CONFLUENTES
MATHEMATICI

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Tome 14, n$^{\text{o}}$ 1 (2022), p. 23–33.
https://doi.org/10.5802/cml.80

Confluentes Math.
14, 1 (2022) 23-33

INFINITE CHARACTERS OF TYPE II ON $\mathrm{SL}_n(\mathbb{Z})$

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Abstract. We construct uncountably many infinite characters of type II for $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 2$.

1. Introduction

Since the work of Bekka [2] it is known that the special linear groups $\mathrm{PSL}_n(\mathbb{Z})$, $n \geqslant 3$, have no characters but the obvious ones. Recall that a character on a group Γ is a positive definite function $\phi : \Gamma \to \mathbb{C}$ which is conjugation invariant, normalized so that $\phi(e) = 1$ and extremal for these properties. Bekka's result states that every character on $\Gamma = \mathrm{PSL}_n(\mathbb{Z})$, $n \geqslant 3$, is either the Dirac function δ_e, or is equal to 1 on a finite index subgroup $\Lambda < \Gamma$, in which case it factors through a character of the finite quotient Γ/Λ. This result was generalized for other higher rank semi-simple lattices by Peterson [9], see also [6, 1] for other proofs and results in this direction.

A classical argument based on the GNS construction shows that, alternatively, a character is of the form $\phi = \tau \circ \pi$, where $\pi : \Gamma \to \mathcal{U}(M)$ is a generating unitary representation into a von Neumann factor M with a faithful normal finite trace τ. Here generating means that $\pi(\Gamma)$ generates M as a von Neumann algebra. Using this characterization, we see that characters admit an *infinite* generalization.

Definition. — A *character* (finite or infinite) on a group Γ is a tracial weight on the universal C*-algebra $C^*(\Gamma)$ of the form $\mathrm{Tr} \circ \pi$, where $\pi : \Gamma \to \mathcal{U}(M)$ is a generating unitary representation into a semi-finite factor M admitting a normal faithful semi-finite trace Tr such that $\pi(C^*(\Gamma))$ contains a non-zero positive operator with finite trace[1]. The *type* of a character will be the von Neumann type of the factor M.

It is natural to ask whether the rigidity results mentioned above for finite characters still hold in the infinite setting. Answering a question of Rosenberg [10], Bekka recently proved that many groups, including the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 2$, admit characters of type I_∞, [3]. He further asked about the existence of characters of type II_∞, see [3, Remark 5]. While this question was raised specifically for $\mathrm{GL}_n(\mathbb{Q})$, which we are unable to treat at the moment, we construct in this note uncountably many characters of type II_∞ for the linear groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 3$. Our approach follows the same ideas as in [3], except that we induce type II representations rather than finite dimensional ones.

The next proposition illustrates the main idea, even though the construction for $\mathrm{SL}_n(\mathbb{Z})$ is a bit more elaborate.

Math. classification: 22D25.
Keywords: Characters on groups.
(*) Research supported by ANR grant AODynG, 19-CE40-0008.

[1] According to the terminology in [4], π is called a *normal* representation.

Following Bekka-Kalantar [5], we say that a subgroup Λ in a group Γ is *a-normal* if $\Lambda \cap g\Lambda g^{-1}$ is amenable for every $g \in \Gamma \setminus \Lambda$. For any group, we generically denote by λ its left regular representation. Classically, $B(H)$ denotes the von Neumann algebra of all bounded operators on a Hilbert space H.

Proposition 1.1. — *Consider a group Γ with a non-amenable, a-normal subgroup Λ.*

(1) *For every factorial representation $\pi_0 : \Lambda \to \mathcal{U}(H_0)$ which is not weakly contained in the regular representation, the induced representation $\pi := \mathrm{Ind}_\Lambda^\Gamma(\pi_0)$ is factorial and $\pi(\Gamma)''$ is naturally isomorphic with $B(\ell^2(\Gamma/\Lambda)) \mathbin{\overline{\otimes}} \pi_0(\Lambda)''$. Moreover, there exists a non-zero $x \in C^*(\pi(\Gamma))$ and a rank one projection $p_0 \in B(\ell^2(\Gamma/\Lambda))$ such that $x(p_0 \otimes 1) = x$.*
(2) *Given two representations π_0, π_0' of Λ, denote by π and π', respectively, the induced Γ-representations. If π is weakly contained in π' then π_0 is weakly contained on $\pi_0' \oplus \lambda$. If π_0 is factorial, this further implies that π_0 is weakly contained in π_0' or in λ.*

Corollary 1.2. — *If Γ contains a proper a-normal non-amenable virtually free subgroup, then it admits uncountably many factorial representations of type II_∞ which are traceable, none of which weakly contains any other.*

In general it is not so easy to construct interesting a-normal subgroups in a given group. For example, $\mathrm{GL}_2(\mathbb{Z})$ can be viewed as an a-normal subgroup of $\mathrm{SL}_3(\mathbb{Z})$ via the top-left embedding, but for $n \geqslant 4$, we do not know if $\mathrm{SL}_n(\mathbb{Z})$ admits a non-amenable a-normal proper subgroup at all. Nevertheless, the same ideas allow to prove that inducing factorial representations of certain products of copies of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_n(\mathbb{Z})$ will still give satisfactory factorial representations of $\mathrm{SL}_n(\mathbb{Z})$.

