CONFLUENTES MATHEMATICI

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Tome 12, nº 1 (2020), p. 93-146.

<http://cml.centre-mersenne.org/item?id=CML_2020__12_1_93_0>

Confluentes Math.
12, 1 (2020) 93-146

UNITARY REPRESENTATIONS OF P-ADIC $U(5)$

CLAUDIA SCHOEMANN

Abstract. We study the parabolically induced complex representations of the unitary group in 5 variables, $U(5)$, defined over a p-adic field.

Let F be a p-adic field. Let $E : F$ be a field extension of degree two. $U(5)$ has three proper standard Levi subgroups, the minimal Levi subgroup $M_0 \cong E^* \times E^* \times E^1$ and the two maximal Levi subgroups $M_1 \cong \mathrm{GL}(2, E) \times E^1$ and $M_2 \cong E^* \times U(3)$.

We consider representations induced from M_0, representations induced from non-cuspidal, not fully-induced representations of M_1 and M_2 and representations induced from cuspidal representations of M_1.

We determine the points and lines of reducibility and the irreducible subquotients of these representations. Further we describe - except several particular cases - the unitary dual in terms of Langlands quotients.

Contents

2020 *Mathematics Subject Classification:* 22E50, 11S85, 20G05, 20C99.

Keywords: Representations, unitary group, unitary, $U(5)$, p-adic groups.

1. Introduction

Determining the irreducible and the unitary dual of a reductive algebraic group is an important problem in representation theory, with numerous applications in harmonic analysis and the theory of automorphic forms. The description of the irreducible and the unitary dual of the unitary group $U(n)$ over a non-archimedean local field is a long-standing open question.

We study the parabolically induced complex representations of the unitary group in 5 variables - $U(5)$ - defined over a non-archimedean local field of characteristic 0, a p-adic field.

A similar example for the composition series for induced representations of $SO(5)$ over a p-adic field can be found in [12] and examples for unitary duals for groups of low rank in [13] for $SO(5)$ and in [15] for the simply connected split simple group of type G_2.

Let F be a p-adic field. Let $E : F$ be a field extension of degree two. Let $\mathrm{Gal}(E : F) = \{\mathrm{id}, \sigma\}$ be the Galois group. We write $\sigma(x) = \overline{x}$ for all $x \in E$. Let $E^* := E \setminus \{0\}$ and let $E^1 := \{x \in E \mid x\overline{x} = 1\}$.

$U(5)$ has three proper standard Levi subgroups, the minimal Levi subgroup $M_0 \cong E^* \times E^* \times E^1$ and the two maximal Levi subgroups $M_1 \cong \mathrm{GL}(2, E) \times E^1$ and $M_2 \cong E^* \times U(3)$.

We consider representations induced from M_0, representations induced from non-cuspidal, not fully-induced representations of M_1 and M_2 and representations induced from cuspidal representations of M_1.

We determine the points and lines of reducibility of the representations of $U(5)$, and we determine the irreducible subquotients. Further we describe - except several particular cases - the unitary dual in terms of Langlands quotients.

Tools of proof include intertwining operator methods by long Weyl group elements, the Jacquet restriction with respect to proper parabolic subgroups and the Frobenius reciprocity. When inducing from cuspidal representations of $\mathrm{GL}(2, E) \times E^1$ methods of proof involve base change lift from $U(2)$ to $\mathrm{GL}(2)$ ([16]) and the poles and zeros of local Asai L-functions ([4]).

The irreducible complex representations of $U(3)$ over a p-adic field obtained as subquotients of parabolically induced representations have been classified by C. D. Keys in [10], the irreducible complex representations of $U(4)$ over a p-adic field obtained as subquotients of parabolically induced representations by K. Konno in [11].

In Section 1 we give some definitions. Section 3 lists results by previous authors that will be used throughout the article. In Section 4 we give the classification of the irreducible non-cuspidal representations of $U(3)$, as has been done in [10]. We reassemble the results for the irreducible unitary representations.

In Section 5 we determine when the induced representations to $U(5)$ are irreducible. It is done for representations induced from the minimal Levi subgroup M_0 and for non-cuspidal, not fully-induced representations of the two maximal Levi subgroups $M_1 \cong \mathrm{GL}(2, E) \times E^1$ and $M_2 \cong E^* \times U(3)$.

For M_1 this means that the representation of the $\mathrm{GL}(2, E)$-part is a proper subquotient of a representation induced from $E^* \times E^*$ to $\mathrm{GL}(2, E)$. For M_2 this means that the representation of the $U(3)$-part of M_2 is a proper subquotient of a representation induced from $E^* \times E^1$ to $U(3)$.

Representations of M_0 are of the form $| \; |_p^{\alpha_1} \chi_1 \otimes | \; |_p^{\alpha_2} \chi_2 \otimes \lambda'$, where $| \; |_p$ denotes the p-adic norm on E, $\alpha_1, \alpha_2 \in \mathbb{R}$, χ_1, χ_2 are unitary characters of E^* and λ' is a unitary character of E^1. Reducibility of the induced representation $| \; |_p^{\alpha_1} \chi_1 \times | \; |_p^{\alpha_2} \chi_2 \rtimes \lambda'$ depends on α_1, α_2 and on the two unitary characters χ_1 and χ_2.

Let $N_{E/F}(E)$ denote the norm map of E with respect to the field extension $E : F$, then $N_{E/F}(x) = x\overline{x}$ for all $x \in E$.

In Theorems 5.1, 5.2 and 5.4 we show that for $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $| \; |_p^{\alpha_1} \chi_1 \times | \; |_p^{\alpha_2} \chi_2 \rtimes \lambda'$ is reducible if and only if at least one of the following cases holds:

(1) $| \alpha_1 - \alpha_2 | = 1$ and $\chi_1 = \chi_2$,
(2) $| \alpha_1 + \alpha_2 | = 1$ and $\chi_1(x) = \chi_2^{-1}(\overline{x}) \; \forall x \in E^*$,
(3) $\exists i \in \{1, 2\}$ s.t.$\alpha_i = 1$ and $\chi_i = 1$,
(4) $\exists i \in \{1, 2\}$ s.t.$\alpha_i = 1/2$ and $\chi_i \mid F^* \neq 1$, but $\chi_i \mid N_{E/F}(E^*) = 1$,
(5) $\exists i \in \{1, 2\}$ s.t.$\alpha_i = 0$ and $\chi_i \neq 1$, but $\chi_i \mid F^* = 1$.

Let χ be a unitary character of E^*. The condition that $\chi(x) = \chi^{-1}(\overline{x})$ for all $x \in E^*$ is equivalent to the condition that $\chi \mid N_{E/F}(E^*) = 1$ and to the fact that χ is a character of a type as in (3),(4) or (5) of the list above.

In 5.3 we consider representations induced from irreducible non-cuspidal representations of M_1 and M_2 that are not fully-induced.

We consider $| \det |_p^{\alpha} \chi \, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $| \det |_p^{\alpha} \chi 1_{\mathrm{GL}_2} \rtimes \lambda'$, where $\alpha \in \mathbb{R}_+$, χ is a unitary character of E^*, $\mathrm{St}_{\mathrm{GL}_2}$ is the Steinberg representation of $\mathrm{GL}(2, E)$, λ' is a unitary character of E^1 and 1_{GL_2} is the trivial representation of $\mathrm{GL}(2, E)$.

In Theorem 5.5 and in Proposition 5.6 we show that for $\alpha \in \mathbb{R}_+$, $| \; |_p^{\alpha} \chi \, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $| \; |_p^{\alpha} \chi 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible unless one of the following cases holds:

(1) $\alpha = 1/2$ or $\alpha = 3/2$ and $\chi = 1$,
(2) $\alpha = 0$, $\alpha = 1/2$ or $\alpha = 1$ and $\chi \mid F^* \neq 1$, but $\chi \mid N_{E/F}(E^*) = 1$,
(3) $\alpha = 1/2$ and $\chi \neq 1$ but $\chi \mid F^* = 1$.

We consider $| \; |^{\alpha} \chi \rtimes \tau$, where $\alpha \in \mathbb{R}_+$, χ is a unitary character of E^* and τ is an irreducible non-cuspidal unitary representation of $U(3)$ that is not fully-induced. We consider all irreducible proper subquotients τ of representations induced to $U(3)$ from its unique proper Levi-subgroup $M \cong E^* \times E^1$, as classified in [10].

In Theorems 5.7, 5.9, 5.11 and in Propositions 5.8, 5.10 and Remark 5.14 we show that these representations are irreducible unless one has a certain combination of $\alpha \in \{0, 1/2, 1, 3/2, 2\}$ and $\chi = 1$, or $\chi \neq 1$ but $\chi \mid \mathrm{F}^* = 1$, or $\chi \mid F^* \neq 1$ but $\chi \mid N_{E/F}(E^*) = 1$.

In the case of reduciblility the irreducible subquotients are determined in the course of Section 5. In several cases the irreducible subquotients are determined in Section 6.

In Section 6 we treat several 'special' reducibility points of representations induced from the minimal parabolic subgroup P_0 with Levi subgroup $M_0 \cong E^* \times E^* \times E^1$. In some cases the induced representation $\mid \mid_p^{\alpha_1} \chi_1 \times \mid \mid_p^{\alpha_2} \chi_2 \rtimes \lambda'$ has more than two irreducible subquotients, more precisely it has four irreducible subquotients.

If this is the case, then $\alpha_1, \alpha_2 \in \{0, 1/2, 1, 3/2, 2\}$ and $\chi_i(x) = \chi_i^{-1}(\overline{x})$ for $i = 1, 2$ and for all $x \in E^*$. Then $\chi_i = 1$, or $\chi_i \neq 1$ but $\chi_i \mid F^* = 1$, or $\chi_i \mid F^* \neq 1$ but $\chi_i \mid N_{E/F}(E^*) = 1$, for $i = 1, 2$.

We determine the irreducible subquotients in terms of Langlands quotients, and we determine whether these Langlands quotients are unitary.

In Section 7 we give a classification of the irreducible unitary representations of $U(5)$ in terms of Langlands quotients. At first we consider the irreducible subquotients obtained by induction from representations of M_0 and from non-cuspidal, not fully-induced representations of M_1 and M_2 (the subquotients determined in Section 5). We then consider the irreducible subquotients of representations induced from cuspidal representations of $M_1 \cong \mathrm{GL}(2, E) \times E^1$.

2. Definitions

Let F be a non-archimedean local field of characteristic 0, that is $\mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$, where p is a prime number.

Let G be a connected reductive algebraic group, defined over F. Let V be a vector space, defined over the complex numbers. Let π be a representation of G on V. We denote it by (π, V) and sometimes by π or V. Let $(\widetilde{\pi}, \widetilde{V})$ denote the dual representation of (π, V).

Let $E : F$ be a field extension of degree two, let $\mathrm{Gal}(E : F) = \{\mathrm{id}, \sigma\}$. We write $\sigma(x) = \overline{x}$ for all $x \in E$ for the non-trivial element of the Galois group.

Let E^* denote the group of invertible elements of E and $E^1 := \{x \in E : x\overline{x} = 1\}$.

Let $N_{E/F}(\ )$ denote the norm on E corresponding to the field extension E/F of degree 2: $N_{E/F}(x) = x\overline{x}$ for all $x \in E$. $N_{E/F}(E^*) \subset F^*$ and $\mid F^*/N_{E/F}(E^*) \mid = 2$.

Let $\omega_{E/F} : F^* \to \mathbb{C}^*$ be the unique non-trivial smooth character with $\omega_{E/F} \mid N_{E/F}(E^*) = 1$. Note that $\omega_{E/F}$ is determined by local class field theory.

Let $X_{\omega_{E/F}}$ be the set of characters χ of E^* such that $\chi \mid F^* = \omega_{E/F}$. Characters in $X_{\omega_{E/F}}$ are unitary.

Let $X_{1_{F^*}}$ be the set of characters χ of E^* that are non-trivial and whose restriction to F^* is trivial: $\chi \neq 1$, $\chi \mid F^* = 1$.

Let $X_{N_{E/F}(E^*)} = \{1\} \cup X_{\omega_{E/F}} \cup X_{1_{F^*}}$. It exhausts all characters χ of E^* trivial on $N_{E/F}(E^*)$, that is verifying $\chi(x) = \chi^{-1}(\overline{x})$ for all $x \in E^*$.

Let $\Phi \in \mathrm{GL}(n, E)$ be a hermitian matrix (that is $\overline{\Phi}^t = \Phi$) and U_Φ the unitary group defined by Φ :

$$U_\Phi = \{g \in \mathrm{GL}(n, E) : g\Phi\overline{g}^t = \Phi\},$$

Let $\Phi_n = (\Phi_{ij})$, where $\Phi_{ij} = (-1)^{i-1}\delta_{i,n+1-j}$ and δ_{ab} is the Kronecker delta.

Let $\zeta \in E^*$ be an element of trace 0, that is $tr(\zeta) = \zeta + \overline{\zeta} = 0$.

If n is odd, then $\Phi_n = \begin{pmatrix} & & & & 1 \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}$ is hermitian. If n is even, $\zeta\Phi_n = \begin{pmatrix} & & & & \zeta \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & -\zeta & & \\ & \zeta & & & \\ -\zeta & & & & \end{pmatrix}$ is hermitian.

Denote by $U(n)$ the **unitary group** corresponding to Φ_n if n is odd or to $\zeta\Phi_n$ if n is even, respectively. It is quasi-split.

Let n be a positive integer. We will call **Levi subgroup** of $U(n)$ a subgroup of block diagonal matrices

$$M := \{ \begin{pmatrix} A_1 & & & & & & - & 0 \\ & A_2 & & & & & & | \\ & & \ddots & & & & & \\ & & & A_k & & & & \\ & & & & B & & & \\ & & & & & {}^t\overline{A_k}^{-1} & & \\ | & & & & & & \ddots & \\ 0 & - & & & & & & {}^t\overline{A_1}^{-1} \end{pmatrix},$$

where $A_i \in \mathrm{GL}_{n_i}(E)$ for $1 \leqslant i \leqslant k$, $B \in U(m)\}$ and $m, n_1, \ldots, n_k$ are strictly positive integers such that $m + 2\sum_{i=1}^{k} n_i = n$. (If $k = 0$, then there are no n_i and $M = U(n)$.)

It is canonically isomorphic to the product $\mathrm{GL}(n_1, E) \times \cdots \times \mathrm{GL}(n_k, E) \times U(m)$. We choose the corresponding **parabolic subgroup** P such that it contains M and the subgroup of upper triangular matrices in $U(n)$. We call a parabolic subgroup P that contains the subgroup of upper triangular matrices standard. Let N be the unipotent subgroup with identity matrices for the block diagonal matrices of M, arbitray entries in E above and $0's$ below. Then one has the Levi decomposition $P = MN$.

We consider representations of the Levi subgroups and extend them to representations of P by trivial extension to the unipotent subgroup N. We perform normalized **parabolic induction** to the whole group $U(n)$.

Let π_i, $i = 1, \ldots k$, be smooth admissible representations of $\mathrm{GL}(n_i, E)$ and σ a smooth admissible representation of $U(m)$. Let $\pi_1 \otimes \ldots \otimes \pi_k \otimes \sigma$ denote the representation of $M = \mathrm{GL}(n_1, E) \times \ldots \times \mathrm{GL}(n_k, E) \times U(m)$ and denote by $\pi := \mathrm{Ind}_P^{U(n)}(\pi_1 \otimes \ldots \otimes \pi_k \otimes \sigma) = \pi_1 \times \ldots \times \pi_k \rtimes \sigma$ the normalized parabolically induced representation, where P is the corresponding standard parabolic subgroup containing M.

Let π be an irreducible representation of $\mathrm{GL}(n, E)$. Then there exist irreducible cuspidal representations $\rho_1, \rho_2, \ldots, \rho_k$ of general linear groups that are, up to isomorphism, uniquely defined by π, such that π is isomorphic to a subquotient of $\rho_1 \times \cdots \times \rho_k$. The multiset of equivalence classes $(\rho_1, \ldots, \rho_k)$ is called the **cuspidal support** of π. It is denoted by $\mathrm{supp}(\pi)$.

Let $n \in \mathbb{N}$ and let τ be an irreducible representation of $U(n)$. Then there exist irreducible cuspidal representations $\rho_1, \ldots, \rho_k$ of general linear groups and an irreducible cuspidal representation σ of some $U(m)$ that are, up to isomorphism and replacement of ρ_i by $\rho_i^{-1}(-)$ for some $i \in \{1, \ldots, k\}$, uniquely defined by τ, such that τ is isomorphic to a subquotient of $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$. The representation σ is called the **partial cuspidal support** of τ and is denoted by τ_{cusp}.

For a parabolically induced representation π of G let $s_P(\pi)$ and $s_{min}(\pi)$ denote the **Jacquet restrictions** with respect to the parabolic subgroup P and with respect to the minimal parabolic subgroup, respectively ([8]).

Let π be a smooth representation of finite length of G. Then $\hat{\pi}$ denotes the **Aubert dual** of π, as defined in [1].

Let $\mathcal{E}_2(\mathrm{G})$ be the set of equivalence classes of irreducible square-integrable representations of G.

Let $\mathrm{Hom}(M, \mathbb{C}^*)^{n.r.}$ denote the group of unramified characters of M.

Let $R(U(n))$ be the Grothendieck group of the category of admissible representations of finite length of $U(n)$ and let $R(U) := \bigoplus_{n \geqslant 0} R(U(n))$.

