε “does not see constants”; therefore, in order to have a priori estimates, one has to “kill the constants”.

3. A DISTRIBUTIONAL APPROACH

A natural generalization of the approach in the previous section (based on the identity (2.4)), consistent with the spirit of the theory of distributions, was initiated in [7]. It consists of taking the *identity* (2.4) as a *definition* of V_ε . More precisely, instead of assuming that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, we assume that (2.4) holds for every $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and some function $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ that is not, a priori, given by (1.4). This is a *distributional version* of V_ε given by (1.4) and, by the proof of (2.4), it coincides with V_ε provided that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. One could even go one step beyond and define the *distribution* V_ε through the formula $V_\varepsilon(\zeta)$ =the right-hand side of (2.4). (See also, for similar approaches in different but related settings, Shieh and Spector [9], Comi and Stefani [4], Bruè, Calzi, Comi, and Stefani [3].)

Repeating the end of the proof of the Propositions 2.2 and 2.3, we obtain, e.g., the following

PROPOSITION 3.1. — Assume (1.1)–(1.3). Let $1 < p < \infty$. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (2.4) holds, $\forall \varepsilon$, $\forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \left(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p} \right). \quad (3.1)$$

And the usual variants for $p = 1$ or $u \in L^1_{loc}(\mathbb{R}^N)$, under the support condition (2.7).

4. HEAVY TAILS KERNELS

The results in this section are in the spirit of [4].

Let ρ_ε satisfy (1.1) and the following variants of (1.2)–(1.3):

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| < 1} \rho_\varepsilon(h) dh = 1, \quad (4.1)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| > \delta} \frac{\rho_\varepsilon(h)}{|h|} dh = 0, \quad \forall \delta > 0. \quad (4.2)$$

Note that these assumptions are weaker than (1.2)–(1.3) and that they allow *heavy tails* kernels, which are not integrable at infinity. Here is a special case, considered, e.g., in [4], of kernels satisfying (4.1)–(4.2):

$$\rho_\varepsilon(h) := \frac{2^{1-\varepsilon}}{N \pi^{N/2}} \frac{\Gamma\left(\frac{N-\varepsilon}{2} + 1\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)} \frac{1}{|h|^{N-\varepsilon}}, \quad 0 < \varepsilon < N + 2. \quad (4.3)$$

Indeed, the validity of (4.2) is straightforward, while (4.1) follows from the fact that

$$\frac{2^{1-\varepsilon}}{N \pi^{N/2}} \frac{\Gamma\left(\frac{N-\varepsilon}{2} + 1\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)} \sim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{N \pi^{N/2}} \Gamma\left(\frac{N}{2} + 1\right) = \frac{\varepsilon}{|\mathbb{S}^{N-1}|},$$

combined with the identity

$$\frac{\varepsilon}{|\mathbb{S}^{N-1}|} \int_{|h| < 1} \frac{1}{|h|^{N-\varepsilon}} dh = 1.$$

The results in the previous sections can be easily adapted to kernels satisfying (4.1)–(4.2). The price to pay is that the natural function setting is $W^{1,p}(\mathbb{R}^N)$, respectively $BV(\mathbb{R}^N)$, rather than $\dot{W}^{1,p}$, respectively $\dot{B}V$. Here are some results from the previous sections adapted to the assumptions (1.1) and (4.1)–(4.2).

PROPOSITION 4.1. — *Assume (1.1) and (4.1)–(4.2). Let $1 \leq p < \infty$. Let $u \in W^{1,p}(\mathbb{R}^N)$. Then*

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

PROPOSITION 4.2. — *Assume (1.1) and (4.1)–(4.2). Let $u \in BV(\mathbb{R}^N)$. Then*

$$V_\varepsilon \rightharpoonup Du \text{ *weakly in } \mathcal{M}(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0 \quad (4.4)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)}. \quad (4.5)$$

In the next two results, we assume that, for some $1 < q \leq \infty$, we have

$$\lim_{\varepsilon \rightarrow 0} \|\rho_\varepsilon(h)/|h|\|_{L^q(|h| > 1)} = 0. \quad (4.6)$$

PROPOSITION 4.3. — *Assume (1.1) and (4.1)–(4.2). Let $1 \leq p < \infty$. Assume that (4.6) holds when q is the conjugate exponent of p . Let $u \in W^{1,p}(\mathbb{R}^N)$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla u(x)$ as $\varepsilon \rightarrow 0$.*

PROPOSITION 4.4. — *Assume (1.1) and (4.1)–(4.2). Assume that (4.6) holds when $q = \infty$. Let $u \in BV(\mathbb{R}^N)$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla^{ac} u(x)$ as $\varepsilon \rightarrow 0$.*

Remark 4.5. — Given any $q > 1$, the kernel in (4.3) satisfies (4.6). Therefore, Propositions 4.3 and 4.4 apply to these kernels.

Remark 4.6. — Propositions 4.3 and 4.4 have straightforward versions, in which the assumption on u is $u \in L^r(\mathbb{R}^N) \cap \dot{W}^{1,p}$, respectively $u \in L^r(\mathbb{R}^N) \cap \dot{B}V$ for some $r \in [1, \infty)$ (and then, in (4.6), q is the conjugate exponent of r).