Theorem 1.3. — *For every $n \geqslant 2$, $\mathrm{SL}_n(\mathbb{Z})$ admits uncountably many factorial representations of type II_∞ which are traceable, none of which weakly contains any other.*

Our construction is ad hoc. We don't know if a similar result holds for, say, co-compact lattices in higher rank semi-simple Lie groups. We point out that a similar argument can be used to produce factorial representations of type III of $\mathrm{SL}_n(\mathbb{Z})$.

2. Preliminaries

2.1. **Factorial representations and weak containment.** By definition, a unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ is called *factorial* if the von Neumann algebra $\pi(\Gamma)''$ is a factor. We sometimes extend this terminology to specify the von Neumann type of $\pi(\Gamma)''$.

Lemma 2.1. — *If $\pi : \Gamma \to \mathcal{U}(H)$ is a factorial unitary representation, then every subrepresentation of π is weakly equivalent to π. More generally, if π is the direct sum $\pi_1 \oplus \cdots \oplus \pi_n$ of finitely many factorial representations $\pi_1, \ldots, \pi_n$ of Γ, then every subrepresentation of π weakly contains one of the π_i's.*

Proof. — Clearly the second statement implies the first one. Assume that $\pi = \pi_1 \oplus \cdots \oplus \pi_n$, for finitely many factorial representations $\pi_1, \ldots, \pi_n$ of Γ. Denote by $p_1, \dots, p_n$ the orthogonal projections on each of these direct summands, so that $\pi_i = p_i\pi$ for $i = 1, \dots, n$.

Consider a non-zero invariant subspace $K \subset H$ and denote by $p \in B(H)$ the orthogonal projection onto K. Denote by $z \in \mathcal{Z}(\pi(\Gamma)'')$ the central support of $p \in \pi(\Gamma)'$. Note that for every $i = 1, \dots, n$, zp_i is either 0 or p_i, because $p_i\pi(\Gamma)'' = \pi_i(\Gamma)''$ is a factor.

Choosing an index i such that $zp_i \neq 0$, we thus get $zp_i = p_i$. Assume that $x \in \pi(\Gamma)''$ is such that $px = 0$. By definition of the central support, this implies that $zx = 0$, and further, $p_ix = p_izx = 0$. So $\pi_i = p_i\pi$ is weakly contained in $p\pi$. □

LEMMA 2.2. — *If $\pi : \Gamma \to \mathcal{U}(H)$ is a unitary representation which is weakly contained in the direct sum of two representations $\pi_1 \oplus \pi_2$, then π is the direct sum of two representations: one weakly contained in π_1 and one weakly contained in π_2. If moreover π is factorial then it is weakly contained in π_1 or π_2.*

Proof. — By assumption the map $\pi_1(g) \oplus \pi_2(g) \mapsto \pi(g)$ extends to a C*-morphism $C^*(\pi_1 \oplus \pi_2) \to C^*(\pi)$. By Arveson extension theorem, this morphism extends to a ucp map $\Phi : B(H_1 \oplus H_2) \to B(H)$. Denote by $p_1, p_2 \in B(H_1 \oplus H_2)$ the orthogonal projections onto H_1, H_2, respectively. By multiplicative domain considerations, $\Phi(p_1), \Phi(p_2) \in \pi(\Gamma)'$. For $i = 1, 2$, denote by $r_i \in \pi(\Gamma)'$ the support projection of $\Phi(p_i) \in \pi(\Gamma)'$.

Claim. For x in the universal C*-algebra $C^*(\Gamma)$, if $\pi_i(x) = 0$, then $r_i\pi(x) = 0$. In particular, $r_i\pi$ is weakly contained in π_i.

Indeed $\pi_i(x) = 0$ means that $p_i(\pi_1 \oplus \pi_2)(x) = 0$. Applying Φ, and using multiplicative domain, we get that $\Phi(p_i)\pi(x) = 0$. This easily implies the claim, by definition of the support projection.

Since $p_1 + p_2 = 1$, we find that $\Phi(p_1), \Phi(p_2), r_1$ and r_2 all commute to each other. Moreover since Φ is a ucp map, we have $0 \leqslant \Phi(p_i) \leqslant 1$, showing that $r_i \geqslant \Phi(p_i)$, for $i = 1, 2$. Therefore $r_1 + r_2 \geqslant \Phi(p_1) + \Phi(p_2) = 1$. So $1 - r_1 \leqslant r_2$. From the claim, it follows that $r_1\pi \prec \pi_1$, while $(1 - r_1)\pi \subset r_2\pi \prec \pi_2$. This gives the desired decomposition $\pi = r_1\pi \oplus (1 - r_1)\pi$.