We define the R-group, a subgroup of the Weyl group W of G. Let λ be a character of the minimal Levi subgroup M_0, and $W_\lambda := \{w \in W : w\lambda = \lambda\}$. Let $a(w, \lambda) : \mathrm{Ind}_P^{U(5)}(\lambda) \to \mathrm{Ind}_P^{U(5)}(w\lambda)$ be the intertwining operator of $\mathrm{Ind}_P^{\mathrm{G}}(\lambda)$ corresponding to w, where $w\lambda(m) := \lambda(w^{-1}mw)$. Let $W' := \{w \in W_\lambda : a(w, \lambda) \text{ is scalar}\}$. Then $W_\lambda = R \ltimes W'$ ([9]).

3. Previous Results

We list results by previous authors that will be used throughout the article.

The group $U(3)$ has one proper parabolic subgroup P with the Levi subgroup $M \cong E^* \times E^1$. For a smooth character $\lambda \in \mathrm{Hom}(M, \mathbb{C}^*)$ there exist unique smooth characters $\lambda_1 \in \mathrm{Hom}(E^*, \mathbb{C}^*)$ and $\lambda' \in \mathrm{Hom}(E^1, \mathbb{C}^*)$ such that $\lambda \cong \lambda_1 \otimes \lambda'$.

By [10] the induced representation $\mathrm{Ind}_P^{U(3)}(\lambda)$ is irreducible except in the following cases:

(1) $\lambda_1 = |\ |_E^{\pm 1}$
(2) $\lambda_1 = |\ |_E^{\pm 1/2} \chi_{\omega_{E/F}}$, where $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$,
(3) $\lambda_1 = \chi_{1_{F^*}}$, where $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

Note that in 1. and 2. changing the sign of the exponent is equivalent to replacing λ by $w\lambda$, where $w \in W$ is the non-trivial element of the Weyl group. Thus the sign of the exponent does not affect the set of irreducible constituents. We give the classification for positive exponent, for negative exponent the irreducible constituents exchange roles.

The classification does not depend on λ'.

In the first case $\mathrm{Ind}_P^{U(3)}(\lambda)$ has exactly two constituents, the character $\psi := \lambda' \circ \det$ and the square-integrable subrepresentation $\mathrm{St}_{U(3)} \cdot \psi$. Both ψ and $\mathrm{St}_{U(3)} \cdot \psi$ are unitary.

In the second case $\mathrm{Ind}_P^{U(3)}(\lambda)$ has exactly two constituents, a square-integrable (and hence unitary) representation $\pi_{1,\chi_{\omega_{E/F}}}$ and a non-tempered unitary representation $\pi_{2,\chi_{\omega_{E/F}}}$.

In the third case $\mathrm{Ind}_P^{U(3)}(\lambda)$ decomposes into the direct sum $\sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$. The two constituents $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are tempered, hence unitary.

Let G be a connected reductive group defined over a p-adic field. Let (π, V) be a representation of G, for a finite dimensional vector space V. The following construction is given in [18]:

Remark 3.1. — Assume one has, on the same vector space V, a continuous family of induced irreducible representations $(\pi_\alpha, V), \alpha \in X$, where X is a connected set, that posses non-trivial hermitian forms (invariant under G). Suppose that some π_α is unitary. If a family of non-degenerate hermitian forms on a finite dimensional space, parametrised by X, is positive definite at one point of X, it is positive definite everywhere. Hence π_α is unitary for all $\alpha \in X$.

Remark 3.2. — One may reduce the argument to finite dimensional spaces by considering spaces $\oplus V(\delta)$, where δ runs over fixed finite subsets of the irreducible unitary representations of the maximal compact subgroup of G.

Let M be a Levi subgroup of G and let π be an irreducible representation of M. The Lemma 5.1(i) of [15] is a special case of Theorem 4.5 in [17]:

Lemma 3.3. — *The set of all $\sigma \in \mathrm{Hom}(M, \mathbb{C}^*)^{n.r.}$ such that $\mathrm{Ind}_P^{\mathrm{G}}(\sigma \otimes \pi)$ has an irreducible unitary subquotient is compact.*

Let $\lambda \in \mathrm{Hom}(M_0, \mathbb{C}^*)$. By [9] Corollary 1, the number of inequivalent irreducible components of $\mathrm{Ind}_P^G(\lambda)$ equals the number of conjugacy classes in R.

Let $\mathrm{G} := \left[\begin{smallmatrix} U(2n) \\ U(2n+1)\end{smallmatrix}\right.$ be the unitary group in $2n$ or $2n+1$ variables, respectively.

For $m \leqslant n$ let $\mathrm{G}(m) := \left[\begin{smallmatrix} U(2m) & \text{if } \mathrm{G}=U(2n) \\ U(2m+1) & \text{if } \mathrm{G}=U(2n+1)\end{smallmatrix}\right.$. By convention $\mathrm{G}(0) = U(1)$.

Let $\sigma_i \in \mathcal{E}_2(\mathrm{GL}_{n_i}(E))$, $i = 1, 2, \ldots$, and $\rho \in \mathcal{E}_2(\mathrm{G}(m))$.

Theorem 3.4 ([5], Thm. 3.4). — *Let $\mathrm{G} = U(2n)$ or $U(2n+1)$. Let $P = MN$ be a parabolic subgroup of* G. *Suppose that $M \cong \mathrm{GL}(n_1, E) \times \ldots \times \mathrm{GL}(n_r, E) \times \mathrm{G}(m)$. Let $\sigma \in \mathcal{E}_2(M)$, with $\sigma \cong \sigma_1 \otimes \ldots \otimes \sigma_r \otimes \rho$. Let d be the number of inequivalent σ_i, such that $\mathrm{Ind}_{\mathrm{GL}_{n_i} \times \mathrm{G}(m)}^{\mathrm{G}(m+n_i)}(\sigma_i \otimes \rho)$ reduces. Then $R \cong (\mathbb{Z}/2\mathbb{Z})^d$.*

We have Lemma 2.1 of [19]:

Lemma 3.5 ([19]). — *Let π be an irreducible representation of $U(m)$ and let ρ be an irreducible cuspidal representation of a general linear group $\mathrm{GL}(p, F)$. Suppose*

(1) $\rho \neq \widetilde{\rho}(-)$.
(2) $\rho \rtimes \pi_{\mathrm{cusp}}$ *is irreducible.*
(3) $\rho \times \rho'$ *and* $\widetilde{\rho}(-) \times \rho'$ *are irreducible for any factor* ρ' *of* π.
(4) *Neither* ρ *nor* $\widetilde{\rho}(-)$ *is a factor of* π.

Then $\rho \rtimes \pi$ is irreducible.

4. The representations of $U(3)$

4.1. **The irreducible representations of $U(3)$.** Let P be the unique standard proper parabolic subgroup of $U(3)$, M the standard Levi subgroup and N the unipotent radical corresponding to P. Then $P = MN$ is the parabolic subgroup of $U(3)$ defined in Section 2 for $m = k = n_1 = 1$. We have

$$M = \left\{ \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \overline{x}^{-1}\end{smallmatrix}\right), x \in E^*, k \in E^1 \right\},\ N = \left\{ \left(\begin{smallmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1\end{smallmatrix}\right), \alpha, \beta \in E, \alpha\overline{\alpha} = \beta + \overline{\beta} \right\} \quad \text{and}$$

$$P = MN = \left\{ \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \overline{x}^{-1}\end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \alpha & \beta \\ 0 & 1 & \overline{\alpha} \\ 0 & 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} x & x\alpha & x\beta \\ 0 & k & k\overline{\alpha} \\ 0 & 0 & \overline{x}^{-1}\end{smallmatrix}\right),\ \begin{array}{l} x \in E^*,\ k \in E^1, \\ \alpha, \beta \in E,\ \alpha\overline{\alpha} = \beta + \overline{\beta}\end{array} \right\}.$$

For a smooth character $\lambda \in \mathrm{Hom}(M, \mathbb{C}^*)$ there exist unique smooth characters $\lambda_1 \in \mathrm{Hom}(E^*, \mathbb{C}^*)$ and $\lambda' \in \mathrm{Hom}(E^1, \mathbb{C}^*)$ such that

$$\lambda\Big(\left(\begin{smallmatrix} x & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \overline{x}^{-1} \end{smallmatrix}\right) \Big) = \lambda_1(x)\lambda'(x\overline{x}^{-1}k), \ \forall x \in E^*, \forall k \in E^1.$$

Remark 4.1. — (1) Every smooth character of E^* can be written in the form $\lambda_1(x) =\mid x \mid^{\alpha_1} \chi_1(x)$, with $\alpha_1 \in \mathbb{R}$ and χ_1 a unitary character.

(2) $\lambda' : E^1 \to \mathbb{C}^*$ is smooth and E^1 is a compact group, hence λ' is unitary.

These are all characters of M. We extend λ from M to P, by taking $\lambda \mid N = 1$. We induce parabolically from P to $U(3)$ and obtain

$$\pi := \mathrm{Ind}_P^{U(3)}(\lambda) = \mathrm{Ind}_P^{U(3)}(\lambda_1 \otimes \lambda') =: \lambda_1 \rtimes \lambda'.$$

The complex vector space V of the representation π is defined as follows:

$$V := \left\{ f : U(3) \to \mathbb{C} : \begin{array}{l} f \text{ smooth and } f(mng) = \delta_P^{1/2}(m)\lambda(m)f(g) \\ \forall m \in M, \ \forall n \in N, \ \forall g \in U(3) \end{array} \right\}.$$

Here $\delta_P^{1/2}$ is the modulus character, and π acts on V by right translations.

Let $\alpha \in \mathbb{R}_+^*$ and χ be a unitary character of E^*. Let λ' be a character of E^1. Let $\lambda =\mid \ \mid^{\alpha} \chi \otimes \lambda'$ be a character of the Levi subgroup M and $\mid \ \mid^{\alpha} \chi \rtimes \lambda'$ the parabolically induced representation to $U(3)$. Then $\mid \ \mid^{\alpha} \chi \rtimes \lambda'$ has a unique irreducible quotient denoted by $\mathrm{Lg}(\mid \ \mid^{\alpha} \chi \rtimes \lambda')$, the Langlands quotient.

Let $T := \{ \left(\begin{smallmatrix} x & & \\ & 1 & \\ & & \overline{x}^{-1} \end{smallmatrix}\right), x \in F^* \}$ be the maximal split torus over F.

Let $N(T)$ be the normaliser and $C(T)$ the centraliser of T in $U(3)$, respectively. The Weyl group is $W := N(T)/C(T) \cong \mathbb{Z}/2\mathbb{Z}$.

By [10] the induced representation $\mathrm{Ind}_P^{U(3)}(\lambda)$ is irreducible except in the three cases:

(1) $\lambda_1 =\mid \ \mid_E^{\pm 1}$,
(2) $\lambda_1 =\mid \ \mid_E^{\pm 1/2} \chi_{\omega_{E/F}}$,
(3) $\lambda_1 = \chi_{1_{F^*}}$.

In the first case $\mathrm{Ind}_P^{U(3)}(\lambda)$ has exactly two constituents, the unitary character $\psi := \lambda' \circ \det = \mathrm{Lg}(\lambda_1; \lambda')$ and the square-integrable (hence unitary) subrepresentation $\mathrm{St}_{U(3)} \cdot \psi$.

In the second case $\mathrm{Ind}_P^{U(3)}(\lambda)$ has exactly two constituents, a square-integrable (hence unitary) representation $\pi_{1,\chi_{\omega_{E/F}}}$ and a non-tempered unitary representation $\pi_{2,\chi_{\omega_{E/F}}} = \mathrm{Lg}(\lambda_1; \lambda')$.

In the third case $\mathrm{Ind}_P^{U(3)}(\lambda)$ decomposes into the direct sum $\sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$. The two constituents $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are tempered, hence unitary.

Remark 4.2. — $\pi_{2,\chi_{\omega_{E/F}}}$ is unitary: Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. $\chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible and unitary, $\mid \ \mid^{\alpha} \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible and unitary for $\alpha \in (0, 1/2)$, by Theorem 4.6, (1.3). By [14] the irreducible subquotients $\pi_{\chi_{1,\chi_{\omega_{E/F}}}}$ and $\pi_{2,\chi_{\omega_{E/F}}}$ of $\mid \ \mid^{1/2} \chi_{\omega_{E/F}} \rtimes \lambda'$ are unitary. See Theorem 4.6.

Remark 4.3. — $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are tempered: In the third case $\lambda_1 =: \chi_{1_{F^*}} \in X_{1_{F^*}}$. Since $\chi_{1_{F^*}}$ is square-integrable, $\chi_{1_{F^*}} \rtimes \lambda'$ is tempered and so are its constituents.

We obtain the following

Corollary 4.4. — *If $\lambda_1 \rtimes \lambda'$ is reducible there are always two distinct irreducible subquotients. They are unitary.*

4.2. The irreducible unitary representations of $U(3)$.

Proposition 4.5. — *The following list exhausts all irreducible hermitian representations of $U(3)$ supported in its parabolic subgroup P. Let $\alpha > 0$, and let χ be a smooth unitary character of E^*.*

(0) $\chi \rtimes \lambda'$, $\chi \notin X_{1_{F^*}}$, $\sigma_{1,\chi_{1_{F^*}}}, \sigma_{2,\chi_{1_{F^*}}}$ as introduced above, tempered,
(1) $\lambda'(\det) = \mathrm{Lg}(|\ |\ 1;\lambda'), \pi_{2,\chi_{\omega_{E/F}}} = \mathrm{Lg}(|\ |^{1/2}\ \chi;\lambda')$, for $\chi \in X_{\omega_{E/F}}$ non-tempered, unitary,
(2) $\lambda'(\det)\,\mathrm{St}_{U(3)}, \pi_{1,\chi_{\omega_{E/F}}}$ square-integrable,
(3) $|\ |^{\alpha}\ 1 \rtimes \lambda', \alpha \neq 1$; $|\ |^{\alpha}\ \chi \rtimes \lambda'$, $\alpha \neq 1/2$, $\chi \in X_{\omega_{E/F}}$; $|\ |^{\alpha}\ \chi \rtimes \lambda'$, $\chi \in X_{1_{F^*}}$.

Proof. — Representations of (0), (1) and (2) are unitary, hence hermitian.

If for $\alpha > 0$, $|\ |^{\alpha}\ \chi \rtimes \lambda'$ is reducible, all subquotients are hermitian and part of the list.

For (3), let $|\ |^{\alpha}\ \chi \rtimes \lambda'$, $\alpha > 0$, be irreducible. By [3], 3.1.2, $|\ |^{\alpha}\ \chi \rtimes \lambda' \cong \overline{\widetilde{|\ |^{\alpha}\ \chi \rtimes \lambda'}}$ iff $w(|\ |^{\alpha}\ \chi \otimes \lambda') \cong \overline{\widetilde{|\ |^{\alpha}\ \chi \otimes \lambda'}}$ for the non-trivial element w of W.

We have $\overline{\widetilde{|\ |^{\alpha}\ \chi \otimes \lambda'}} = |\ |^{-\alpha}\ \chi \otimes \lambda' = w(|\ |^{\alpha}\ \chi \otimes \lambda') = |\ |^{-\alpha}\ \chi^{-1}(-) \otimes \lambda' \Leftrightarrow \chi \cong \chi^{-1}(-)$, that is $\chi \in X_{N_{E/F}}$. □

For $\alpha \in \mathbb{R}_+^*$, let $\pi_\alpha = |\ |^{\alpha}\ \chi \rtimes \lambda'$ be a representation of $U(3)$ and V be the corresponding vector space. We give, on the same vector space V, a family of $U(3)$-invariant hermitian forms, parametrised by $\alpha \in \mathbb{R}_+^*$:

$$\langle\ ,\ \rangle_\alpha : V \times V \to \mathbb{C}, (f,h) \mapsto \int_{U(3,\mathcal{O})} A(w,\lambda) f(k)\overline{h(k)}dk.$$

w is the non-trivial and the longest element of W, and $A(w,\lambda) : |\ |^{\alpha}\ \chi \rtimes \lambda' \to |\ |^{-\alpha}\ \chi^{-1}(-) \rtimes \lambda'$ is the intertwining operator for $|\ |^{\alpha}\ \chi \rtimes \lambda'$ corresponding to w. $\mathcal{O}$ is the ring of integers of E. For $\alpha \in \mathbb{R}_-^*$, one can equivalently define such an intertwining operator.

Let $\alpha \in \mathbb{R}$ and χ be a smooth unitary character of E^*. Like before we set $\lambda = \lambda_1 \otimes \lambda'$, where $\lambda_1 = |\ |^{\alpha}\ \chi$.

If $\mathrm{Ind}_P^{U(3)}(\lambda)$ reduces we have seen that all subquotients are unitary.

Theorem 4.6. — (1) $\mathrm{Ind}_P^{U(3)}(\lambda)$ *is irreducible and unitary if and only if*
- (1.1) $\chi \notin X_{1_{F^*}}$ *and* $\alpha = 0$,
- (1.2) $\chi = 1$ *and* $\alpha \in]-1,0[\cup]0,1[$,
- (1.3) $\chi \in X_{\omega_{E/F}}$ *and* $\alpha \in]-1/2,0[\cup]0,1/2[$.

(2) $\mathrm{Ind}_P^{U(3)}(\lambda)$ *is irreducible and non-unitary if and only if*
- (2.1) $\chi_1 \neq 1$, $\chi_1 \notin X_{\omega_{E/F}}$ $\forall\alpha \in \mathbb{R}^*$.
- (2.2) $\chi_1 = 1$ *and* $\alpha \in]-\infty,-1[\cup]1,\infty[$,
- (2.3) $\chi_1 \in X_{\chi_{\omega_{E/F}}}$ *and* $\alpha \in]-\infty,-1/2[\cup]1/2,\infty[$.

Proof. — We use Remark 4.1 1.