PROPOSITION 4.7. — Assume (1.1) and (4.1)–(4.2). Let $1 < p < \infty$. If $u \in L^p(\mathbb{R}^N)$ and $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ for every ε , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \left(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N) \right). \quad (4.7)$$

PROPOSITION 4.8. — Assume (1.1) and (4.1)–(4.2). If $u \in L^1(\mathbb{R}^N)$ and $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ for every ε , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \quad \left(\text{with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N) \right). \quad (4.8)$$

PROPOSITION 4.9. — Assume (1.1) and (4.1)–(4.2). Let $1 < p < \infty$. If $u \in L^p(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (2.4) holds, $\forall \varepsilon, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \left(\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N) \right). \quad (4.9)$$

PROPOSITION 4.10. — Assume (1.1) and (4.1)–(4.2). If $u \in L^1(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (2.4) holds, $\forall \varepsilon, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \quad \left(\text{with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N) \right). \quad (4.10)$$

We prove only Propositions 4.1 and 4.3; the other results are obtained from the corresponding ones in the previous sections using similar arguments.

Proof of Proposition 4.1. — Set

$$\rho_\varepsilon^1 := \rho_\varepsilon \chi_{B(0,1)}, \quad \rho_\varepsilon^2 := \rho_\varepsilon - \rho_\varepsilon^1, \quad V_\varepsilon^1 := V_{\varepsilon,u,\rho_\varepsilon^1}, \quad V_\varepsilon^2 := V_{\varepsilon,u,\rho_\varepsilon^2}, \quad \kappa_\varepsilon(h) := \frac{\rho_\varepsilon^2(h)}{|h|}.$$

By Proposition 1.7 and the assumptions (1.1) and (4.1)–(4.2), we have $V_\varepsilon^1 \rightarrow \nabla u$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. On the other hand, we have the straightforward inequality

$$|V_\varepsilon^2(x)| \leq |u| * \kappa_\varepsilon(x) + |u(x)| \|\kappa_\varepsilon\|_{L^1(\mathbb{R}^N)}, \quad \forall x \in \mathbb{R}^N. \quad (4.11)$$

Combining (4.2), (4.11) with $\delta = 1$, and the Young inequality, we find that $V_\varepsilon^2 \rightarrow 0$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. \square

Proof of Proposition 4.3. — It suffices to note that (by (4.11) and (4.6)) we have $V_\varepsilon^2 \rightarrow 0$ pointwise as $\varepsilon \rightarrow 0$. \square

Remark 4.11. — The fact that ρ is radial when $|h| > 1$ is not relevant for the above results.

5. Γ -CONVERGENCE

One can associate Γ -convergence results with the above convergence statements. We present one result of this type, in the spirit of [7, Theorem 1.7].

Let $1 \leq p < \infty$. Set, for $0 < \varepsilon < \varepsilon_0$ and $u \in L^1_{loc}(\mathbb{R}^N)$,

$$\begin{aligned} J_{\varepsilon,p}(u) &:= \begin{cases} \|V_{\varepsilon,u}\|_{L^p(\mathbb{R}^N)}, & \text{if } W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N), \\ \infty, & \text{otherwise} \end{cases}, \\ J_{0,p}(u) &:= \begin{cases} \|\nabla u\|_{L^p(\mathbb{R}^N)}, & \text{if } u \in \dot{W}^{1,p}, \\ \infty, & \text{otherwise} \end{cases}, \quad \forall 1 < p < \infty, \\ J_{0,1}(u) &:= \begin{cases} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}, & \text{if } u \in \dot{B}V, \\ \infty, & \text{otherwise} \end{cases}. \end{aligned}$$

PROPOSITION 5.1. — Assume (1.1)–(1.3). Let $1 \leq p < \infty$.

- (1) For $1 \leq q < \infty$, $J_{\varepsilon,p}$ Γ -converges to $J_{0,p}$ in $L^q(\mathbb{R}^N)$.
- (2) Under the support assumption (2.7), $J_{\varepsilon,p}$ Γ -converges to $J_{0,p}$ in $L^1_{loc}(\mathbb{R}^N)$.

Proof of item (2). — Let $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0} \subset L^1_{loc}(\mathbb{R}^N)$ be a family such that $u_\varepsilon \rightarrow u$ in $L^1_{loc}(\mathbb{R}^N)$ and $\liminf_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon) < \infty$. By Proposition 2.5 and the proof of Propositions 2.2 and 2.3 (see, more specifically, (2.9) and (2.10)), we find that $u \in \dot{W}^{1,p}$ if $1 < p < \infty$ (respectively $u \in \dot{B}V$ if $p = 1$) and $J_{0,p}(u) \leq \liminf_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon)$.

In the opposite direction, we do not need the support assumption (2.7). Let $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in \dot{B}V$ if $p = 1$. Let η be a normalized bump function. By Proposition 1.7 applied to $u * \eta_{1/j}$, where $j \geq 1$ is an integer, there exists a sequence $(\varepsilon_j)_{j \geq 1}$ such that

$$J_{\varepsilon,p}(u * \eta_{1/j}) \leq J_{0,p}(u * \eta_{1/j}) + \frac{1}{j}, \quad \forall j \geq 1, \quad \forall 0 < \varepsilon < \varepsilon_j.$$

With no loss of generality, we may assume that $\varepsilon_j \rightarrow 0$ and $\varepsilon_{j+1} < \varepsilon_j$. If we set

$$u_\varepsilon := u * \eta_{1/j}, \quad \forall \varepsilon_{j+1} \leq \varepsilon < \varepsilon_j,$$

then $u_\varepsilon \rightarrow u$ in $L^1_{loc}(\mathbb{R}^N)$ when $\varepsilon \rightarrow 0$ and

$$\limsup_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon) \leq \limsup_{j \rightarrow \infty} J_{0,p}(u * \eta_{1/j}) = J_{0,p}(u).$$

Proof of item (1). — The proof is similar to the one of item (2). It suffices to note that Propositions 2.2, 2.3, and 2.5 still hold if we replace $(L^1 + L^\infty)(\mathbb{R}^N)$ with $L^q(\mathbb{R}^N)$. \square

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