The moreover part follows from Lemma 2.1. □

2.2. **Induced representations.** In this section we are given two groups $\Lambda < \Gamma$ and a unitary representation π_0 of Λ.

Denote by $s : \Gamma/\Lambda \to \Gamma$ a section to the natural projection map, and by $c : \Gamma \times \Gamma/\Lambda \to \Lambda$ the cocycle given by the formula $c(g, x) = s(gx)^{-1}gs(x)$, for $g \in \Gamma$, $x \in \Gamma/\Lambda$. By definition, the representation $\pi = \mathrm{Ind}_\Lambda^\Gamma(\pi_0)$ is defined on the Hilbert space $H = \ell^2(\Gamma/\Lambda) \otimes H_0$ by the formula

$$\pi_g(\delta_x \otimes \xi) = \delta_{gx} \otimes (\pi_0)_{c(g,x)}\xi, \text{ for all } g \in \Gamma, x \in \Gamma/\Lambda, \xi \in H.$$

The following easy lemma is a special case of a result of Mackey [8]. It is given in this form in [3, Proposition 9].

LEMMA 2.3. — *Given another subgroup $\Sigma < \Gamma$, denote by $S \subset \Gamma$ a system of representatives for the double coset space $\Sigma\backslash\Gamma/\Lambda$. For each $s \in S$, we denote by π_s the representation of $s\Lambda s^{-1}$ given by $\pi_s(sgs^{-1}) = \pi_0(g)$, for all $g \in \Lambda$. Then the* *restriction of π to Σ is equivalent to the direct sum*

$$\bigoplus_{s\in S} \mathrm{Ind}_{s\Lambda s^{-1}\cap\Sigma}^{\Sigma}(\pi_s|_{s\Lambda s^{-1}\cap\Sigma}).$$

2.3. **Finite index considerations.** We will sometimes need to induce representations from normal finite index subgroups. In this context we prove the following stability result.

Lemma 2.4. — *Consider a group Γ with a finite index normal subgroup $\Lambda < \Gamma$. Consider a factorial unitary representation $\pi : \Lambda \to \mathcal{U}(H)$ and denote by ρ the induced representation $\mathrm{Ind}_\Lambda^\Gamma(\pi)$.*

1. *Then ρ is the direct sum of finitely many factorial representations. If π is of type II_1, so is ρ.*
2. *Consider another factorial representation π' of Λ and its induced representation ρ'. Take an automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(\Lambda) = \Lambda$. If a subrepresentation of ρ' is weakly contained in $\rho \circ \alpha$, then π' is weakly contained in $\pi \circ \mathrm{Ad}_g \circ\alpha$ for some $g \in \Gamma$ (and Ad_g denotes the automorphism of Λ given by g-conjugation inside Γ).*

Proof. — (1) By definition ρ is acting on $\ell^2(\Gamma/\Lambda) \mathbin{\overline{\otimes}} H$. Since Λ is normal in Γ, its restriction to Λ is given by

$$\rho_h(\delta_{g\Lambda} \otimes \xi) = \delta_{g\Lambda} \mathbin{\overline{\otimes}} \pi_{g^{-1}hg}(\xi), \text{ for every } h \in \Lambda, g \in \Gamma, \xi \in H.$$

In other words, $\rho|_\Lambda$ is equivalent to $\bigoplus_{g\Lambda\in\Gamma/\Lambda} \pi \circ \mathrm{Ad}_{g^{-1}}$. Here we note that the equivalence class of $\pi \circ \mathrm{Ad}_{g^{-1}}$ does not depend on the choice of the representative $g \in \Gamma$ in the class $g\Lambda \in \Gamma/\Lambda$.

Consider the von Neumann algebras $N \subset M \subset \widetilde{M}$ defined by $N = \rho(\Lambda)''$, $M = \rho(\Gamma)''$ and $\widetilde{M} = B(\ell^2(\Gamma/\Lambda))\overline{\otimes}\pi(\Lambda)''$. For every $a \in \Gamma/\Lambda$, denote by $p_a \in N'\cap\widetilde{M}$ the orthogonal projection onto $\delta_a \otimes H$. Then we see that $p_aN = \delta_a \otimes \pi(\Lambda)'' = p_a\widetilde{M}p_a$.

In particular, $p_a(N'\cap\widetilde{M})p_a = \mathbb{C}p_a$ for every $a \in \Gamma/\Lambda$. So $N'\cap\widetilde{M}$ admits a finite partition of unity consisting of minimal projections; it must be finite dimensional. In particular $\mathcal{Z}(M)$ is finite dimensional, which precisely means that ρ is the direct sum of finitely many factorial representations.

Assume that $\pi(\Lambda)''$ is of type II_1. Then $\widetilde{M}$ is also of type II_1, and hence M is a finite von Neumann algebra. Moreover it contains N, which is of type II by our description of $\rho|_\Lambda$. So M has no type I direct summand, which proves that it is of type II_1.