(1.1)$\Leftarrow$) If $\alpha = 0$, then $\lambda = \chi \otimes \lambda'$ is unitary, hence $\mathrm{Ind}_P^{U(3)}(\lambda)$ is unitary. The representation $\mathrm{Ind}_P^{U(3)}(\lambda)$ is irreducible unless $\chi \in X_{1_{F^*}}$ [10].

(1.2) and (1.3)$\Leftarrow$: For $\alpha = 0$ and $\chi = 1$ or $\chi \in X_{\omega_{E/F}}$, $\mathrm{Ind}_P^{U(3)}(\lambda)$ is irreducible and unitary, hence by Remark 3.1 $\mathrm{Ind}_P^{U(3)}(\lambda)$ is unitary for $\chi = 1$ and $\alpha \in]-1, 1[$ and for $\chi \in X_{\omega_{E/F}}$ and $\alpha \in]-1/2, 1/2[$. The hermitian forms are given above.

If $\alpha \neq 0$ and $\chi \notin X_{N_{E/F}(E^*)}$, $\mathrm{Ind}_P^{U(3)}(\lambda)$ is irreducible and not hermitian and hence not unitarisable.

It remains to show that $\mathrm{Ind}_P^{U(3)}(\lambda)$ is non-unitary if $\alpha \in \mathbb{R}^*$ and $\chi \in X_{1_{F^*}}$ (if $\alpha = 0$ and $\chi \in X_{1_{F^*}}$ then $\mathrm{Ind}_P^{U(3)}(\lambda)$ is reducible). Further it remains to show that $\mathrm{Ind}_P^{U(3)}(\lambda)$ is non-unitary if $\chi = 1$ and $\alpha \in]-\infty, -1[\cup]1, \infty[$ and if $\chi \in X_{\omega_{E/F}}$ and $\alpha \in]-\infty, -1/2[\cup]1/2, \infty[$.

This will show (1)$\Rightarrow$ and (2)$\Leftarrow$; (2)$\Rightarrow$ is shown by (1)$\Leftarrow$.

We use the Lemma 3.3, here $|\ |^{\alpha} \otimes 1 \in \mathrm{Hom}(M, \mathbb{C}^*)^{n.r.}$ and $\chi \otimes \lambda'$ is an irreducible representation of M.

(1)$\Rightarrow$ and (2)$\Leftarrow$: $\mathrm{Ind}_P^{U(3)}(\lambda)$ is irreducible for $\chi = 1$ and $\alpha \in]1, \infty[$ (or $\alpha \in]-\infty, -1[$, or for $\chi \in X_{\omega_{E/F}}$ and $\alpha \in]-\infty, -1/2[\cup]1/2, \infty[$, or for $\chi \in X_{1_{F^*}}$ and $\alpha \in \mathbb{R}^*$, respectively). If there existed $\alpha \in]1, \infty[$ (or in one of the other intervals or in $\mathbb{R}^*$, respectively) such that $|\ |^{\alpha} \chi \rtimes \lambda'$ is unitary, with χ chosen appropriately, then by Remark 3.1 all representations $|\ |^{\alpha} \chi \rtimes \lambda'$ with $\alpha \in]1, \infty[$ (or in one of the other intervalls or in $\mathbb{R}^*$) would be unitary, in contradiction to Lemma 3.3. $\square$

The induced representations of $U(4)$ over a p-adic field have been classified by K. Konno [11].

5. The irreducible representations of $U(5)$

5.1. Levi decomposition for $U(5)$.

Recall the Levi decomposition $P = MN$, where P is a standard parabolic subgroup, M is the standard Levi subgroup corresponding to P and N is the unipotent subgroup corresponding to P and to M.

The standard Levi subgroups of $U(5)$ are the following three:

$M_0 := E^* \times E^* \times E^1$	(the Levi-group corresponding to the minimal parabolic subgroup),
$M_1 := \mathrm{GL}(2, E) \times E^1$	and
$M_2 := E^* \times U(3)$	(the two Levi-groups corresponding to the maximal parabolic subgroups).

We obtain the parabolic subgroups

$$P_0 = M_0 N_0 = \left\{ \begin{pmatrix} x & & & & * \\ & y & & & \\ & & k & & \\ & & & \overline{y}^{-1} & \\ 0 & & & & \overline{x}^{-1} \end{pmatrix},\ x, y, \in E^*, k \in E^1, * \in E \right\} \cap U(5),$$

$$P_1 = M_1 N_1 = \left\{ \begin{pmatrix} a & & * \\ & k & \\ 0 & & \overline{a}^{-1} \end{pmatrix},\ a \in \mathrm{GL}(2, E), k \in E^1, * \in E \right\} \cap U(5), \text{ and}$$

$$P_2 = M_2 N_2 = \left\{ \begin{pmatrix} x & & * \\ & u & \\ 0 & & \overline{x}^{-1} \end{pmatrix}, x \in E^*, u \in U(3), * \in E \right\} \cap U(5).$$

5.2. **Representations with cuspidal support in M_0, fully-induced.** The irreducible representations of M_0 are characters. Let $\lambda_1, \lambda_2 \in \mathrm{Hom}(E^*, \mathbb{C}^*)$ and $\lambda' \in \mathrm{Hom}(E^1, \mathbb{C}^*)$ be smooth characters. One may write $\lambda_i = \mid \ \mid_E^{\alpha_i} \chi_i$, $i = 1, 2$, where $\alpha_i \in \mathbb{R}$ and χ_i is a unitary character of E^*. λ' is unitary.

Then each character λ of M_0 can be written as

$$\lambda(m) = \mid x \mid_E^{\alpha_1} \chi_1(x) \mid y \mid_E^{\alpha_2} \chi_2(y) \lambda'(x\overline{x}^{-1} y \overline{y}^{-1} k),$$
$$m = \begin{pmatrix} x & & & & 0 \\ & y & & & \\ & & k & & \\ & & & \overline{y}^{-1} & \\ 0 & & & & \overline{x}^{-1} \end{pmatrix}, \ x, y \in E^*, \ k \in E^1.$$

By $\lambda := \lambda_1 \otimes \lambda_2 \otimes \lambda'$ we denote the characters of M_0 and by $\lambda_1 \times \lambda_2 \rtimes \lambda' := \mathrm{Ind}_P^{U(5)}(\lambda_1 \otimes \lambda_2 \otimes \lambda')$ the induced representations to $U(5)$.

We start with the case where $\lambda_1 = \chi_1$ and $\lambda_2 = \chi_2$ are unitary characters, i.e. $\alpha_1 = \alpha_2 = 0$.

5.2.1. *Irreducible subquotients of* $\chi_1 \times \chi_2 \rtimes \lambda'$. Let P_0 be the minimal parabolic subgroup of $U(5)$ (the upper triangular matrices in $U(5)$) with Levi subgroup M_0 and unipotent subgroup N_0, such that $P_0 = M_0 N_0$).

Theorem 5.1. — *Let χ_1, χ_2 be unitary characters of E^* and let λ' be a (unitary) character of E^1.*

The induced representation $\chi_1 \times \chi_2 \rtimes \lambda'$ is reducible if and only if there exists $i \in \{1, 2\}$ such that $\chi_i \in X_{1_{F^*}}$.

Proof. — By [9] Corollary 1, the number of inequivalent irreducible components of $\mathrm{Ind}_{P_0}^{U(5)}(\lambda)$ equals the number of conjugacy classes in the R-group. We apply the Theorem 3.4 with $\mathrm{G} = U(5)$, $P = P_0$ the minimal parabolic subgroup and $M = M_0 \cong \mathrm{GL}_1(E) \times \mathrm{GL}_1(E) \times G(0) \cong E^* \times E^* \times E^1$.

Then $\sigma_1 = \chi_1$, $\sigma_2 = \chi_2$, and $\rho = \lambda'$.

Recall that for a unitary character χ of E^*, $\chi \rtimes \lambda'$ is reducible if and only if $\chi \in X_{1_{F^*}}$. Then $\chi \rtimes \lambda' = \sigma_{1,\chi} \oplus \sigma_{2,\chi}$, where $\sigma_{1,\chi}$ and $\sigma_{2,\chi}$ are tempered.

By the Theorem 3.4, for $\lambda = \chi_1 \otimes \chi_2 \otimes \lambda'$ and $W \cong S_2 \times (\mathbb{Z}/2\mathbb{Z})^2$ the integer d may equal $0, 1$ or 2.

(0) Let $d = 0$. $\chi_i \rtimes \lambda'$, $i \in \{1, 2\}$ is irreducible for $i \in \{1, 2\}$, and $R \cong \{1\}$, $\chi_1 \times \chi_2 \rtimes \lambda'$ is irreducible and unitary.
(1) Let $d = 1$. Then there exist $i, j \in \{1, 2\}$, $i \neq j$, such that $\chi_i \in X_{1_{F^*}}$ and $\chi_j \notin X_{1_{F^*}}$ or $\chi_i \in X_{1_{F^*}}$ and $\chi_j \cong \chi_i$. Hence $R \cong \mathbb{Z}/2\mathbb{Z}$, and $\chi_1 \times \chi_2 \rtimes \lambda'$ has two irreducible inequivalent constituents: $\chi_j \rtimes \sigma_{1,\chi_i}$ and $\chi_j \rtimes \sigma_{2,\chi_i}$. They are tempered and hence unitary.
(2) Let $d = 2$, χ_1 and χ_2 are two inequivalent characters and $\chi_i \in X_{1_{F^*}}$ for $i = 1, 2$.

$R \cong (\mathbb{Z}/2\mathbb{Z})^2$, and $\chi_1 \times \chi_2 \rtimes \chi'$ has four irreducible inequivalent unitary constituents. By [5, Theorem 4.3] they are tempered and elliptic. $\square$

5.2.2. *Irreducible subquotients of* $\mid \ \mid^{\alpha_1} \chi_1 \times \mid \ \mid^{\alpha_2} \chi_2 \rtimes \lambda'$, $\alpha_1, \alpha_2 > 0$ *and of* $\mid \ \mid^{\alpha} \chi_1 \times \chi_2 \rtimes \lambda'$, $\alpha > 0$. Let $M_0 \cong E^* \times E^* \times E^1$ be the minimal Levi subgroup, and let $P_0 = M_0 N_0$ be the corresponding minimal parabolic subgroup.

Let $\lambda := \lambda_1 \otimes \lambda_2 \otimes \lambda' = \mid \ \mid^{\alpha_1} \chi_1 \otimes \mid \ \mid^{\alpha_2} \chi_2 \otimes \lambda'$ be a character of M_0, where $\alpha_1, \alpha_2 \in \mathbb{R}$ and χ_1, χ_2 are unitary characters of E^*.

Let $\alpha_1 \geqslant \alpha_2 > 0$. If $\alpha_2 > \alpha_1$ we change parameters. The case $\alpha_2 = 0$ is treated seperately.

Recall that $X_{\omega_{E/F}}$ is the set of characters of E^* whose restriction to F^* is the character $\omega_{E/F}$, that is whose restriction to F^* is non-trivial on F^* but trivial on $N_{E/F}(E^*)$. $X_{1_{F^*}}$ is the set of non-trivial characters of E^* whose restriction to F^* is trivial. $X_{N_{E/F}(E^*)} = 1 \cup X_{\omega_{E/F}} \cup X_{1_{F^*}}$. The set $X_{N_{E/F}(E^*)}$ contains all characters χ of E^* satisfying $\chi(x) = \chi^{-1}(\overline{x})$. They are unitary.

From now on, the lack of an entry at position ij in a matrix means that the entry equals 0.

Theorem 5.2. — *Let χ_1, χ_2 be unitary characters of E^* and let λ' be a character of E^1. Let $\alpha_1, \alpha_2 \in \mathbb{R}_+^*$ such that $\alpha_1 \geqslant \alpha_2$. Then*

$$| \, |^{\alpha_1} \chi_1 \times | \, |^{\alpha_2} \chi_2 \rtimes \lambda'$$

is reducible if and only if at least one of the following conditions holds:

(1) $\alpha_1 - \alpha_2 = 1$ *and* $\chi_1 = \chi_2$,
(2) $\alpha_1 + \alpha_2 = 1$ *and* $\chi_1(x) = \chi_2^{-1}(\overline{x})$,
(3) $\alpha_1 = 1$ *and* $\chi_1 = 1$ *or* $\alpha_1 = 1/2$ *and* $\chi_1 \in X_{\omega_{E/F}}$,
(4) $\alpha_2 = 1$ *and* $\chi_2 = 1$ *or* $\alpha_2 = 1/2$ *and* $\chi_2 \in X_{\omega_{E/F}}$.

Proof. — Let $\lambda := | \, |^{\alpha_1} \chi_1 \otimes | \, |^{\alpha_2} \chi_2 \otimes \lambda'$ be a character of M_0, and let

$$A(w,\lambda): \begin{array}{c} \mathrm{Ind}_{P_0}^{U(5)}(\lambda) = \\ | \, |^{\alpha_1} \chi_1 \times | \, |^{\alpha_2} \chi_2 \rtimes \lambda' \end{array} \to \begin{array}{c} \mathrm{Ind}_{P_0}^{U(5)}(w\lambda) = \\ | \, |^{-\alpha_1} \chi_1^{-1}(\overline{-}) \times | \, |^{-\alpha_2} \chi_2^{-1}(\overline{-}) \rtimes \lambda' \end{array}$$

be a standard long intertwining operator for the representation $| \, |^{\alpha_1} \chi_1 \times | \, |^{\alpha_2} \chi_2 \rtimes \lambda'$.

Remark 5.3. — $w = \begin{pmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{pmatrix}$ is the longest element of the Weyl group, and for $m \in M_0$ it is

$$w\lambda(m) := \lambda\left(\begin{pmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{pmatrix}\begin{pmatrix} x&&&&\\ &y&&&\\ &&k&&\\ &&&\overline{y}^{-1}&\\ &&&&\overline{x}^{-1} \end{pmatrix}\begin{pmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{pmatrix}\right) = \lambda\left(\begin{pmatrix} \overline{x}^{-1}&&&&\\ &\overline{y}^{-1}&&&\\ &&k&&\\ &&&y&\\ &&&&x \end{pmatrix}\right).$$

Hence $\mathrm{Ind}_{P_0}^{U(5)}(w\lambda)$ equals $| \, |^{-\alpha_1} \chi_1^{-1}(\overline{-}) \times | \, |^{-\alpha_2} \chi_2^{-1}(\overline{-}) \rtimes \lambda'$.

If $A(w,\lambda)$ is either not injective or not surjective it follows that $\mathrm{Ind}_{P_0}^{U(5)}(\lambda)$ is reducible. The decomposition of the long intertwining operator into short operators shows for which α_1, α_2 and unitary characters χ_1 and χ_2 the long intertwining operator is not an isomorphism.

Let $w_1 := \begin{pmatrix} 0&1&&&\\ 1&0&&&\\ &&1&&\\ &&&0&1\\ &&&1&0 \end{pmatrix}$ and $w_2 := \begin{pmatrix} 1&&&&\\ &&&1&\\ &&1&&\\ &1&&&\\ &&&&1 \end{pmatrix}$.

We have

$$w = \begin{pmatrix} &&&&1\\ &&&1&\\ &&1&&\\ &1&&&\\ 1&&&& \end{pmatrix} = \begin{pmatrix} 0&1&&&\\ 1&0&&&\\ &&1&&\\ &&&0&1\\ &&&1&0 \end{pmatrix}\begin{pmatrix} 1&&&&\\ &&&1&\\ &&1&&\\ &1&&&\\ &&&&1 \end{pmatrix}\begin{pmatrix} 0&1&&&\\ 1&0&&&\\ &&1&&\\ &&&0&1\\ &&&1&0 \end{pmatrix}\begin{pmatrix} 1&&&&\\ &&&1&\\ &&1&&\\ &1&&&\\ &&&&1 \end{pmatrix} = w_1 w_2 w_1 w_2.$$

The following diagram gives the decomposition of $A(w,\lambda)$.

$$\begin{array}{ccc}
|\,|^{\alpha_1}\chi_1\times|\,|^{\alpha_2}\chi_2\rtimes\lambda' & \overset{\cong}{\to} & |\,|^{\alpha_1}\chi_1\times|\,|^{\alpha_2}\chi_2\rtimes\lambda' \\
 & & A(w_2,\lambda)\downarrow w_2 \\
 & & |\,|^{\alpha_1}\chi_1\times|\,|^{-\alpha_2}\chi_2^{-1}(\overline{})\rtimes\lambda' \\
A(w,\lambda)\downarrow w & & A(w_1,w_2\lambda)\downarrow w_1 \\
 & & |\,|^{-\alpha_2}\chi_2^{-1}(\overline{})\times|\,|^{\alpha_1}\chi_1\rtimes\lambda' \\
 & & A(w_2,w_1w_2\lambda)\downarrow w_2 \\
 & & |\,|^{-\alpha_2}\chi_2^{-1}(\overline{})\times|\,|^{-\alpha_1}\chi_1^{-1}(\overline{})\rtimes\lambda' \\
 & & A(w_1,w_2w_1w_2\lambda)\downarrow w_1 \\
|\,|^{-\alpha_1}\chi_1^{-1}(\overline{})\times|\,|^{-\alpha_2}\chi_2^{-1}(\overline{})\rtimes\lambda' & \overset{\cong}{\to} & |\,|^{-\alpha_1}\chi_1^{-1}(\overline{})\times|\,|^{-\alpha_2}\chi_2^{-1}(\overline{})\rtimes\lambda'.
\end{array}$$

If $A(w,\lambda)$ is not an isomorphism, then at least one of the operators $A(w_2\lambda)$, $A(w_1,w_2\lambda)$, $A(w_2,w_1w_2\lambda)$ or $A(w_1,w_2w_1w_2\lambda)$ is not an isomorphism.