(2) Take a subrepresentation σ of ρ'. By Lemma 2.1, we find that $\sigma|_\Lambda$ weakly contains a representation of the form $\pi' \circ \mathrm{Ad}_h$ for some $h \in \Gamma$. So if σ is weakly contained in $\rho\circ\alpha$, then restricting to Λ, we find that $\pi' \circ \mathrm{Ad}_h$ is weakly contained in $\bigoplus_{g\Lambda\in\Gamma/\Lambda} \pi \circ \mathrm{Ad}_g \circ\alpha$. Since π' is factorial, Lemma 2.2 shows that $\pi' \circ \mathrm{Ad}_h$ is weakly contained in some $\pi \circ \mathrm{Ad}_g \circ\alpha$. Hence π' is weakly contained in $\pi \circ \mathrm{Ad}_{g'} \circ\alpha$, for $g' = g\alpha(h)^{-1}$. □

2.4. **Many representations of virtually free groups.** We record here some obvious facts about the abundance of representations of free groups.

Lemma 2.5. — *There exists $n \geqslant 2$ such that for every $k \geqslant n$, the free group on k generators admits uncountably many factorial representations π_i, $i \in I$, of* *hyperfinite type II_1 such that π_i is not weakly contained in $\pi_j \circ \alpha$ for any distinct indices $i, j \in I$ and any $\alpha \in \mathrm{Aut}(F_n)$.*

Proof. — There are uncountably many pairwise non-isomorphic finitely generated simple groups. By Juschenko-Monod's theorem [7] we can even choose these groups to be all amenable. So we may find some n large enough so that F_n has uncountably many non-isomorphic simple quotients Λ_i, $i \in I$, which are amenable (and infinite). The regular representation of each Λ_i yields by composition a unitary representation π_i of F_n. Since Λ_i is simple infinite, it is ICC, so this representation is factorial, and by amenability it generates the hyperfinite II_1-factor.

Note that the kernel of π_i is precisely the kernel of the quotient map $p_i : F_n \to \Lambda_i$. Given two indices $i \neq j$ and an automorphism $\alpha \in \mathrm{Aut}(F_n)$, if $\pi_i \prec \pi_j \circ \alpha$, then the quotient map $p_i : F_n \to \Lambda_i$ factors through $p_j \circ \alpha$. By simplicity of Λ_j, the factorized map $\Lambda_j \to \Lambda_i$ must be an isomorphism. This gives $i = j$.

If $k \geqslant n$, then any representation of F_n gives a representation of F_k by composition with the natural surjection $F_k \to F_n$. □

Lemma 2.6. — *Consider a finitely generated group Γ containing a non-abelian free group of finite index. Then Γ admits uncountably many unitary representations π_i, $i \in I$ of hyperfinite type II_1, such that π_i is not weakly contained in $\pi_j \circ \alpha$ for any distinct indices $i, j \in I$ and any $\alpha \in \mathrm{Aut}(\Gamma)$.*

Proof. — Consider a free subgroup F of finite index in Γ. Taking F smaller if necessary, we can assume that $F = F_k$, for k large enough so that the previous lemma holds true. We can also assume that F is normal inside Γ. In fact, since Γ is finitely generated, it admits only finitely many subgroups of a given finite index. So we may assume that in fact F is characteristic in Γ, i.e. invariant under every automorphism of Γ. Then the result follows from combining the above lemma with Lemma 2.4. □

We observe that a representation π of a group into the hyperfinite II_1-factor is amenable in the sense of Bekka, meaning that $\pi \otimes \overline{\pi}$ has almost invariant vectors. So, if the group is non-amenable, such an amenable representation π is not weakly contained in the regular representation.

3. Proof of the main results

3.1. General results.

Proof of Proposition 1.1. —

(1) Take a non-amenable a-normal subgroup $\Lambda < \Gamma$ and a representation π_0 of Λ. Denote by $\pi = \mathrm{Ind}_\Lambda^\Gamma(\pi_0)$ the induced representation.

Set $I := (\Gamma/\Lambda) \setminus \{\Lambda\}$. Then $\Gamma/\Lambda = \{\Lambda\} \sqcup I$ is a Λ-invariant partition, so that H is the direct sum of two Λ-invariant subspaces, $H_1 = \delta_\Lambda \otimes H_0$ and $H_2 = \ell^2(I) \otimes H_0$. Denote by π_1 and π_2 the two Λ-representations obtained this way.

We see that π_1 is canonically isomorphic with π_0 while Lemma 2.3 describes π_2 as a direct sum of representations $\mathrm{Ind}_\Sigma^\Lambda(\sigma)$, where $\Sigma < \Gamma$ is of the form $\Lambda \cap g\Lambda g^{-1}$ for some $g \in \Gamma \setminus \Lambda$, and σ is a representation of Σ. Since Λ is a-normal in Γ, each such Σ is amenable and thus σ is weakly contained in the regular representation. After inducing to Λ and taking the sirect sum, we find that π_2 is weakly contained in the regular representation of Λ.

By definition, we have an inclusion $\pi(\Gamma)'' \subset B(\ell^2(\Gamma/\Lambda)) \overline{\otimes} \pi_0(\Lambda)''$, and our goal is to prove that this is an equality. By taking commutants, we need to prove that $\pi(\Gamma)' \subset 1 \otimes \pi_0(\Lambda)'$. Denote by $p \in B(H)$ the orthogonal projection onto $H_1 = \delta_\Lambda \otimes H_0$. As explained above, $p \in \pi(\Lambda)'$.

Claim. $pT(1-p) = 0$ for every $T \in \pi(\Lambda)'$.

Otherwise, by classical von Neumann algebra theory, we could find a nonzero partial isometry $u \in \pi(\Lambda)'$ such that $uu^* \leqslant p$ and $u^*u \leqslant 1-p$. Then u implements a conjugation between a subrepresentation of $\pi_1 \simeq \pi_0$ and a subrepresentation of π_2. By Lemma 2.1, this implies that π_0 is weakly contained in π_2, and further, in the regular representation. This is excluded by assumption.

Fix $T \in \pi(\Gamma)'$. In particular, T commutes with $\pi(\Lambda)$ so the claim implies that p commutes with T: there exists $T_0 \in B(H_0)$ such that $T(\delta_\Lambda \otimes \xi) = \delta_\Lambda \otimes (T_0\xi)$ for every $\xi \in H_0$. Further, we observe that $T_0 \in \pi_0(\Lambda)'$. Hence for every $g \in \Gamma$:

$$\begin{aligned} T(\delta_{g\Lambda} \otimes \xi) &= T\pi(g)(\delta_\Lambda \otimes (\pi_0)_{c(g,\Lambda)^{-1}}\xi) \\ &= \pi(g)(\delta_\Lambda \otimes T_0(\pi_0)_{c(g,\Lambda)^{-1}}\xi) \\ &= \pi(g)(\delta_\Lambda \otimes (\pi_0)_{c(g,\Lambda)^{-1}}T_0\xi) \\ &= \delta_{g\Lambda} \otimes T_0\xi. \end{aligned}$$

Therefore, $T = \mathrm{id} \otimes T_0 \in 1 \otimes \pi_0(\Lambda)'$, as desired.

For the moreover part, observe that π_1 is not weakly contained in π_2, as Λ-representations. Hence, there exists a in the universal C*-algebra $C^*(\Lambda)$ such that $\pi_1(a) \neq 0$ while $\pi_2(a) = 0$. This implies that $\pi(a) = \pi_1(a) + \pi_2(a) = \pi_1(a) = \pi(a)p$ and indeed, $p = p_0 \otimes 1$ for some rank one projection p_0.

(2) Assume that π is weakly contained in π'. Then this is also true for the restriction to Λ of these representations. In particular, π_0 is weakly contained in $\pi'|_\Lambda$ which is weakly contained in $\pi'_0 \oplus \lambda_\Lambda$, as we observed in the proof of (1). The factorial case follows from Lemma 2.2. □

Proof of Corollary 1.2. — Assume that Γ contains a proper a-normal non-amenable subgroup Λ which is virtually free. Note that Λ is necessarily of infinite index inside Γ.

Lemma 2.6 provides us with an uncountable family of unitary representations π_i, $i \in I$, of Λ which are all amenable, factorial of type II_1, and none of which is weakly contained in any other. In particular no such π_i is weakly contained in the regular representation.

We may then induce these representations to Γ and Proposition 1.1 gives that the representations ρ_i, $i \in I$ that we get are all factorial of type II_∞, traceable, and none of them is weakly contained in any other. □

Corollary 1.2 raises the question whether virtually free groups themselves admit many traceable representations of type II_∞. This is indeed the case, as follows from the next lemma.

Lemma 3.1. — *Consider a group Γ and a finite index normal subgroup $\Lambda < \Gamma$. Assume that Λ admits a non-amenable a-normal proper subgroup. Then Γ admits a non-amenable a-normal proper subgroup Γ_0 such that $\Gamma_0 \cap \Lambda$ has finite index inside Γ_0.*

Proof. — Take a non-amenable a-normal proper subgroup $\Lambda_0 < \Lambda$. Consider the quotient group $F = \Gamma/\Lambda$, with projection map $p : \Gamma \to F$, and take a maximal subset $I \subset F$ such that there exist lifts $g_i \in \Gamma$, $i \in I$, for which $p(g_i) = i$ for every $i \in I$ and $\bigcap_{i\in I} g_i\Lambda_0 g_i^{-1}$ is non amenable. Denote by Λ_1 this non-amenable subgroup.

Claim. For every $g \in \Gamma$, either $g\Lambda_1 g^{-1} \cap \Lambda_1$ is amenable or g normalizes Λ_1.

Assume that $g\Lambda_1 g^{-1} \cap \Lambda_1$ is non-amenable. Then by maximality of I, we must have that $p(g)I = I$. Then for every index $i \in I$, we find $j \in I$ such that $p(g)i = j$. This means that $gg_i\Lambda_0 = g_j\Lambda_0$, and hence $gg_i\Lambda_0 g_i^{-1} g^{-1} = g_j\Lambda_0 g_j^{-1}$. Applying this observation for every $i \in I$ and intersecting over I we find that indeed g normalizes Λ_1, as claimed.