$A(w_1,\lambda)$ is no isomorphism if and only if the induced representation $|\,|^{\alpha_2}\chi_2\rtimes\lambda'$ is reducible. This is the case if and only if $\alpha_2=1$ and $\chi_2=1$ or $\alpha_2=1/2$ and $\chi_2\in X_{\omega_{E/F}}$.

$A(w_1,w_2\lambda)$ is no isomorphism if and only if the corresponding representation $|\,|^{\alpha_1}\chi_1\times|\,|^{-\alpha_2}\chi_2^{-1}(\overline{})$ is reducible if and only if $\alpha_1+\alpha_2=1$ and $\chi_1(x)=\chi_2^{-1}(\overline{x})$ for all $x\in E^*$.

$A(w_2,w_1w_2\lambda)$ is no isomorphism if and only if $|\,|^{\alpha_1}\chi_1\rtimes\lambda'$ is reducible if and only if $\alpha_1=1$ and $\chi_1=1$ or $\alpha_1=1/2$ and $\chi_1\in X_{\omega_{E/F}}$.

$A(w_1,w_2w_1w_2\lambda)$ is no isomorphism if and only if $|\,|^{-\alpha_2}\chi_2^{-1}(\overline{})\times|\,|^{-\alpha_1}\chi_1^{-1}(\overline{})$ is reducible if and only if $\alpha_1-\alpha_2=1$ and $\chi_1=\chi_2$.

In all other cases the short intertwining operators are holomorphic and isomorphisms, hence $A(w,\lambda)$ is an isomorphism and the representation $|\,|^{\alpha_1}\chi_1\times|\,|^{\alpha_2}\chi_2\rtimes\lambda'$ is irreducible.

On the other hand, if at least one of the short intertwining operators is no isomorphism, $|\,|^{\alpha_1}\chi_1\times|\,|^{\alpha_2}\chi_2\rtimes\lambda'$ is reducible by induction in stages; in these cases we determine the irreducible constituents in Theorems 5.5, 5.7, 5.9, 6.2, 6.3, 6.4 and 6.6. ☐

Let $\alpha_1>\alpha_2=0$.

Theorem 5.4. — *Let χ_1,χ_2 be unitary characters of E^*, let λ' be a (unitary) character of E^1. Let $\alpha_1\in\mathbb{R}_+^*$. The induced representation*

$$|\,|^{\alpha_1}\chi_1\times\chi_2\rtimes\lambda'$$

is reducible if and only if

(1) $\alpha_1=1$ *and* $\chi_1=\chi_2$,
(2) $\alpha_1=1$ *and* $\chi_1(x)=\chi_2^{-1}(\overline{x}\,)$,
(3) $\alpha_1=1$ *and* $\chi_1=1$ *or* $\alpha_1=1/2$ *and* $\chi_1\in X_{\omega_{E/F}}$,
(4) $\chi_2\in X_{1_{F^*}}$.

Proof. — (1)$\Rightarrow$: We apply the Lemma 3.5 ([19], Lemma 2.1) with $\pi\cong\chi_2\rtimes\lambda'$ and $\rho\cong|\,|^{\alpha_1}\chi_1$.

If none of the four cases in Theorem 5.4 holds we are in the position to apply Lemma 3.5, hence $|\,|^{\alpha_1}\chi_1\times\chi_2\rtimes\lambda'$ is irreducible.

$\Leftarrow$: If at least one of the four cases holds, clearly $|\,|^{\alpha_1}\chi_1\times\chi_2\rtimes\lambda'$ is reducible. ☐

In those cases where $|\ |^{\alpha_1}\ \chi_1 \times \chi_2 \rtimes \lambda'$, $\alpha > 0$, is reducible, the irreducible constituents will be investigated in 5.5, 5.8, 5.10 and in 5.11.

5.3. **Representations induced from M_1 and M_2, with cuspidal support in M_0.** We consider representations induced from the maximal parabolic subgroups with Levi groups M_1 and M_2, whose cuspidal support is in M_0 and that are not fully induced. We begin with $M_1 = \mathrm{GL}(2, E) \times E^1$.

5.3.1. *Representations $|\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^{\alpha}\ \chi 1_{\mathrm{GL}_2} \rtimes \lambda'$, $\alpha > 0$.* Let $\alpha \in \mathbb{R}_+^*$ and χ be a unitary character of E^*. We study $|\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ that is a subrepresentation of the induced representation $|\ |^{\alpha+1/2}\ \chi \times |\ |^{\alpha-1/2}\ \chi \rtimes \lambda'$, and its Aubert dual $|\ |^{\alpha}\ \chi 1_{\mathrm{GL}_2} \rtimes \lambda'$.

Theorem 5.5. — *Let $\alpha \in \mathbb{R}_+^*$ and χ be a unitary character of E^*. The representations $|\ |^{\alpha}\ \chi(\det)\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^{\alpha}\ \chi(\det) 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible, unless one of the following cases holds:*

(1) *$\alpha = 1/2$ and $\chi \in X_{N_{E/F}(E^*)}$,*
(2) *$\alpha = 3/2$ and $\chi = 1$,*
(3) *$\alpha = 1$ and $\chi \in X_{\omega_{E/F}}$.*

Proof. — In $R(U)$ we have

$$|\ |^{\alpha+1/2}\ \chi \times |\ |^{\alpha-1/2}\ \chi \rtimes \lambda' = |\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + |\ |^{\alpha}\ \chi 1_{\mathrm{GL}_2} \rtimes \lambda'.$$

$|\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^{\alpha}\ \chi 1_{\mathrm{GL}_2} \rtimes \lambda'$ are dual in the sense of Aubert and have the same number of irreducible constituents. We give the proof for $|\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ as subrepresentation of $|\ |^{\alpha+1/2}\ \chi \times |\ |^{\alpha-1/2}\ \chi \rtimes \lambda'$.

Let $\lambda := |\ |^{\alpha+1/2}\ \chi \otimes |\ |^{\alpha-1/2}\ \chi \otimes \lambda'$, and let

$$A(w', \lambda) : |\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' \to |\ |^{-\alpha}\ \chi^{-1}(\overline{\det})\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$$

be the long intertwining operator for the representation $|\ |^{\alpha}\ \chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$, where w' is the longest element of W respecting $M_1 \cong \mathrm{GL}(2, E) \times E^1$.

We have $w' := \begin{pmatrix} &&&1&0\\ &&&0&1\\ &&1&&\\ 1&0&&&\\ 0&1&&& \end{pmatrix} = \begin{pmatrix} 1&&&&\\ &&&1&\\ &&1&&\\ &1&&&\\ &&&&1 \end{pmatrix} \begin{pmatrix} 0&1&&&\\ 1&0&&&\\ &&1&&\\ &&&0&1\\ &&&1&0 \end{pmatrix} \begin{pmatrix} 1&&&&\\ &&&1&\\ &&1&&\\ &1&&&\\ &&&&1 \end{pmatrix} = w_2 w_1 w_2.$

The decomposition of $A(w', \lambda)$ into short intertwining operators gives information for which $\alpha > 0$ and unitary characters χ of E^* this operator is an isomorphism. The following diagram shows the decomposition of $A(w', \lambda)$, where i_1 and i_2 are inclusions that depend holomorphically on α.

$$\begin{array}{ccc}
|\ |^{\alpha}\ \chi(\det)\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' & \overset{i_1}{\hookrightarrow} & |\ |^{\alpha+1/2}\ \chi \times |\ |^{\alpha-1/2}\ \chi \rtimes \lambda' \\
 & & A(w_2, \lambda) \downarrow w_2 \\
 & & |\ |^{\alpha+1/2}\ \chi \times |\ |^{-\alpha+1/2}\ \chi^{-1}(-) \rtimes \lambda' \\
A(w', \lambda) \downarrow w' & & A(w_1, w_2\lambda) \downarrow w_1 \\
 & & |\ |^{-\alpha+1/2}\ \chi^{-1}(-) \times |\ |^{\alpha+1/2}\ \chi \rtimes \lambda' \\
 & & A(w_2, w_1 w_2 \lambda) \downarrow w_2 \\
|\ |^{-\alpha}\ \chi^{-1}(\overline{\det})\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' & \overset{i_2}{\hookrightarrow} & |\ |^{-\alpha+1/2}\ \chi^{-1}(-) \times |\ |^{-\alpha-1/2}\ \chi^{-1}(-) \rtimes \lambda'
\end{array}$$

If $\alpha \neq 1/2$, $A(w_2, \lambda)$ is no isomorphism if and only if $|\ |^{\alpha-1/2}\ \chi \rtimes \lambda'$ reduces, if and only if $\alpha = 3/2$ and $\chi = 1$ or $\alpha = 1$ and $\chi \in X_{\omega_{E/F}}$.

If $\alpha = 1/2$ and $\chi \in X_{1_{F^*}}$, then $\chi \rtimes \lambda'$ reduces.

$A(w_1, w_2\lambda)$ is no isomorphism if and only if $|\ |^{\alpha+1/2}\ \chi \times |\ |^{-\alpha+1/2}\ \chi^{-1}(-)$ reduces, if and only if $\alpha = 1/2$ and $\chi \in X_{N_{E/F}(E^*)}$.

$A(w_2, w_1 w_2\lambda)$ is no isomorphism if and only if $|\ |^{\alpha+1/2}\ \chi \rtimes \lambda'$ reduces if and only if $\alpha = 1/2$ and $\chi = 1$.

In all other cases $A(w_2, \lambda), A(w_1, w_2\lambda)$ and $A(w_2, w_1 w_2\lambda)$ are holomorphic and isomorphisms and $A(w', \lambda)$ is also an isomorphism. Hence the representations $|\ |^\alpha\ \chi(\det) \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^\alpha\ \chi(\det) 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible.

If one of the three cases in Theorem 1.5 holds, reducibility of $|\ |^\alpha\ \chi(\det) \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^\alpha\ \chi(\det) 1_{\mathrm{GL}_2} \rtimes \lambda'$ has to be investigated. This is done in 5.8, 6.2, 6.4, 6.5 and in 6.6. □

5.3.2. *Representations* $\chi \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ *and* $\chi 1_{\mathrm{GL}_2} \rtimes \lambda'$. Let $0 < \alpha_2 \leqslant \alpha_1$, $\alpha > 0$. Let χ_1, χ_2, χ and χ' be unitary characters of E^*. Let λ' be a unitary character of E^1. Let $\chi \notin X_{1_{F^*}}$ (hence $\chi \rtimes \lambda'$ is irreducible by [10]). Let τ_1 be a tempered representation of $\mathrm{GL}(2, E)$, let τ_2 be a tempered representation of $U(3)$ and let τ be a tempered representation of $U(5)$.

The representations

$$|\ |^{\alpha_1}\ \chi_1 \times |\ |^{\alpha_2}\ \chi_2 \rtimes \lambda', \quad |\ |^\alpha\ \chi_1 \times \chi \rtimes \lambda', \quad |\ |^\alpha\ \tau_1 \rtimes \lambda', \quad |\ |^\alpha\ \chi' \rtimes \tau_2 \quad \text{and} \quad \tau$$

have a unique irreducible quotient, the Langlands quotient, denoted by

$$\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2\ \lambda'),\ \mathrm{Lg}(|\ |^\alpha\ \chi_1; \chi \rtimes \lambda'),\ \mathrm{Lg}(|\ |^\alpha\ \tau_1; \lambda'),\ \mathrm{Lg}(|\ |^\alpha\ \chi'; \tau_2) \text{ and } \tau,$$

respectively.

Proposition 5.6. — *Let χ be a unitary character of E^*, let λ' be a (unitary) character of E^1. The representations $\chi \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $\chi 1_{\mathrm{GL}_2} \rtimes \lambda'$ are reducible if and only if $\chi \in X_{\omega_{E/F}}$.*

Let $\chi =: \chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique irreducible square-integrable subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [10]. Then

$$\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') + \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}),$$
$$\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' = \tau_1 + \tau_2,$$

where τ_1 and τ_2 are tempered such that $\tau_1 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}; |\ |^{1/2}}\ \chi_{\omega_{E/F}}; \lambda')$ and $\tau_2 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}})$. All subquotients are unitary.

Proof. — We consider the Jacquet restriction of $\chi 1_{\mathrm{GL}_2} \rtimes \lambda'$ with respect to the minimal parabolic subgroup [8]:

$$\begin{aligned} s_{\min}(\chi(\det) 1_{\mathrm{GL}_2} \rtimes \lambda') = &\ |\ |^{-1/2}\ \chi \otimes |\ |^{1/2}\ \chi \otimes \lambda' \\ &+ |\ |^{-1/2}\ \chi^{-1}(-) \otimes |\ |^{1/2}\ \chi^{-1}(-) \otimes \lambda' \\ &+ |\ |^{-1/2}\ \chi \otimes |\ |^{-1/2}\ \chi^{-1}(-) \otimes \lambda' \\ &+ |\ |^{-1/2}\ \chi^{-1}(-) \otimes |\ |^{-1/2}\ \chi \otimes \lambda'. \end{aligned}$$

Hence all subquotients of $\chi 1_{\mathrm{GL}_2} \rtimes \lambda'$ are non-tempered.

$\chi \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $\chi 1_{\mathrm{GL}_2} \rtimes \lambda'$ are subquotients of $|\ |^{1/2}\ \chi \times |\ |^{1/2}\ \chi \rtimes \lambda'$.

For $w = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$ we have

$$w(|\ |^{1/2}\chi \otimes |\ |^{-1/2}\chi \otimes \lambda') = |\ |^{1/2}\chi \otimes |\ |^{1/2}\chi^{-1}(-) \otimes \lambda',$$

and $|\ |^{1/2}\chi \times |\ |^{-1/2}\chi \rtimes \lambda'$ and $|\ |^{1/2}\chi \times |\ |^{1/2}\chi^{-1}(-) \rtimes \lambda'$ have the same irreducible constituents. Therefore we consider the reducibility of $|\ |^{1/2}\chi \times |\ |^{1/2}\chi^{-1}(-) \rtimes \lambda'$.

If $\chi \notin X_{\omega_{E/F}}$, then $\mathrm{Lg}(|\ |^{1/2}\chi; |\ |^{1/2}\chi^{-1}(-); \lambda')$ is the only non-tempered Langlands quotient supported in $|\ |^{1/2}\chi \otimes |\ |^{1/2}\chi^{-1}(-) \otimes \lambda'$. Hence $\chi 1_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{1/2}\chi; |\ |^{1/2}\chi^{-1}(-); \lambda')$ is irreducible. $\chi \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible by the Aubert duality, it is tempered.

Let $\chi =: \chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$.

$$\mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; |\ |^{1/2}\chi_{\omega_{E/F}}; \lambda') \quad \text{and} \quad \mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$$

are the only non-tempered Langlands quotients supported in

$$|\ |^{1/2}\chi_{\omega_{E/F}} \otimes |\ |^{1/2}\chi_{\omega_{E/F}} \otimes \lambda'.$$

$\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is tempered and so are its subquotients. Hence

$$\mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; |\ |^{1/2}\chi_{\omega_{E/F}}; \lambda') \quad \text{and} \quad \mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$$

are the subquotients of $\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$. By the Aubert duality $\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ has the two irreducible subquotients

$$\tau_1 := \widehat{\mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; |\ |^{1/2}\chi_{\omega_{E/F}}; \lambda')} \quad \text{and} \quad \tau_2 := \widehat{\mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})}.$$

We consider the restriction of $\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ with respect to the parabolic subgroup P_1 :

$$\begin{aligned} s_{P_1}(\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda') = {} & \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \otimes \lambda' + \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \otimes \lambda' \\ & + |\ |^{1/2}\chi_{\omega_{E/F}} \times |\ |^{1/2}\chi_{\omega_{E/F}} \otimes \lambda'. \end{aligned}$$

$\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is unitary, hence $\tau_1 \hookrightarrow \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $\tau_2 \hookrightarrow \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$. By Frobenius reciprocity,

$$s_{P_1}(\tau_1) \twoheadrightarrow \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \otimes \lambda' \quad \text{and} \quad s_{P_1}(\tau_2) \twoheadrightarrow \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \otimes \lambda'.$$

Now $\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \otimes \lambda'$ is irreducible and has multiplicity 2 in $s_{P_1}(\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda')$. Hence τ_1 and τ_2 have multiplicities 1, and $\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is a representation of length 2. By the Aubert duality

$$\mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; |\ |^{1/2}\chi_{\omega_{E/F}}; \lambda') \quad \text{and} \quad \mathrm{Lg}(|\ |^{1/2}\chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$$

have multiplicities 1, and $\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ is of length 2.

$\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ are unitary, hence all subquotients are unitary. $\square$

5.3.3. *Representations* $|\ |^{\alpha}\ \chi \rtimes \tau$ *and* $\chi \rtimes \tau$*,* $\alpha > 0$*,* τ *irreducible non-cuspidal of* $U(3)$*, not fully-induced.* We now look at representations induced from the maximal parabolic subgroup P_2, whose cuspidal support is in M_0 and that are not fully induced.

Recall that $P_2 = M_2 N_2$, where $M_2 \cong E^* \times U(3)$ is a maximal standard Levi subgroup and N_2 the unipotent subgroup corresponding to P_2 and M_2.

Let χ be a unitary character of E^*.

Let $\beta \in \mathbb{R}_+$. Recall from 4.1 the irreducible subquotients of the induced representations to $U(3)$ in the cases that $|\ |^{\beta}\ \chi \rtimes \lambda'$ is reducible: $\lambda'(\det) \operatorname{St}_{U(3)}$, $\lambda'(\det) 1_{U(3)}$, $\pi_{1,\chi\omega_{E/F}}$, $\pi_{2,\chi\omega_{E/F}}$, $\sigma_{1,\chi 1_{F^*}}$, $\sigma_{2,\chi 1_{F^*}}$. All irreducible subquotients are unitary.