Note that Λ_1 is a-normal inside Λ, being an intersection of a-normal subgroups of Λ. So it is equal to its own normalizer inside Λ. Furthermore, since Λ has finite index inside Γ, the normalizer $N_\Lambda(\Lambda_1)$ of Λ_1 inside Λ has finite index in the normalizer $N_\Gamma(\Lambda_1)$ inside Γ. So we conclude that $\Lambda_1 = N_\Lambda(\Lambda_1)$ has finite index inside $\Lambda_2 := N_\Gamma(\Lambda_1)$.

Let us check that Λ_2 is a-normal in Γ. Take $g \in \Gamma$ such that $g\Lambda_2 g^{-1} \cap \Lambda_2$ is non-amenable. Since Λ_1 has finite index inside Λ_2, we find that $g\Lambda_1 g^{-1} \cap \Lambda_1$ is non-amenable as well. By the claim this implies that $g \in \Lambda_2$. □

Corollary 3.2. — *If Γ is virtually free, non-amenable, then it admits uncountably many factorial representations of type II_∞ which are traceable, and none of which is weakly contained in any other.*

3.2. **The case of** $\mathrm{SL}_n(\mathbb{Z})$**.** The case $n = 2$ is a special case of Corollary 3.2.

Fix $n \geqslant 3$ and denote by $\Gamma := \mathrm{SL}_n(\mathbb{Z})$. Denote by d the integer part of $n/2$, so that $n = 2d$ or $n = 2d + 1$. Denote by $\Sigma < \Gamma$ the copy of $\mathrm{SL}_2(\mathbb{Z})^d$ given by block diagonal matrices $\mathrm{diag}(A_1, \dots, A_d, 1_{\mathrm{odd}})$, where $A_1, \dots, A_d \in \mathrm{SL}_2(\mathbb{Z})$ and 1_{odd} is the empty matrix if n is even and equals the 1×1-matrix with entry 1 if n is odd.

Denote by $e_1, \dots, e_n$ the canonical basis in $V = \mathbb{R}^n$. For the natural action of Γ on V, Σ fixes the planes $V_k := \mathrm{span}(\{e_{2k-1}, e_{2k}\})$, for $k = 1, \dots, d$. It also fixes the space V_{odd}, defined to be $\mathbb{R}e_n$ if n is odd and 0 otherwise. For every $k = 1, \dots, d$, denote by $\Sigma_k < \Sigma$ the set of elements which preserve V_k. Then Σ_k is a copy of $\mathrm{SL}_2(\mathbb{Z})$ and $\Sigma = \Sigma_1 \times \cdots \times \Sigma_d$.

Although Σ is not a-normal in Γ, the family of subgroups $\{\Sigma_i, , i = 1, \dots, d\}$ satisfies a property of this kind (up to a finite index normalizer). The next lemma specifies this property. The task will be to extend Proposition 1.1 to this setting.

Lemma 3.3. — *The following facts are true :*

(1) *Σ has finite index in its normalizer $\Lambda := N_\Gamma(\Sigma)$;*
(2) *Λ coincides with the set of elements in Γ which globally preserve the direct sum decomposition $V = V_1 \oplus \cdots \oplus V_d \oplus V_{\mathrm{odd}}$. In other words it is the set of elements which permute the spaces V_k, $k = 1, \dots, d$.*
(3) *For every $g \in \Gamma \setminus \Lambda$, there exists $1 \leqslant k \leqslant d$ such that $\Sigma_k \cap g\Sigma g^{-1}$ is amenable.*

(4) *For every* $g \in \Gamma \setminus \Lambda$*, there exists* $1 \leqslant k \leqslant d$ *such that* $\Sigma_k \cap g\Lambda g^{-1}$ *is amenable.*

Proof. —

(1) Note that Σ has finite index inside the set of elements $g \in \Gamma$ such that $gV_i = V_i$ for all $i = 1, \ldots, d$. So (2) is easily seen to imply (1).

(2) Denote by Λ' the set of elements which permute the spaces V_k, $k = 1, \ldots, d$. It is easy to see that elements of Λ' normalize Σ. So $\Lambda' \subset \Lambda$. For the converse inclusion, it suffices to check that property (3) holds for every $g \in \Gamma \setminus \Lambda'$, since clearly the conclusion of (3) prevents g to normalize Σ.

(3) Take $g \in \Gamma \setminus \Lambda'$. Then there exists $1 \leqslant k \leqslant d$ such that V_k is not equal to some gV_i, $i = 1, \ldots, d$. Let us prove that $\Sigma_0 := \Sigma_k \cap g\Sigma g^{-1}$ is amenable.

Denote by $W_k := \sum_{i \neq k} V_i$, and by p the projection onto V_k parallel to W_k and $q = 1-p$ the projection onto W_k. These projections are Σ_0-equivariant. Let us take some $1 \leqslant i \leqslant d$ such that $p(gV_i) \neq 0$. Since $g(V_i)$ is globally Σ_0-invariant, $p(gV_i)$ is Σ_0-invariant as well. If $p(gV_i)$ is one dimensional, then we have found a Σ_0-invariant line inside the plane V_k, proving that Σ_0 acts amenably on V_k. Since it acts trivially on W_k, this implies that Σ_0 is amenable.

Assume on the contrary that $p(gV_i) = V_k$. Then p implements a conjugation between gV_i and V_k. But by assumption, $gV_i \neq V_k$. So must also have $q(gV_i) \neq 0$. If $q|_{gV_i}$ is injective, then we find that Σ_0 acts trivially on gV_i. Otherwise the kernel of q intersects gV_i into a line, which is globally Σ_0-invariant. In both cases we find a Σ_0-invariant line in gV_i, hence in V_k, and we conclude again that Σ_0 is amenable.

(4) follows obviously from (1) and (3). □

Proposition 3.4. — *Consider a type* II_1 *factorial representation* σ *of* Σ*, whose restriction to each* Σ_k*,* $k = 1, \ldots, d$*, is factorial and not weakly contained in the regular representation. The following facts hold true.*

(1) *The induced representation* $\rho = \mathrm{Ind}_\Sigma^\Gamma(\sigma)$ *is a direct sum of finitely many factorial representations of type II. At least one of them is tracial*[2]*.*

(2) *Take* σ' *another representation of* Σ *and* ρ' *denotes its induced* Γ*-representation. If a subrepresentation of* ρ *is weakly contained in* ρ' *then* $\sigma \circ \mathrm{Ad}(g)$ *is weakly contained in* σ'*, for some* $g \in \Lambda$ *(and* $\mathrm{Ad}(g)$ *denotes the automorphism of* Σ *obtained by* g *conjugation).*

Proof. —

(1) In fact we we can give a more precise statement using the induction by stages principle. Denote by $\pi := \mathrm{Ind}_\Sigma^\Lambda(\sigma)$, so that ρ is conjugate to the induced representation of π. Then by Lemma 2.4, we know that π is a direct sum of finitely many factorial representations of type II_1. We will prove that $\rho(\Gamma)'' = B(\ell^2(\Gamma/\Lambda)) \mathbin{\overline{\otimes}} \pi(\Lambda)''$ and that $C^*(\rho(\Gamma))$ contains a non-zero element x such that $(p_\Lambda \otimes 1)x = x$, proving the traceability property.

Denote by H_0 the Hilbert space on which π acts. Set $I := (\Gamma/\Lambda) \setminus \Lambda$. Then $\ell^2(\Gamma/\Lambda) \otimes H_0$ is the direct sum of $H_1 := \delta_\Lambda \otimes H_0$ and $H_2 := \ell^2(I) \otimes H_0$,

[2] In fact all of them, but we don't need this fact.

both of which are Λ-invariant. Denote by π_1 and π_2 the representations of Λ obtained this way.

The result will follow exactly as in Proposition 1.1 once we prove that no sub-representation of π_1 is weakly contained in π_2. Let us restrict further the discussion to Σ.

The restriction $\pi_1|_\Sigma$ is the direct sum of finitely many representations of the form $\sigma \circ \mathrm{Ad}(g)$ for elements $g \in \Lambda$. Indeed this follows by the proof of Lemma 2.4. In particular each such representation is factorial. Note that $\mathrm{Ad}(g)$ permutes the groups Σ_k, $k = 1, \dots, d$. It follows that the restriction of each such representation $\sigma \circ \mathrm{Ad}(g)$ to Σ_k is factorial and not weakly contained in the regular representation for every $k = 1, \dots, d$.

Claim. No subrepresentation of π_1 is weakly contained in π_2.

Assume the contrary. Then by restricting to Σ, we find that some Σ-subrepresentation of $\pi_1|_\Sigma$ is weakly contained in $\pi_2|_\Sigma$. By Lemma 2.1, we find that $\sigma \circ \mathrm{Ad}(g)$ is weakly contained in $\pi_2|_\Sigma$ for some $g \in \Lambda$. For simplicity denote by $\sigma' := \sigma \circ \mathrm{Ad}(g)$ and H' the Hilbert space on which it acts.

Denote by $\Phi : C^*(\pi_2(\Sigma)) \to C^*(\sigma'(\Sigma))$ the C*-morphism such that $\Phi(\pi_2(h)) = \pi'(h)$, for all $h \in \Sigma$. Use Arveson extension theorem to extend Φ to a ucp map $E : B(\ell^2(I) \otimes H_0) \to B(H')$. For every k, denote by $I_k \subset I$ the set of cosets $g\Lambda \in I$ such that $\Sigma_k \cap g\Lambda g^{-1}$ is amenable. By Lemma 3.3, we have $\bigcup_{k=1}^d I_k = I$. Denote by $p_k \in B(\ell^2(I) \otimes H_0)$ the orthogonal projection onto $\ell^2(I_k) \otimes H_0$. Since I_k is Σ_k-invariant, p_k commutes with $\pi_2(\Sigma_k)$. Denote by $r_k \in B(H')$ the support projection of $E(p_k)$.