Let $\alpha \in \mathbb{R}_+^*$. We study the representations

$$\begin{array}{lll} |\ |^{\alpha}\ \chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}, & |\ |^{\alpha}\ \chi \rtimes \lambda'(\det) 1_{U(3)}, & |\ |^{\alpha}\ \chi \rtimes \pi_{1,\chi\omega_{E/F}}, \\ |\ |^{\alpha}\ \chi \rtimes \pi_{2,\chi\omega_{E/F}}, & |\ |^{\alpha}\ \chi \rtimes \sigma_{1,\chi 1_{F^*}}, \text{ and } & |\ |^{\alpha}\ \chi \rtimes \sigma_{2,\chi 1_{F^*}}. \end{array}$$

Further we study representations

$$\begin{array}{lll} \chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}, & \chi \rtimes \lambda'(\det) 1_{U(3)}, & \chi \rtimes \pi_{1,\chi\omega_{E/F}}, \\ \chi \rtimes \pi_{2,\chi\omega_{E/F}}, & \chi \rtimes \sigma_{1,\chi 1_{F^*}} \text{ and } & \chi \rtimes \sigma_{2,\chi 1_{F^*}}. \end{array}$$

5.3.3.1 *Representations* $|\ |^{\alpha}\ \chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}$ *and* $|\ |^{\alpha}\ \chi \rtimes \lambda'(\det) 1_{U(3)}$, $\alpha > 0$.

THEOREM 5.7. — *Let* $\alpha \in R_+^*$ *and* χ *be a unitary character of* E^**. The representations* $|\ |^{\alpha}\ \chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}$ *and* $|\ |^{\alpha}\ \chi \rtimes \lambda'(\det) 1_{U(3)}$ *are irreducible unless one of the following conditions holds:*

(1) $\alpha = 2$ *and* $\chi = 1$*,*
(2) $\alpha = 1$ *and* $\chi = 1$*,*
(3) $\alpha = 1/2$ *and* $\chi \in X_{\omega_{E/F}}$*.*

Proof. — The proof is similar to the proof of Theorem 5.5. If (1), (2) or (3) holds, then the reducibility of $|\ |^{\alpha}\ \chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}$ and $|\ |^{\alpha}\ \chi \rtimes \lambda'(\det) 1_{U(3)}$ has to be investigated. It is done in 6.2 and in 6.6. □

5.3.3.2 *Representations* $\chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}$ *and* $\chi \rtimes \lambda'(\det) 1_{U(3)}$. Let $\chi 1_{F^*} \in X_{1_{F^*}}$. Recall that $\chi 1_{F^*} \rtimes \lambda' = \sigma_{1,\chi 1_{F^*}} \oplus \sigma_{2,\chi 1_{F^*}}$, where $\sigma_{1,\chi 1_{F^*}}$ and $\sigma_{2,\chi 1_{F^*}}$ are tempered [10].

PROPOSITION 5.8. — *Let* χ *be a unitary character of* E^**, let* λ' *be a (unitary) character of* E^1*. The representations* $\chi \rtimes \lambda'(\det) \operatorname{St}_{U(3)}$ *and* $\chi \rtimes \lambda'(\det) 1_{U(3)}$ *are reducible if and only if* $\chi = 1$ *or* $\chi \in X_{1_{F^*}}$*.*

- *Let* $\chi = 1$*.*
  $1 \rtimes \lambda'(\det) 1_{U(3)} = \operatorname{Lg}(|\ |\ 1; 1 \rtimes \lambda') + \operatorname{Lg}(|\ |^{1/2} \operatorname{St}_{\mathrm{GL}_2}; \lambda')$ *and* $1 \rtimes \lambda'(\det) \operatorname{St}_{U(3)} = \tau_3 + \tau_4$*, where* τ_3 *and* τ_4 *are tempered such that* $\tau_3 = \widehat{\operatorname{Lg}(|\ |\ 1; 1 \rtimes \lambda')}$ *and* $\tau_4 = \widehat{\operatorname{Lg}(|\ |^{1/2} \operatorname{St}_{\mathrm{GL}_2}; \lambda')}$*.*
- *Let* $\chi =: \chi 1_{F^*} \in X_{1_{F^*}}$*.*
  $\chi 1_{F^*} \rtimes \lambda'(\det) 1_{U(3)} = \operatorname{Lg}(|\ |\ 1; \sigma_{1,\chi 1_{F^*}}) + \operatorname{Lg}(|\ |\ 1; \sigma_{2,\chi 1_{F^*}})$ *and* $\chi 1_{F^*} \rtimes \lambda'(\det) \operatorname{St}_{U(3)} = \tau_5 + \tau_6$*, where* τ_5 *and* τ_6 *are tempered, such that* $\tau_5 = \widehat{\operatorname{Lg}(|\ |\ 1; \sigma_{1,\chi 1_{F^*}})}$ *and* $\tau_6 = \widehat{\operatorname{Lg}(|\ |\ 1; \sigma_{2;\chi 1_{F^*}})}$*.*

All subquotients are unitary.

Proof. — The proof follows similar lines to the proof of Proposition 5.6. □

5.3.3.3 *Representations* $|\ |^\alpha\ \chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ *and* $|\ |^\alpha\ \chi \rtimes \pi_{2,\chi_{\omega_{E/F}}}$, $\alpha > 0$. Let $\alpha \in \mathbb{R}_+^*$ and let χ be a unitary character of E^*. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$, that is $\chi_{\omega_{E/F}}$ is a (unitary) character of E^* whose restriction to F^* equals the character $\omega_{E/F}$.

Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique square-integrable subquotient and let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique irreducible non-tempered subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [10].

Theorem 5.9. — *Let $\alpha \in \mathbb{R}_+^*$ and χ be a unitary character of E^*. The representations $|\ |^\alpha\ \chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ and $|\ |^\alpha\ \chi \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ are irreducible unless*

(1) *$\alpha = 1/2$ or $\alpha = 3/2$ and $\chi = \chi_{\omega_{E/F}}$,*
(2) *$\alpha = 1$ and $\chi = 1$,*
(3) *$\alpha = 1/2$ and $\chi \in X_{\omega_{E/F}}$.*

Proof. — The proof is similar to the proof of Theorem 5.5. If (1), (2) or (3) holds, then reducibility of $|\ |^\alpha\ \chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ and $|\ |^\alpha\ \chi \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ needs to be investigated. This is done in 5.6 and 6.3 (for $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ and $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$), in 6.4, 6.6 and in 6.9. □

5.3.3.4 *Representations* $\chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ *and* $\chi \rtimes \pi_{2,\chi_{\omega_{E/F}}}$. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique square-integrable subquotient and $\pi_{2,\chi_{\omega_{E/F}}}$ the unique non-tempered irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

Proposition 5.10. — *Let χ be a unitary character of E^* and let λ' be a (unitary) character of E^1. The representations $\chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ and $\chi \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ are reducible if and only if $\chi \in X_{1_{F^*}}$.*

Let $\chi =: \chi_{1_{F^*}} \in X_{1_{F^*}}$. Then

$$\begin{aligned}\chi_{1_{F^*}} \rtimes \pi_{2,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \sigma_{1,\chi_{1_{F^*}}}) + \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \sigma_{2,\chi_{1_{F^*}}}),\\ \chi_{1_{F^*}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} &= \tau_7 + \tau_8,\end{aligned}$$

where τ_7 and τ_8 are tempered representations with $\tau_7 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}}; \sigma_{1,\chi_{1_{F^*}}})$ and $\tau_8 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}}; \sigma_{2,\chi_{1_{F^*}}})$.

All subquotients are unitary.

Proof. — The proof is similar to the proof of Proposition 5.6. □

5.3.3.5 *Representations* $|\ |^\alpha\ \chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ *and* $|\ |^\alpha\ \chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$, $\alpha > 0$. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

Theorem 5.11. — *Let χ be a unitary character of E^*. Let $\alpha \in \mathbb{R}_+^*$. The representations $|\ |^\alpha\ \chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha\ \chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible unless one of the following cases holds:*

(1) *$\alpha = 1$ and $\chi = \chi_{1_{F^*}}$,*
(2) *$\alpha = 1$ and $\chi = 1$,*
(3) *$\alpha = 1/2$ and $\chi \in X_{\omega_{E/F}}$.*

Proof. — The proof is similar to the proof of Proposition 5.5. If (1), (2) or (3) holds, then the reducibility of $|\ |^\alpha\ \chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha\ \chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$ has to be investigated. It is done in 6.6, 6.7 and in 6.8. □

5.3.3.6 *Representations* $\chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ *and* $\chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$, $\alpha > 0$. Let χ be a unitary character of E^*. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

Remark 5.12. — By Theorem 5.1 the representations $\chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are reducible if and only if $\chi \in X_{1_{F^*}}$ such that $\chi \ncong \chi_{1_{F^*}}$.

6. 'Special' Reducibility points of representations of $U(5)$ with cuspidal support in M_0

We determine the irreducible subquotients of the representations whose reducibility has not been examined in Chapter 5.

Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique irreducible square-integrable subquotient and let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique irreducible non-tempered subquotient of $|\,|^{1/2}\,\chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Recall that $\chi_{1_{F^*}} = \sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$, where $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are tempered [10].

In Theorem 5.5 the irreducible subquotients of the following representations are left to be examined:

$$\begin{array}{lll}
|\,|^{1/2}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda', & |\,|^{1/2}\,1_{\mathrm{GL}_2} \rtimes \lambda', & |\,|^{3/2}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda', \\
|\,|^{3/2}\,1_{\mathrm{GL}_2} \rtimes \lambda', & |\,|^{1/2}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda', & |\,|^{1/2}\,\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda', \\
|\,|^{1/2}\,\chi_{1_{F^*}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda', & |\,|^{1/2}\,\chi_{1_{F^*}} 1_{\mathrm{GL}_2} \rtimes \lambda', & |\,|\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' \\
|\,|\,\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'. & &
\end{array}$$

In Theorem 5.7 the irreducible subquotients of the following representations are left to be examined:

$$\begin{array}{lll}
|\,|^2\,1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}, & |\,|^2\,1 \rtimes \lambda'(\det) 1_{U(3)}, & |\,|\,1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}, \\
|\,|\,1 \rtimes \lambda'(\det) 1_{U(3)}, & |\,|^{1/2}\,\chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}, & |\,|^{1/2}\,\chi_{\omega_{E/F}} \rtimes \lambda'(\det) 1_{U(3)}.
\end{array}$$

Theorem 5.9 leaves the following representations to be examined:

$$\begin{array}{lll}
|\,|^{1/2}\,\chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}, & |\,|^{1/2}\,\chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}, & |\,|^{3/2}\,\chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}, \\
|\,|^{3/2}\,\chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}, & |\,|\,1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}, & |\,|\,1 \rtimes \pi_{2,\chi_{\omega_{E/F}}}, \\
|\,|^{1/2}\,\chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}, & |\,|^{1/2}\,\chi \rtimes \pi_{2,\chi_{\omega_{E/F}}}, &
\end{array}$$

for $\chi \in X_{\omega_{E/F}}$, $\chi \ncong \chi_{\omega_{E/F}}$.

Theorem 5.11 leaves the following representations to be examined:

$$\begin{array}{lll}
|\,|\,\chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}, & |\,|\,\chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}, & |\,|\,1 \rtimes \sigma_{1,\chi_{1_{F^*}}}, \\
|\,|\,1 \rtimes \sigma_{2,\chi_{1_{F^*}}}, & |\,|^{1/2}\,\chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}, & |\,|\,\chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}.
\end{array}$$

All representations are treated in this section. We determine whether the irreducible subquotients are unitary.

6.1. $|\,|\,1 \times 1 \rtimes \lambda'$.

In the Grothendieck group of the category of admissible representations of finite length one has

$$|\,|\,1 \times 1 \rtimes \lambda' = |\,|^{1/2}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + |\,|^{1/2}\,1_{\mathrm{GL}_2} \rtimes \lambda' = 1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)} + 1 \rtimes \lambda'(\det) 1_{U(3)}.$$

Theorem 6.1. —

$$
\begin{aligned}
|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda') + \tau_3,\\
|\ |^{1/2}\,1_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(|\ |\,1; 1\rtimes\lambda') + \tau_4,\\
\rtimes\lambda'(\det)\,\mathrm{St}_{U(3)} &= \tau_3+\tau_4,\\
1\rtimes\lambda'(\det)1_{U(3)} &= \mathrm{Lg}(|\ |\,1;1\rtimes\lambda') + \mathrm{Lg}(|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda'),
\end{aligned}
$$

where $\tau_3 = \mathrm{Lg}(\widehat{|\ |\,1;1\rtimes}\,\lambda')$ *and* $\tau_4 = \mathrm{Lg}(|\ \widehat{|^{1/2}\,\mathrm{St}}_{\mathrm{GL}_2};\lambda')$. *Note that* τ_3 *and* τ_4 *are tempered. All irreducible subquotients are unitary.*

Proof. — We have seen in Proposition 5.8 that

$$
\begin{aligned}
1\rtimes\lambda'(\det)\,\mathrm{St}_{U(3)} &= \tau_3+\tau_4,\\
1\rtimes\lambda'(\det)1_{U(3)} &= \mathrm{Lg}(|\ |\,1;1\rtimes\lambda') + \mathrm{Lg}(|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda'),
\end{aligned}
$$

where $\tau_3 = \mathrm{Lg}(\widehat{|\ |\,1;1\rtimes}\,\lambda')$ and $\tau_4 = \mathrm{Lg}(|\ \widehat{|^{1/2}\,\mathrm{St}}_{\mathrm{GL}_2};\lambda')$ are both tempered.

$|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2}\rtimes\lambda'$ is a subrepresentation of $|\ |\,1\times 1\rtimes\lambda'$, whereas $|\ |^{1/2}\,1_{\mathrm{GL}_2}\rtimes\lambda'$ is a quotient. Hence $\mathrm{Lg}(|\ |\,1;1\rtimes\lambda')$ is a subquotient of $|\ |^{1/2}\,1_{\mathrm{GL}_2}\rtimes\lambda'$. $|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2}\rtimes\lambda'$ is the Aubert dual of $|\ |^{1/2}\,1_{\mathrm{GL}_2}\rtimes\lambda'$, hence $\tau_3 = \mathrm{Lg}(\widehat{|\ |\,1;1\rtimes}\,\lambda')$ is a subquotient of $|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2}\rtimes\lambda'$. Now $\mathrm{Lg}(|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is a subquotient of $|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2}\rtimes\lambda'$, hence $\tau_4 = \mathrm{Lg}(|\ \widehat{|^{1/2}\,\mathrm{St}}_{\mathrm{GL}_2};\lambda')$ is a subquotient of $|\ |^{1/2}\,1_{\mathrm{GL}_2}\rtimes\lambda'$.

$1\rtimes\lambda'(\det)\,\mathrm{St}_{U(3)}$ and $1\rtimes\lambda'(\det)1_{U(3)}$ are unitary, hence all irreducible subquotients are unitary. □

6.2. $|\ |^2\,1\times|\ |\,1\rtimes\lambda'$. In the Grothendieck group of admissible representations of finite length one has

$$
\begin{aligned}
|\ |^2\,1\times|\ |\,1\rtimes\lambda' &= |\ |^{3/2}\,\mathrm{St}_{\mathrm{GL}_2}\rtimes\lambda' + |\ |^{3/2}\,1_{\mathrm{GL}_2}\rtimes\lambda'\\
&= |\ |^2\,1\rtimes\lambda'(\det)\,\mathrm{St}_{U(3)} + |\ |^2\,1\rtimes\lambda'(\det)1_{U(3)}.
\end{aligned}
$$

Theorem 6.2. — *The representation* $|\ |^2\,1\times|\ |\,1\rtimes\lambda'$ *is reducible and we have*

$$
\begin{aligned}
|\ |^2\,1\times|\ |\,1\rtimes\lambda' =& \lambda'(\det)\,\mathrm{St}_{U(5)} + \mathrm{Lg}(|\ |^{3/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')\\
&+\lambda'(\det)1_{U(5)} + \mathrm{Lg}(|\ |^2\,1;\lambda'(\det)\,\mathrm{St}_{U(3)}).
\end{aligned}
$$

$\lambda'(\det)\,\mathrm{St}_{U(5)}$ *and* $\lambda'(\det)1_{U(5)}$ *are unitary,*

$$
\mathrm{Lg}(|\ |^{3/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda') \quad \textit{and} \quad \mathrm{Lg}(|\ |^2\,1;\lambda'(\det)\,\mathrm{St}_{U(3)})
$$

are non-unitary.

Proof. — By [3], $|\ |^2\,1\times|\ |\,1\rtimes\lambda'$ is a representation of length 4.

$\lambda'(\det)1_{U(5)} = \mathrm{Lg}(|\ |^2\,1;|\ |\,1;\lambda')$, $\mathrm{Lg}(|\ |^{3/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ and $\mathrm{Lg}(|\ |^2\,1;\lambda'(\det)\,\mathrm{St}_{U(3)})$ are all non-tempered Langlands quotients supported in $|\ |^2\,1\otimes|\ |\,1\otimes\lambda'$. The subrepresentation $\lambda'(\det)\,\mathrm{St}_{U(5)} = \lambda'\widehat{(\det)1}_{U(5)}$ is square-integrable.