Since $\bigcup_{k=1}^d I_k = I$, we get $\sum_k p_k \geqslant 1$, and thus we may find k such that $E(p_k) \neq 0$. In this case r_k is a non-zero projection invariant under $\sigma'(\Sigma_k)$. Moreover, the map E restricted to $p_k B(\ell^2(I) \otimes H_0) p_k$ witnesses that $p_k \pi_2|_{\Sigma_k}$ weakly contains $r_k \sigma'|_{\Sigma_k}$. Since $\sigma'|_{\Sigma_k}$ is factorial, Lemma 2.1 implies that $r_k \sigma'|_{\Sigma_k}$ is weakly equivalent to $\sigma'|_{\Sigma_k}$. Moreover, our choice of I_k and p_k and Lemma 2.3 imply that $p_k \pi_2|_{\Sigma_k}$ is weakly contained in the regular representation of Σ_k. So we arrive at the conclusion that $\sigma'|_{\Sigma_k}$ is weakly contained in the regular representation. But we observed that this was impossible. This contradiction finishes the proof of the claim, and the rest of (1) follows as in the proof of Proposition 1.1.

(2) Assume that a subrepresentation of ρ is weakly contained in ρ'. By (1), we know that ρ is the direct sum of finitely many factorial representations of the form $\mathrm{Ind}_\Lambda^\Gamma(\pi_0)$ for some factorial representation π_0 of Λ. In fact π_0 is a factorial subrepresentation of $\pi = \mathrm{Ind}_\Sigma^\Lambda(\sigma)$.

By Lemma 2.1, we deduce that one such factorial summand $\mathrm{Ind}_\Lambda^\Gamma(\pi_0)$ is weakly contained in ρ'. Restricting to Σ we deduce that in particular $\pi_0|_\Sigma$ is weakly contained in $\rho'|_\Sigma$. But $\pi_0|_\Sigma$ is a subrepresentation of $\pi|_\Sigma$, which is the direct sum of finitely many factorial representations of the form $\sigma \circ \mathrm{Ad}(g)$. So applying again Lemma 2.1, we find that some $\sigma \circ \mathrm{Ad}(g)$ is weakly contained in $\rho'|_\Sigma$.

Now we apply the analysis made in (1) to $\rho'|_\Sigma$. It is the direct sum of finitely many representations $\sigma' \circ \mathrm{Ad}(h)$, for some elements $h \in \Lambda$, together with one representation σ_2, which is the restriction to Σ of the representation on $\ell^2(I) \otimes H_0'$. The proof of the claim above shows that $\sigma \circ \mathrm{Ad}(g)$ is not weakly contained in σ_2. By Lemma 2.2, we then deduce that $\sigma \circ \mathrm{Ad}(g)$ is weakly contained in some $\sigma' \circ \mathrm{Ad}(h)$ for some $h \in \Lambda$, and we get the desired conclusion. □

Proof of Theorem 1.3. — By Lemma 2.6, we may find an uncountable family π_i, $i \in I$, of factorial representations of type II_1 of $\mathrm{SL}_2(\mathbb{Z})$ which are all amenable and such that π_i is not weakly contained in $\pi_j \circ \alpha$ for any $i \neq j$ and $\alpha \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{Z}))$. Since I is uncountable, we may cut it in d copies of itself, $I \simeq I^d$. So in fact, we may find uncountably many d-tuples of representations $(\pi_i^1, \dots, \pi_i^d)$, $i \in I$, such that π_i^k is not weakly contained in $\pi_j^\ell \circ \alpha$ for every $(i,k) \neq (j,\ell)$, and $\alpha \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{Z}))$.

Each such tuple gives a representation σ_i of Σ, defined by

$$\sigma_i(g_1, \dots, g_d) = \pi_i^1(g_1) \otimes \cdots \otimes \pi_i^d(g_d), \text{ for every } (g_1, \dots, g_d) \in \Sigma.$$

Since the action of Λ on Σ permutes the factors Σ_i, we find that σ_i is not weakly contained in $\sigma_j \circ \mathrm{Ad}(g)$ for every $i \neq j$ in I and $g \in \Lambda$.

Moreover, each σ_i is factorial since $\sigma_i(\Sigma)'' = \pi_i^1(\Sigma_1)'' \overline{\otimes} \cdots \overline{\otimes} \pi_i^d(\Sigma_d)''$. Likewise, its restriction to Σ_k, $k = 1, \dots, d$, is factorial and amenable (hence not weakly contained in the regular representation).

We may apply Proposition 3.4 to find in the induced Γ-representation of each σ_i a direct summand ρ_i which is factorial and traceable. The family ρ_i is uncountable and none of them is weakly contained in any other. □

Acknowledgement

We thank Cyril Houdayer for inspiring discussions on this question and Bachir Bekka and the anonymous referee for useful comments.

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Manuscript received January 6, 2022,
revised March 28, 2022,
accepted March 28, 2022.

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