By results of Casselman [2, page 915], $\lambda'(\det)\,\mathrm{St}_{U(5)}$ and its Aubert dual

$$
\lambda'(\det)1_{U(5)} = \mathrm{Lg}(|\ |^2\,1;|\ |\,1;\lambda')
$$

are unitary, $\mathrm{Lg}(|\ |^{3/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ and $\mathrm{Lg}(|\ |^2\,1;\lambda'(\det)\,\mathrm{St}_{U(3)})$ are not unitary. □

6.3. $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times\ |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique irreducible square-integrable subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$, and let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique irreducible non-tempered subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [10].

In the Grothendieck group of admissible representations of finite length one has

$$\begin{aligned}|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda' &= \chi_{\omega_{E/F}}\, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' \\ &= |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} + |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}.\end{aligned}$$

Theorem 6.3. — *The representation $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ is reducible and we have*

$$\begin{aligned}|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda' &= \chi_{\omega_{E/F}}\, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' \\ &= |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} + |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}.\end{aligned}$$

Moreover we have

$$\begin{aligned}&\chi_{\omega_{E/F}}\, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' = \tau_1 + \tau_2, \\ &\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') + \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}),\end{aligned}$$

$$\begin{aligned}|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}) + \tau_1, \\ |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') + \tau_2,\end{aligned}$$

where τ_1 and τ_2 are tempered such that $\tau_1 = \mathrm{Lg}(|\ |^{1/2}\ \chi_{\widehat{\omega_{E/F}}; |\ |}^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ and $\tau_2 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}}; \pi_{1,\chi_{\omega_{E/F}}})$. All irreducible subquotients are unitary.

Proof. — In Proposition 5.6 we have seen that

$$\begin{aligned}&\chi_{\omega_{E/F}}\, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' = \tau_1 + \tau_2, \\ &\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') + \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}),\end{aligned}$$

where τ_1 and τ_2 are tempered such that $\tau_1 = \mathrm{Lg}(|\ |^{1/2}\ \chi_{\widehat{\omega_{E/F}}; |\ |}^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ and $\tau_2 = \mathrm{Lg}(|\ \widehat{|^{1/2}\ \chi_{\omega_{E/F}}}; \pi_1)$.

$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ is a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $\pi_{2,\chi_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [Ke], hence $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$.

$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is the irreducible Langlands quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times\ |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$, hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$. Hence $\tau_1 = \mathrm{Lg}(|\ |^{1/2}\ \chi_{\widehat{\omega_{E/F}}; |\ |}^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ and $\tau_2 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}}; \pi_{1,\chi_{\omega_{E/F}}})$ is a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$.

$\chi_{\omega_{E/F}}\, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $\chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ are unitary, hence all irreducible subquotients are unitary. $\square$

6.4. $|\ |^{3/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$.

Theorem 6.4. — *Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique irreducible square-integrable subquotient and let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique irreducible non-tempered subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. The representation $|\ |^{3/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ is reducible, and we have*

$$\begin{aligned}|\ |^{3/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda' =& |\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + |\ |\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' \\ =& |\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} + |\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}.\end{aligned}$$

We have

$$\begin{aligned}|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' =& \mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') + \delta, \\ |\ |\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' =& \mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') \\ &+ \mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}), \\ |\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} =& \mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_1) + \delta, \\ |\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}} =& \mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') \\ &+ \mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda'),\end{aligned}$$

where $\delta = \mathrm{Lg}(|\ |^{3/2}\ \chi_{\widehat{\omega_{E/F}}; |\ |}^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is square-integrable.

$$\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') \quad \textit{and} \quad \mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$$

are not unitary.

Proof. — $|\ |^{3/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ has only the following non-tempered irreducible subquotients: $\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$, $\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ and $\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$.

$\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is a subquotient of $|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$.

$\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ is a subquotient of $|\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$.

We consider the Jacquet restrictions of $|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and of $|\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ with respect to the minimal parabolic subgroup:

$$\begin{aligned}&s_{\min}(|\ |\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda') = \\ &\quad |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' \\ &\quad + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-3/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes \lambda'. \\ &s_{\min}(|\ |^{3/2}\chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}) = \\ &\quad |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' \\ &\quad + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-3/2}\ \chi_{\omega_{E/F}} \otimes \lambda'.\end{aligned}$$

$|\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda'$ is the only common irreducible subquotient in the restrictions of $|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and of $|\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$. Hence these representations have exactly one subquotient in common, denoted by δ. By the Casselman square-integrability criterion δ is square-integrable [3].

We have

$$| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + | \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' = | \, |^{3/2} \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} + | \, |^{3/2} \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}.$$

Therefore $\mathrm{Lg}(| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is a subquotient of $| \, |^{3/2} \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$, and $\mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ is a subquotient of $| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$.

$| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2}$ is the Langlands quotient of $| \, |^{3/2} \chi_{\omega_{E/F}} \times | \, |^{1/2} \chi_{\omega_{E/F}}$.

$| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \otimes \lambda'$ is a quotient of $| \, |^{3/2} \chi_{\omega_{E/F}} \times | \, |^{1/2} \chi_{\omega_{E/F}} \otimes \lambda'$.

Hence $| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ is a quotient of $| \, |^{3/2} \chi_{\omega_{E/F}} \times | \, |^{1/2} \chi_{\omega_{E/F}} \rtimes \lambda'$.

$\mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; | \, |^{1/2} \chi_{\omega_{E/F}}; \lambda')$ is the unique irreducible quotient of $| \, |^{3/2} \chi_{\omega_{E/F}} \times | \, |^{1/2} \chi_{\omega_{E/F}} \rtimes \lambda'$, in particular it is a quotient of $| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$. In the same manner $\mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; | \, |^{1/2} \chi_{\omega_{E/F}}; \lambda')$ is a quotient of $| \, |^{3/2} \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$.

So far we have shown:

$$\begin{aligned}
| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') + \delta + A_1, \\
| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; | \, |^{1/2} \chi_{\omega_{E/F}}; \lambda') \\
&\quad + \mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}) + A_2, \\
| \, |^{3/2} \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}) + \delta + A_3, \\
| \, |^{3/2} \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(| \, |^{3/2} \chi_{\omega_{E/F}}; | \, |^{1/2} \chi_{\omega_{E/F}}; \lambda') \\
&\quad + \mathrm{Lg}(| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') + A_4,
\end{aligned}$$

where A_1, A_2, A_3, A_4 are sums of tempered representations. We will prove that $A_1 = A_2 = A_3 = A_4 = 0$.

A tempered representation is the subquotient of a representation induced from a square-integrable representation. Here, for each proper Levi subgroup M_i, $i = 0, 1, 2$ of $U(5)$, $\mathrm{Ind}_{M_0}^{M_i}(| \, |^{3/2} \chi_{\omega_{E/F}} \otimes | \, |^{1/2} \chi_{\omega_{E/F}} \otimes \lambda')$ does not contain any square-integrable subquotient. Hence all tempered subquotients of $| \, |^{3/2} \chi_{\omega_{E/F}} \times | \, |^{1/2} \chi_{\omega_{E/F}} \rtimes \lambda'$ are square-integrable.

Assume there existed a square-integrable subquotient β of $| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$.

We consider the Jacquet restrictions of $| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$, β and $\hat{\beta}$ with respect to the minimal parabolic subgroup.

$$\begin{aligned}
s_{\min}(| \, | \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda') = {} & \\
& | \, |^{1/2} \chi_{\omega_{E/F}} \otimes | \, |^{3/2} \chi_{\omega_{E/F}} \otimes \lambda' + | \, |^{1/2} \chi_{\omega_{E/F}} \otimes | \, |^{-3/2} \otimes \lambda' \\
& + | \, |^{-3/2} \chi_{\omega_{E/F}} \otimes | \, |^{-1/2} \chi_{\omega_{E/F}} \otimes \lambda' + | \, |^{-3/2} \chi_{\omega_{E/F}} \otimes | \, |^{1/2} \chi_{\omega_{E/F}} \otimes \lambda',
\end{aligned}$$

hence by the Casselman square-integrability criterion [3]

$$s_{\min}(\beta) = | \, |^{1/2} \chi_{\omega_{E/F}} \otimes | \, |^{3/2} \chi_{\omega_{E/F}} \otimes \lambda'.$$

By [1] Théorème 1.7, $s_{\min}(\hat{\beta}) = | \, |^{-1/2} \chi_{\omega_{E/F}} \otimes | \, |^{-3/2} \chi_{\omega_{E/F}} \otimes \lambda'$.

$\hat{\beta}$ is an irreducible subquotient of $| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$. As its restriction is negative and as $\mathrm{Lg}(| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is the only non-tempered subquotient of $| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$, $\hat{\beta}$ must equal $\mathrm{Lg}(| \, | \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda')$.

We have seen that

$$\begin{aligned} &s_{\min}(|\ |\chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda') = \\ &\qquad |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1/2} \otimes \lambda' \\ &\qquad + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-3/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes \lambda'. \end{aligned}$$

So $s_{\min}(\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda'))$ must contain at least the two negative irreducible subquotients $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-3/2}\ \chi_{\omega_{E/F}} \otimes \lambda'$ and $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{3/2}\ \chi_{\omega_{E/F}} \otimes \lambda'$. Hence $\hat{\beta} \neq \mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$.

We obtain that $\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') = \mathrm{Lg}(|\ |^{3/2}\ \widehat{\chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}})$, and $\delta = \mathrm{Lg}(|\ |^{3/2}\ \widehat{\chi_{\omega_{E/F}}; |\ |}^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$.

$\delta = \mathrm{Lg}(|\ |^{3/2}\ \widehat{\chi_{\omega_{E/F}}; |\ |}^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is square-integrable, hence unitary. $\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is the dual of a square-integrable representation. It should be unitary, but we have no proof for it. See [6], where the proof for the unitarisability of the Aubert dual of a strongly positive square-integrable representation is given for orthogonal and symplectic groups. Applying Theorem 1.1 and Remark 4.7 of [7] to the representation $|\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ we see that $\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ and $\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ are non-unitary. □

6.5. $|\ |\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$. In the Grothendieck group of admissible representations of finite length one has

$$|\ |\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda' = |\ |^{1/2}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'.$$

We have no proof that $|\ |^{1/2}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible. See [18], Proposition 6.3, where a proof is given for symplectic and special orthogonal groups and when the representation of the GL_{2p}-part, $p \geqslant 1$, of the inducing representation, is essentially square-integrable.

Remark 6.5. — If we assume that $|\ |^{1/2}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and by the Aubert duality $|\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible, then

$$\begin{aligned} |\ |^{1/2}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') \quad \text{and} \\ |\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda'), \end{aligned}$$

and both subquotients are non-unitary.

Further we are able to prove that $|\ |^{\alpha}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ and $|\ |^{\alpha}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda')$ are non-unitary for $0 < \alpha < 1/2$, $1/2 < \alpha_1 < 1$, $\alpha_2 = 1 - \alpha_1$. See Remarks 7.7 and 7.19.

6.6. $|\ |\ \chi_{1_{F^*}} \times \chi_{1_{F^*}} \rtimes \lambda'$. We can not give a complete decomposition of

$$|\ |\ \chi_{1_{F^*}} \times \chi_{1_{F^*}} \rtimes \lambda'$$

into irreducible subquotients. We have the following result:

Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. By [10] $\chi_{1_{F^*}} \rtimes \lambda' = \sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$, where $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are irreducible tempered. The representation $|\ |\ \chi_{1_{F^*}} \times \chi_{1_{F^*}} \rtimes \lambda'$ is reducible and we have

$$\begin{aligned} | \, | \chi_{1_{F^*}} \times \chi_{1_{F^*}} \rtimes \lambda' &= | \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + | \, |^{1/2} \chi_{1_{F^*}} 1_{\mathrm{GL}_2} \rtimes \lambda' \\ &= | \, | \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}} + | \, | \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}. \end{aligned}$$

Moreover,

$$\begin{aligned} | \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') + \delta + A_1, \\ | \, |^{1/2} \chi_{1_{F^*}} 1_{\mathrm{GL}_2} \rtimes \lambda' &= \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}}) + \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}}) + A_2, \\ | \, | \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}} &= \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}}) + \delta + A_2, \\ | \, | \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}} &= \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}}) + \mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda') + A_1, \end{aligned}$$

where δ is square-integrable.

$$\mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda'), \quad \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}}) \quad \text{and} \quad \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$$

are unitary. A_1 and A_2 are either both equal to 0, or A_1 is equal to δ or δ', where δ' is square-integrable and $\delta \neq \delta'$, and A_2 is either equal to $\mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$ or to $\mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$.

We now prove the assertion that $\mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda'), \mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$ are the only non-tempered irreducible subquotients of *

$$| \, | \chi_{1_{F^*}} \times \chi_{1_{F^*}} \rtimes \lambda'.$$

$\mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is a subquotient of $| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$. We consider the Jacquet restriction of $| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ with respect to the minimal parabolic subgroup.

$$\begin{aligned} s_{\min}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda') &= | \, | \chi_{1_{F^*}} \otimes \chi_{1_{F^*}} \otimes \lambda' + | \, | \chi_{1_{F^*}} \otimes \chi_{1_{F^*}} \otimes \lambda' \\ &\quad + \chi_{1_{F^*}} \otimes | \, |^{-1} \chi_{1_{F^*}*} \otimes \lambda' + \chi_{1_{F^*}} \otimes | \, | \chi_{1_{F^*}} \otimes \lambda' \\ &= 2 \, | \, | \chi_{1_{F^*}} \otimes \chi_{1_{F^*}} \otimes \lambda' + \chi_{1_{F^*}} \otimes | \, |^{-1} \chi_{1_{F^*}} \otimes \lambda' \\ &\quad + \chi_{1_{F^*}} \otimes | \, | \chi_{1_{F^*}} \otimes \lambda'. \end{aligned}$$

By the Casselman square-integrability criterion [3], $\mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is the only non-tempered irreducible subquotient of $| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$. A tempered representation is the subquotient of a representation induced from a square-integrable representation. $| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \otimes \lambda'$ is not square-integrable, hence any other subquotient of $| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ must be square-integrable. Therefore $\mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(| \, | \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$ are subquotients of $| \, |^{1/2} \chi_{1_{F^*}} 1_{\mathrm{GL}_2} \rtimes \lambda'$. Let δ be a square-integrable subquotient of $| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$.

$| \, | \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $| \, | \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ have the same Jacquet restrictions with respect to the minimal parabolic subgroup:

$$\begin{aligned} s_{\min}(| \, | \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}) &= s_{\min}(| \, | \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}) \\ &= | \, | \chi_{1_{F^*}} \otimes \chi_{1_{F^*}} \otimes \lambda' + | \, |^{-1} \chi_{1_{F^*}} \otimes \chi_{1_{F^*}} \otimes \lambda' \\ &\quad + \chi_{1_{F^*}} \otimes | \, | \chi_{1_{F^*}*} \otimes \lambda' + \chi_{1_{F^*}} \otimes | \, |^{-1} \chi_{1_{F^*}} \otimes \lambda'. \end{aligned}$$

We chose $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ such that δ is a subquotient of $| \, | \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\mathrm{Lg}(| \, |^{1/2} \chi_{1_{F^*}} \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is a subquotient of $| \, | \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$.

We have no contradiction that $|\ |^{1/2}\ \chi 1_{F^*}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ contains a second irreducible square-integrable subquotient δ' that would be a subquotient of $|\ |\ \chi 1_{F^*} \rtimes \sigma_{2,\chi 1_{F^*}}$, and that $|\ |^{1/2}\ \chi 1_{F^*} 1_{\mathrm{GL}_2} \rtimes \lambda'$ either contains $\mathrm{Lg}(|\ |\ \chi 1_{F^*}; \sigma_{2,\chi 1_{F^*}})$ with multiplicity 2 or $\mathrm{Lg}(|\ |^{1/2}\ \chi 1_{F^*}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda')$. $\mathrm{Lg}(|\ |\ \chi 1_{F^*}; \sigma_{2,\chi 1_{F^*}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi 1_{F^*}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda')$ would then be subquotients of $|\ |\ \chi 1_{F^*} \rtimes \sigma_{1,\chi 1_{F^*}}$.

Let $0 < \alpha < 1$. By Theorem 5.11 the representations $|\ |^{\alpha}\ \chi 1_{F^*} \rtimes \sigma_{1,\chi 1_{F^*}}$ and $|\ |^{\alpha}\ \chi 1_{F^*} \rtimes \sigma_{2,\chi 1_{F^*}}$ are irreducible, they are equal to $\mathrm{Lg}(|\ |^{\alpha}\ \chi 1_{F^*}; \sigma_{1,\chi 1_{F^*}})$ and to $\mathrm{Lg}(|\ |^{\alpha}\ \chi 1_{F^*}; \sigma_{2,\chi 1_{F^*}})$, respectively. By Theorem 7.23 (2) they are unitary. For $\alpha = 1$, by [14] the irreducible subquotients of $|\ |\ \chi 1_{F^*} \rtimes \sigma_{1,\chi 1_{F^*}}$ and of $|\ |\ \chi 1_{F^*} \rtimes \sigma_{2,\chi 1_{F^*}}$ are unitary.

6.7. $|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Recall that $\lambda'(\det)\,\mathrm{St}_{U(3)}$ is the unique irreducible square-integrable subquotient and that $\lambda'(\det) 1_{U(3)}$ is the unique irreducible non-tempered subquotient of $|\ |\ 1 \rtimes \lambda'$. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique irreducible square-integrable subquotient and let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique irreducible non-tempered subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [10].

Theorem 6.6. — *The representation $|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ is reducible, and we have*

$$\begin{aligned}|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda' &= |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)} + |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det) 1_{U(3)} \\ &= |\ |\ 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}} + |\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}}.\end{aligned}$$

Moreover we have

$$\begin{aligned}|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}) + \delta, \\ |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det) 1_{U(3)} &= \mathrm{Lg}(|\ |\ 1; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') + \mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}}), \\ |\ |\ 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}}) + \delta, \\ |\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |\ 1; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') + \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}),\end{aligned}$$

where δ is square-integrable. $\delta = \mathrm{Lg}(|\ |\ 1; \widehat{|\ |^{1/2}\ \chi_{\omega_{E/F}}}; \lambda')$, and

$$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}) = \mathrm{Lg}(|\ |\ \widehat{1; \pi_{1,\chi_{\omega_{E/F}}}}).$$

The representations

$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |\ 1; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ *and δ are all unitary.*

Proof. — $|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ has only the following irreducible non-tempered subquotients:

$$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}),\ \mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}}) \text{ and } \mathrm{Lg}(|\ |\ 1; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda').$$

$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$.

We consider the Jacquet restriction of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ with respect to the minimal parabolic subgroup:

$$\begin{aligned}s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}) &= |\ |\ 1 \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' \\ &+ |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda' + |\ |\ 1 \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda'.\end{aligned}$$

By the Casselman square-integrability criterion [3],

$$s_{\min}(\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}))$$

must contain the irreducible subquotient $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda'$. We have

$$\begin{aligned} s_{\min}(|\ |\ 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}) = &|\ |\ 1 \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda' \\ &+ |\ |^{-1}\ 1 \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1}\ 1 \otimes \lambda', \end{aligned}$$

and

$$\begin{aligned} s_{\min}(|\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}}) = &|\ |\ 1 \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda' \\ &+ |\ |^{-1}\ 1 \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1}\ 1 \otimes \lambda'. \end{aligned}$$

The irreducible subquotient $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda'$ appears in $s_{\min}(|\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}})$, not in $s_{\min}(|\ |\ 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}})$. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}))$ is a subquotient of $|\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}}$.

$\mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}})$ is the unique irreducible quotient of $|\ |\ 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}$,

$$\begin{aligned} &s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)}) = \\ &\qquad |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1}\ 1 \otimes \lambda' + |\ |^{-1}\ 1 \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' \\ &\qquad + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1}\ 1 \otimes \lambda' + |\ |^{-1}\ 1 \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda'. \end{aligned}$$

Looking at Jacquet modules, we see that $\mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}})$ is a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)}$.

$\lambda'(\det)1_{U(3)}$ is a quotient of $|\ |\ 1 \rtimes \lambda'$ [10]. Hence $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)}$ is a quotient of $|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. $\pi_{2,\chi_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Hence $|\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ is also a quotient of $|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$.

$\mathrm{Lg}(|\ |\ 1; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$ is the unique irreducible quotient of $|\ |\ 1 \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Hence it is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)}$ and of $|\ |\ 1 \rtimes \pi_{2,\chi_{\omega_{E/F}}}$. It is of multiplicity one.

Each irreducible subquotient in $s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)})$ is of multiplicity 1. Hence $\mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}})$ is of multiplicity 1. We have seen that $s_{\min}(\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}))$ contains $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda'$, with multiplicity 1. $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |\ 1 \otimes \lambda'$ does not appear in $s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)}))$. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$ equally has multiplicity 1.

By the Casselman square-integrability criterion [3] any subquotient of

$$|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$$

other than $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is square-integrable.

$|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)1_{U(3)}$ has the two irreducible subquotients

$$\mathrm{Lg}(|\ |\ 1; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') \quad \text{and} \quad \mathrm{Lg}(|\ |\ 1; \pi_{1,\chi_{\omega_{E/F}}}).$$

By the Aubert duality $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ has exactly one square-integrable irreducible subquotient, denoted by δ.

Looking at Jacquet modules, we see that δ is a subquotient $|\ |\,1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}$. $\delta = \mathrm{Lg}(|\ |\,1; \widehat{|\ |^{1/2}\,\chi_{\omega_{E/F}}}; \lambda')$, and

$$\mathrm{Lg}(|\ |^{1/2}\,\chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)}) = \mathrm{Lg}(|\ |\,\widehat{1; \pi_{1,\chi_{\omega_{E/F}}}}).$$

$1 \times \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. For $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1/2$, representations $|\ |^{\alpha_1}\,1\times |\ |^{\alpha_2}\,\chi_{\omega_{E/F}} \rtimes \lambda'$ are irreducible by Theorem 5.2 and unitary by Theorem 7.10 (1). By [Mi], all irreducible subquotients of $|\ |\,1\times |\ |^{1/2}\,\chi_{\omega_{E/F}} \rtimes \lambda'$ are unitary. □

6.8. $|\ |\,1 \times \chi_{1_{F^*}} \rtimes \lambda'$. Recall that $\lambda'(\det)\,\mathrm{St}_{U(3)}$ is the unique irreducible square-integrable subquotient and that $\lambda'(\det)1_{U(3)}$ is the unique irreducible non-tempered subquotient of $|\ |\,1 \rtimes \lambda'$ ([10]). Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

Theorem 6.7. — *The representation $|\ |\,1 \times \chi_{1_{F^*}} \rtimes \lambda'$ is reducible and we have*

$$\begin{aligned}|\ |\,1 \times \chi_{1_{F^*}} \rtimes \lambda' &= \chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)} + \chi_{1_{F^*}} \rtimes \lambda'(\det)1_{U(3)}\\ &= |\ |\,1 \rtimes \sigma_{1,\chi_{1_{F^*}}} + |\ |\,1 \rtimes \sigma_{2,\chi_{1_{F^*}}}.\end{aligned}$$

Furthermore

$$\begin{aligned}\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)} &= \tau_5 + \tau_6,\\ \chi_{1_{F^*}} \rtimes \lambda'(\det)1_{U(3)} &= \mathrm{Lg}(|\ |\,1; \sigma_{1,\chi_{1_{F^*}}}) + \mathrm{Lg}(|\ |\,1; \sigma_{2,\chi_{1_{F^*}}}),\\ |\ |\,1 \rtimes \sigma_{1,\chi_{1_{F^*}}} &= \mathrm{Lg}(|\ |\,1; \sigma_{1,\chi_{1_{F^*}}}) + \tau_6,\\ |\ |\,1 \rtimes \sigma_{2,\chi_{1_{F^*}}} &= \mathrm{Lg}(|\ |\,1; \sigma_{2,\chi_{1_{F^*}}}) + \tau_5,\end{aligned}$$

where τ_5 and τ_6 are tempered with $\tau_5 = \mathrm{Lg}(|\ |\,\widehat{1; \sigma_{1,\chi_{1_{F^*}}}})$ and $\tau_6 = \mathrm{Lg}(|\ |\,\widehat{1; \sigma_{2,\chi_{1_{F^*}}}})$. All irreducible subquotients are unitary.

Proof. — $\mathrm{Lg}(|\ |\,1; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |\,1; \sigma_{2,\chi_{1_{F^*}}})$ are the only non-tempered subquotients of $|\ |\,1 \times \chi_{1_{F^*}} \rtimes \lambda'$.

Moreover, $\mathrm{Lg}(|\ |\,1; \sigma_{1,\chi_{1_{F^*}}})$ is the unique irreducible quotient of $|\ |\,1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$, and $\mathrm{Lg}(|\ |\,1; \sigma_{2,\chi_{1_{F^*}}})$ is the unique irreducible quotient of $|\ |\,1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$.

$\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ is tempered, hence all irreducible subquotients of $\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ are tempered. Hence $\mathrm{Lg}(|\ |\,1; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |\,1; \sigma_{2,\chi_{1_{F^*}}})$ are subquotients of $\chi_{1_{F^*}} \rtimes \lambda'(\det)1_{U(3)}$.

Let τ_5 and τ_6 be two tempered subquotients of $\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, such that τ_5 is a subquotient of $|\ |\,1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$ and τ_6 is a subquotient of $|\ |\,1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$.

We now show that no other irreducible subquotients of $|\ |\,1 \times \chi_{1_{F^*}} \rtimes \lambda'$ exist. Assume there exists a tempered subquotient τ_7 of $\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$. Consider the Jacquet restrictions of $\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ and τ_i for $i \in \{5,6,7\}$ with respect to the minimal parabolic subgroup:

$$\begin{aligned}s_{\min}(\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}) =& \chi_{1_{F^*}} \otimes |\ |\,1 \otimes \lambda' + |\ |\,1 \otimes \chi_{1_{F^*}} \otimes \lambda'\\ &+ \chi_{1_{F^*}} \otimes |\ |\,1 \otimes \lambda' + |\ |\,1 \otimes \chi_{1_{F^*}} \otimes \lambda'\\ =& 2\,\chi_{1_{F^*}} \otimes |\ |\,1 \otimes \lambda' + 2\,|\ |\,1 \otimes \chi_{1_{F^*}} \otimes \lambda'.\end{aligned}$$

Hence $\exists\, i \in \{5,6,7\}$ such that $s_{\min}(\tau_i)$ does not contain the irreducible subquotient $\chi_{1_{F^*}} \otimes |\ |\,1 \otimes \lambda'$. The Casselman square-integrability criterion [3] implies that τ_i is square-integrable. This can not be the case. Hence τ_7 does not exist, and τ_5

and τ_6 are of multiplicity 1. By the Aubert duality, $\chi 1_{F^*} \rtimes \lambda'(\det)1_{U(3)}$ does not have any subquotients other than $\mathrm{Lg}(|\ | 1; \sigma_{1,\chi 1_{F^*}})$ and $\mathrm{Lg}(|\ | 1; \sigma_{2,\chi 1_{F^*}})$, both of multiplicity 1.

We obtain $\tau_5 = \mathrm{Lg}(|\ |\widehat{1; \sigma_{1,\chi 1_{F^*}}})$, and $\tau_6 = \mathrm{Lg}(|\ |\widehat{1; \sigma_{2,\chi 1_{F^*}}})$.

$1 \rtimes \sigma_{1,\chi 1_{F^*}}$ and $1 \rtimes \sigma_{2,\chi 1_{F^*}}$ are irreducible by Theorem 5.1.1, and by [10] $\sigma_{1,\chi 1_{F^*}}$ and $\sigma_{2,\chi 1_{F^*}}$ are unitary. Hence $1 \rtimes \sigma_{1,\chi 1_{F^*}}$ and $1 \rtimes \sigma_{2,\chi 1_{F^*}}$ are unitary. For $0 < \alpha < 1$, $|\ |^{\alpha}\ 1 \rtimes \sigma_{1,\chi 1_{F^*}}$ and $|\ |^{\alpha}\ 1 \rtimes \sigma_{2,\chi 1_{F^*}}$ are irreducible by Theorem 5.11 and unitary by Theorem 7.23 (2). By [14], all irreducible subquotients of $|\ |\ 1 \times \chi 1_{F^*} \rtimes \lambda'$ are unitary. $\square$

6.9. $|\ |^{1/2}\ \chi\omega_{E/F} \times \chi 1_{F^*} \rtimes \lambda'$. Let $\chi\omega_{E/F} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi\omega_{E/F}}$ be the unique square-integrable irreducible subquotient and let $\pi_{2,\chi\omega_{E/F}}$ be the unique non-tempered irreducible subquotient of $|\ |^{1/2}\ \chi\omega_{E/F} \rtimes \lambda'$ ([10]). Let $\chi 1_{F^*} \in X_{1_{F^*}}$.

Theorem 6.8. — *The representation $|\ |^{1/2}\ \chi\omega_{E/F} \times \chi 1_{F^*} \rtimes \lambda'$ is reducible and we have*

$$\begin{aligned}|\ |^{1/2}\ \chi\omega_{E/F} \times \chi 1_{F^*} \rtimes \lambda' &= \chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}} + \chi 1_{F^*} \rtimes \pi_{2,\chi\omega_{E/F}} \\ &= |\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{1,\chi 1_{F^*}} + |\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{2,\chi 1_{F^*}}.\end{aligned}$$

Furthermore

$$\begin{aligned}\chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}} &= \tau_7 + \tau_8, \\ \chi 1_{F^*} \rtimes \pi_{2,\chi\omega_{E/F}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{1,\chi 1_{F^*}}) + \mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{2,\chi 1_{F^*}}), \\ |\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{1,\chi 1_{F^*}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{1,\chi 1_{F^*}}) + \tau_8, \\ |\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{2,\chi 1_{F^*}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{2,\chi 1_{F^*}}) + \tau_7,\end{aligned}$$

where τ_7 and τ_8 are tempered such that $\tau_7 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi\omega_{E/F}}; \sigma_{1,\chi 1_{F^*}})$ and $\tau_8 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi\omega_{E/F}}; \sigma_{2,\chi 1_{F^*}})$. All irreducible subquotients are unitary.

Proof. — $\mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{1,\chi 1_{F^*}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{2,\chi 1_{F^*}})$ are the only non-tempered subquotients of $|\ |^{1/2}\ \chi\omega_{E/F} \times \chi 1_{F^*} \rtimes \lambda'$.

$\mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{1,\chi 1_{F^*}})$ is the irreducible Langlands quotient of $|\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{1,\chi 1_{F^*}}$, $\mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{2,\chi 1_{F^*}})$ is the irreducible Langlands quotient of $|\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{2,\chi 1_{F^*}}$.

$\chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}}$ is tempered. Hence all irreducible subquotients of $\chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}}$ are tempered. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{1,\chi 1_{F^*}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi\omega_{E/F}; \sigma_{2,\chi 1_{F^*}})$ are subquotients of $\chi 1_{F^*} \rtimes \pi_{2,\chi\omega_{E/F}}$.

Let τ_7 and τ_8 be two tempered subquotients of $\chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}}$, such that τ_7 is a subquotient of $|\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{2,\chi 1_{F^*}}$ and τ_8 is a subquotient of $|\ |^{1/2}\ \chi\omega_{E/F} \rtimes \sigma_{1,\chi 1_{F^*}}$.

We now show that no other irreducible subquotients of $|\ |^{1/2}\ \chi\omega_{E/F} \times \chi 1_{F^*} \rtimes \lambda'$ exist. Assume there exists a tempered subquotient τ_9 of $\chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}}$. Consider the Jacquet restrictions of $\chi 1_{F^*} \rtimes \pi_{1,\chi\omega_{E/F}}$ and τ_i for $i \in \{7, 8, 9\}$ with respect to the minimal parabolic subgroup:

$$\begin{aligned}s_{\min}(\chi_{1_{F^*}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}) =& \chi_{1_{F^*}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \chi_{1_{F^*}} \otimes \lambda' \\ &+ \chi_{1_{F^*}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \chi_{1_{F^*}} \otimes \lambda' \\ =& 2\ \chi_{1_{F^*}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + 2\ |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \chi_{1_{F^*}} \otimes \lambda'.\end{aligned}$$

Hence $\exists\ i \in \{7,8,9\}$ such that $s_{\min}(\tau_i)$ does not contain the irreducible subquotient $\chi_{1_{F^*}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda'$. The Casselman square-integrability criterion [3] implies that τ_i is square-integrable. This can not be the case. Hence τ_9 does not exist, and τ_7 and τ_8 are of multiplicity 1. By the Aubert duality, $\chi_{1_{F^*}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ does not have any subquotients other than $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \sigma_{2,\chi_{1_{F^*}}})$, both of multiplicity 1.

We obtain $\tau_7 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}; \sigma_{1,\chi_{1_{F^*}}}})$, and $\tau_8 = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}; \sigma_{2,\chi_{1_{F^*}}}})$.

$\chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible by Theorem 5.1.$\chi_{\omega_{E/F}}$ and by [10] $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are unitary. Hence $\chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are unitary. For $0 < \alpha < 1/2$, $|\ |^{\alpha}\ \chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^{\alpha}\ \chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ are irreducible by Theorem 5.11 and unitary by Theorem 7.23 (3). By [14], all irreducible subquotients of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times \chi_{1_{F^*}} \rtimes \lambda'$ are unitary. □

6.10. $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{\omega_{E/F}}, \chi'_{\omega_{E/F}} \in X_{\omega_{E/F}}$ be such that $\chi_{\omega_{E/F}} \neq \chi'_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique square-integrable subquotient and let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique non-tempered irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\pi_{1,\chi_{\omega'_{E/F}}}$ be the unique square-integrable irreducible subquotient and let $\pi_{2,\chi_{\omega'_{E/F}}}$ be the unique non-tempered irreducible subquotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$ [10].

Theorem 6.9. — *The representation $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$ is reducible. We have*

$$\begin{aligned}|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda' &= |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}} + |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}} \\ &= |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} + |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}.\end{aligned}$$

Furthermore

$$\begin{aligned}|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}}) + \delta, \\ |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda') \\ &\quad + \mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}), \\ |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}) + \delta, \\ |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}} &= \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda') \\ &\quad + \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi'_{1,\chi_{\omega_{E/F}}}),\end{aligned}$$

where $\delta = \mathrm{Lg}(\widehat{|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |}^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$ *is square-integrable. Moreover,* $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ *and* $\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ *are unitary.*

Proof. — Clearly $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi'_{1,\chi_{\omega_{E/F}}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ are all the non-tempered irreducible subquotients of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$.

$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ is a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}}$.

Consider the Jacquet restrictions of

$$|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}},\ |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} \text{ and } |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$$

with respect to the minimal parabolic subgroup:

$$\begin{aligned} s_{\min}(|\ |^{1/2}\chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}}) = \\ &|\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' \\ &+ |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda'. \end{aligned}$$

As $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ is non-tempered, $s_{\min}(\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}}))$ must contain the irreducible subquotient $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda'$.

$$\begin{aligned} s_{\min}(|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}) = \\ &|\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda' \\ &+ |\ |^{-1/2}\ \chi'_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda', \text{ and} \\ s_{\min}(|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}) = \\ &|\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda' \\ &+ |\ |^{-1/2}\ \chi'_{\omega_{E/F}} \otimes |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes \lambda' + |\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{-1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda'. \end{aligned}$$

The irreducible subquotient $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda'$ does appear in $s_{\min}(|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}})$, but not in $s_{\min}(|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}})$. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ is also a subquotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$.

$\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ is a subquotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$. In the same manner as above we find that $\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ is also a subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}}$.

$\pi_{2,\chi_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Hence $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$. $\pi_{2,\chi'_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$, hence $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}}$ is a quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$. $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$ is the unique irreducible quotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$ is a quotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ and of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}}$.

A tempered representation is the subquotient of a representation induced from a square-integrable representation of a parabolic subgroup. Here, for $i = 0, 1, 2$, $\mathrm{Ind}_{M_0}^{M_i}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda')$ does not contain any square-integrable subquotient. Hence any irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$ other than $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi'_{1,\chi_{\omega_{E/F}}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ must be square-integrable.

$s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}})$ contains only one negative subquotient, $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda'$. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ is the only non-tempered irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}}$.

Let δ denote a square-integrable irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}}$. Looking at Jacquet modules we find that δ is also a subquotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$.

So far we have seen:

$$\begin{aligned}
|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}} =& \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}}) + \delta + A_1, \\
|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}} =& \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda') \\
&+ \mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}) + A_2, \\
|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}} =& \mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}}) + \delta + A_3, \\
|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}} =& \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda') \\
&+ \mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}}) + A_4,
\end{aligned}$$

where A_1, A_2, A_3 and A_4 are sums of tempered representations. We will now show that A_1, A_2, A_3 and A_4 are equal to 0.

$s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}})$ does not contain any non-negative subquotients. Hence by the Casselman square-integrability criterion [3], all irreducible subquotients of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}}$ are non-tempered. Since each subquotient in $s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}})$ is of multiplicity one, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ are of multiplicity one in $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}}$. The irreducible subquotient $|\ |^{-1/2}\ \chi_{\omega_{E/F}} \otimes |\ |^{1/2}\ \chi'_{\omega_{E/F}} \otimes \lambda'$ in $s_{\min}(\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}}))$ does not appear in $s_{\min}(|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}})$. Hence $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ is no subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi'_{\omega_{E/F}}}$.

Equivalently we obtain that all irreducible subquotients of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ are non-tempered, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ are of multiplicity one and $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}})$ is no subquotient of $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$. By the Aubert duality, $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi'_{\omega_{E/F}}}$ and $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ do not have any other subquotients.

We obtain that $\delta = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi_{\omega_{E/F}}; |\ |}^{1/2}\ \chi'_{\omega_{E/F}}; \lambda')$ and

$$\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi'_{\omega_{E/F}}}) = \mathrm{Lg}(|\ |^{1/2}\ \widehat{\chi'_{\omega_{E/F}};} \pi_{1,\chi_{\omega_{E/F}}}).$$

$\chi_{\omega_{E/F}} \times \chi'_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. For $0 < \alpha_1, \alpha_2 < 1/2$, $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.2 and unitary by Theorem 7.13 (1). By [14], all irreducible subquotients of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \lambda'$ are unitary. ☐

7. Irreducible unitary representations of $U(5)$, in terms of Langlands quotients

7.1. **Representations with cuspidal support in M_0, fully-induced.** For any $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$, let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique irreducible square-integrable subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Recall that $\chi_{1_{F^*}} = \sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$, where $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are tempered [10].

Proposition 7.1. — *Let $0 < \alpha_2 \leqslant \alpha_1$, $\alpha > 0$. Let χ_1, χ_2 and χ be unitary characters of E^*. The following list exhausts all irreducible hermitian representations of $U(5)$ with cuspidal support in M_0:*

(0) *tempered representations of $U(5)$,*
(1) $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ *where $\chi_1, \chi_2 \in X_{N_{E/F}(E^*)}$ or $\alpha_1 = \alpha_2$ and $\chi_1(x) = \chi_2^{-1}(\overline{x})\ \forall x \in E^*$,*
(2) $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \chi_2 \rtimes \lambda')$ *where $\chi_1 \in X_{N_{E/F}(E^*)}$ and $\chi_2 \notin X_{1_{F^*}}$,*
(3) $\mathrm{Lg}(|\ |^{\alpha}\ \chi\, \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ *where $\chi \in X_{N_{E/F}(E^*)}$,*
(4) $\mathrm{Lg}(|\ |^{\alpha}\ \chi\,; \lambda'\,(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^{\alpha}\ \chi\,; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^{\alpha}\ \chi\,; \sigma_{1,\chi_{1_{F^*}}})$ *and* $\mathrm{Lg}(|\ |^{\alpha}\ \chi; \sigma_{2,\chi_{1_{F^*}}})$, *where $\chi \in X_{N_{E/F}(E^*)}$.*

Outline of the proof: 0. Tempered representations are unitary, hence hermitian.

1.-4. Let λ_i, for $i = 0, 1, 2$, be representations of the Levi subgroups M_0, M_1 and M_2. By [3], $\mathrm{Ind}_{M_i}^{U(5)}(\widetilde{\lambda_i}) \cong \mathrm{Ind}_{M_i}^{U(5)}(\lambda_i)$, for $i = 0, 1, 2$, is equivalent to the existence of $w \in W$ such that $\widetilde{\lambda_i} = w\lambda_i$ for $i = 0, 1, 2$. This holds also for the Langlands quotients, and the proof is immediate. ☐

7.1.1. *Irreducible subquotients of $\chi_1 \times \chi_2 \rtimes \lambda'$.* Let χ_1, χ_2 be unitary characters of E^*.

- All irreducible subquotients of $\chi_1 \times \chi_2 \rtimes \lambda'$ are tempered, hence unitary.
- $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$, $0 < \alpha_2 \leqslant \alpha_1$, $\chi_1 \notin X_{N_{E/F}(E^*)}$ or $\chi_2 \notin X_{N_{E/F}(E^*)}$.

Theorem 7.2. — *Let χ_1, χ_2 be unitary characters of E^* with $\chi_1 \notin X_{N_{E/F}(E^*)}$ or $\chi_2 \notin X_{N_{E/F}(E^*)}$.*

(1) *Let $0 < \alpha_2 \leqslant \alpha_1$. Let $\alpha_1 \neq \alpha_2$ or $\exists x \in E^*$ s. t.$\chi_1(x) \neq \chi_2^{-1}(\overline{x})$.* $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ *is non-unitary.*
(2) *Let $0 < \alpha_1 = \alpha_2$ and $\chi_1(x) = \chi_2^{-1}(\overline{x})\ \forall x \in E^*$. Then* $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ *is unitary for $0 < \alpha_1 = \alpha_2 \leqslant 1/2$, and* $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ *is non-unitary for $\alpha > 1/2$.*

Proof. —

(1) Let $\alpha_1 \neq \alpha_2$ or there exists $x \in E^*$ such that $\chi_1(x) \neq \chi_2^{-1}(\overline{x})$. The representations $|\ |^{\alpha_1}\ \chi_1 \times |\ |^{\alpha_2}\ \chi_2 \rtimes \lambda'$ are not hermitian, by [3] 3.1.2. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ is not hermitian, hence not unitary.

(2) Let $\alpha_1 = \alpha_2$ and $\chi_1(x) = \chi_2^{-1}(\overline{x})$ for all $x \in E^*$. Representations $|\ |^{\alpha_1}\ \chi_1 \times |\ |^{\alpha_2}\ \chi_2 \rtimes \lambda'$ are hermitian. Let $\alpha_1 = \alpha_2 < 1/2$ and $\chi_1(x) = \chi_2^{-1}(\overline{x})$ for all $x \in E^*$. Representations $|\ |^{\alpha_1}\ \chi_1 \times |\ |^{\alpha_2}\ \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.2 and equal to their Langlands quotients $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$. $\chi_1 \times \chi_2 \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. For $\alpha_1 = \alpha_2 < 1/2$, representations $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ form a continuous 1-parameter family of irreducible hermitian representations that we realize on the same vector space V (for a detailed version of this argument in a similar case see the proof of Theorem 7.4). By Remark 3.1, $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ is unitary for $\alpha_1 = \alpha_2 < 1/2$. For $\alpha_1 = \alpha_2 = 1/2$ and $\chi_1(x) = \chi_2^{-1}(\overline{x})$ for all $x \in E^*$, $|\ |^{1/2}\ \chi_1 \times |\ |^{1/2}\ \chi_2 \rtimes \lambda'$ is reducible by Theorem 5.2. By [14], $\mathrm{Lg}(|\ |^{1/2}\ \chi_1; |\ |^{1/2}\ \chi_2; \lambda')$ is unitary.

Let $\alpha_1 = \alpha_2 > 1/2$ and $\chi_1(x) = \chi_2^{-1}(\overline{x})$ for all $x \in E^*$. Representations $|\ |^{\alpha_1}\ \chi_1 \times |\ |^{\alpha_2}\ \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.2 and equal to their Langlands quotients $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$. By Remark 3.1 and Lemma 3.3 $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_1; |\ |^{\alpha_2}\ \chi_2; \lambda')$ is non-unitary for $\alpha_1 = \alpha_2 > 1/2$. □

7.1.2. *$\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \chi_2 \rtimes \lambda')$, $\alpha > 0$, $\chi_1 \notin X_{N_{E/F}(E^*)}$* ***or*** *$\chi_2 \notin X_{N_{E/F}(E^*)}$.* Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Recall that $X_{N_{E/F}(E^*)} = 1 \cup X_{\omega_{E/F}} \cup X_{1_{F^*}}$.

Theorem 7.3. — (1) *Let $\alpha > 0$. Let $\chi_1 \notin X_{N_{E/F}(E^*)}$ and $\chi_2 \notin X_{1_{F^*}}$. Then $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \chi_2 \rtimes \lambda')$ is non-unitary.*

(2) *Let $\alpha > 0$. Let $\chi_1 \notin X_{N_{E/F}(E^*)}$. Then $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \sigma_{2,\chi_{1_{F^*}}})$ are non-unitary.*

(3) *Let $\alpha > 0$. Let $\chi_1 \in X_{N_{E/F}(E^*)}$ and $\chi_2 \notin X_{N_{E/F}(E^*)}$.*
   (3.1) *Let $\chi_1 = 1$. Let $0 < \alpha \leqslant 1$. $\mathrm{Lg}(|\ |^{\alpha}\ 1; \chi_2 \rtimes \lambda')$ is unitary.*
   Let $\alpha > 1$. $\mathrm{Lg}(|\ |^{\alpha}\ 1; \chi_2 \rtimes \lambda')$ is non-unitary.
   (3.2) *Let $\chi_1 \in X_{\omega_{E/F}}$. Let $0 < \alpha \leqslant 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_2 \rtimes \lambda')$ is unitary.*
   Let $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_2 \rtimes \lambda')$ is non-unitary.
   (3.3) *Let $\chi_1 \in X_{1_{F^*}}$. Let $\alpha > 0$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}; \chi_2 \rtimes \lambda')$ is non-unitary.*

Proof. —

(1) For $\alpha > 0$, representations $|\ |^{\alpha}\ \chi_1 \times \chi_2 \rtimes \lambda'$ are not hermitian. By [3] 3.1.2, $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \chi_2 \rtimes \lambda')$ is not hermitian, hence not unitary.

(2) For $\alpha > 0$, $|\ |^{\alpha}\ \chi_1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^{\alpha}\ \chi_1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are not hermitian. By [3] $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{\alpha}\ \chi_1; \sigma_{2,\chi_{1_{F^*}}})$ are not hermitian and hence non-unitary.

(3) (3.1) $1 \times \chi_2 \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. For $0 < \alpha < 1$, representations $|\ |^{\alpha}\ 1 \times \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.4 and equal to their Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\ 1; \chi_2 \rtimes \lambda')$. By Remark 3.1 these Langlands quotients are unitary. For $\alpha = 1$, $|\ |\ 1 \times \chi_2 \rtimes \lambda'$ reduces for the first time, by Theorem 5.4. By [14] $\mathrm{Lg}(|\ |\ 1; \chi_2 \rtimes \lambda')$ is unitary. For $\alpha > 1$, representations $|\ |^{\alpha}\ 1 \times \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.4 and equal to their Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\ 1; \chi_2 \rtimes \lambda')$.

By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary.

(3.2) $\chi_{\omega_{E/F}} \times \chi_2 \rtimes \lambda'$ is irreducibly by Theorem 5.1 and unitary. For $0 < \alpha < 1/2$, representations $| \, |^{\alpha} \chi_{\omega_{E/F}} \times \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.4 and equal to their Langlands quotient $\mathrm{Lg}(| \, |^{\alpha} \chi_{\omega_{E/F}}; \chi_2 \rtimes \lambda')$. By Remark 3.1 these Langlands quotients are unitary. For $\alpha = 1/2$, $| \, |^{1/2} \chi_{\omega_{E/F}} \times \chi_2 \rtimes \lambda'$ reduces for the first time, by Theorem 5.4. By [14] $\mathrm{Lg}(| \, |^{1/2} \chi_{\omega_{E/F}}; \chi_2 \rtimes \lambda')$ is unitary. For $\alpha > 1/2$, representations $| \, |^{\alpha} \chi_{\omega_{E/F}} \times \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.4 and equal to their Langlands quotient $\mathrm{Lg}(| \, |^{\alpha} \chi_{\omega_{E/F}}; \chi_2 \rtimes \lambda')$. By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary.

(3.3) For $\alpha > 0$, representations $| \, |^{\alpha} \chi_{1_{F^*}} \times \chi_2 \rtimes \lambda'$ are irreducible by Theorem 5.4 and equal to their Langlands quotient $\mathrm{Lg}(| \, |^{\alpha} \chi_{1_{F^*}}; \chi_2 \rtimes \lambda')$. By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary. □

We now take $\chi_1, \chi_2 \in X_{N_{E/F}(E^*)}$.

7.1.3. $\mathrm{Lg}(| \, |^{\alpha_1} 1; | \, |^{\alpha_2} 1; \lambda')$, $0 < \alpha_2 \leqslant \alpha_1$, $\mathrm{Lg}(| \, |^{\alpha} 1; 1 \rtimes \lambda')$, $\alpha > 0$.

Theorem 7.4. — (1) *Let* $0 < \alpha_2 \leqslant \alpha_1 \leqslant 1$ *and* $\alpha_1 + \alpha_2 \leqslant 1$. *Then* $\mathrm{Lg}(| \, |^{\alpha_1} 1; | \, |^{\alpha_2} 1; \lambda')$ *is unitary.*

(2) *Let* $0 < \alpha_2 \leqslant \alpha_1$, $\alpha_1 + \alpha_2 > 1$, $(\alpha_1, \alpha_2) \neq (2, 1)$. *If* $\alpha_1 = 1$, *then let* $\alpha_2 \notin (0, 1]$. *Then* $\mathrm{Lg}(| \, |^{\alpha_1} 1; | \, |^{\alpha_2} 1; \lambda')$ *is non-unitary.*

(2.1) $\mathrm{Lg}(| \, |^{2} 1; | \, | 1; \lambda') = 1_{U(5)}$ *is unitary.*

(3) *For* $0 < \alpha \leqslant 1$, $\mathrm{Lg}(| \, |^{\alpha} 1; 1 \rtimes \lambda')$ *is unitary.*

(4) *For* $\alpha > 1, \mathrm{Lg}(| \, |^{\alpha} 1; 1 \rtimes \lambda')$ *is non-unitary.*

Proof. —

(1) $1 \times 1 \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. Let $0 < \alpha_2 \leqslant \alpha_1$. For $\alpha_1 + \alpha_2 < 1$, representations $| \, |^{\alpha_1} 1 \times | \, |^{\alpha_2} 1 \rtimes \lambda'$ are irreducible by Theorem 5.2, hence equal to $\mathrm{Lg}(| \, |^{\alpha_1} 1; | \, |^{\alpha_2} 1; \lambda')$.

We will now construct a continuous one-parameter family of hermitian representations. Let $0 < \alpha_2 \leqslant \alpha_1$ be such that $\alpha_1 + \alpha_2 \leqslant 1$. Let π_{α_1,α_2} denote the two-parameter family of hermitian representations $| \, |^{\alpha_1} 1 \times | \, |^{\alpha_2} 1 \rtimes \lambda'$. Let V_{α_1,α_2} be the vector space of π_{α_1,α_2}.

Recall the Levi decomposition $P_0 = M_0 N_0$, where P_0 is the minimal parabolic subgroup of $U(5)$, $M_0 = \{ \begin{pmatrix} x & & & & 0 \\ & y & & & \\ & & k & & \\ & & & \overline{y}^{-1} & \\ 0 & & & & \overline{x}^{-1} \end{pmatrix}, \, x, y \in E^*, \, k \in E^1 \}$ is the minimal Levi subgroup and N_0 the unipotent radical of P_0.

Let (π, V) be the extension to P_0 of the representation $1 \otimes 1 \otimes \lambda'$ of M_0. Let δ_{P_0} denote the modulus character of P_0, and put

$$
\begin{aligned}
V_{0,0} &= \{ f : \mathrm{G} \to V : f \text{ is smooth and } f(pg) = \delta_{P_0}(p)\pi(p)f(g) \forall g \in \mathrm{G} \}, \\
V_{\alpha_1,\alpha_2} &= \{ h : \mathrm{G} \to V : h \text{ is smooth and for all } g \in G \\
&\qquad h(pg) = \delta_{P_0}(p) \mid x \mid^{\alpha_1} \mid y \mid^{\alpha_2} \pi(p)h(g) \},
\end{aligned}
$$

where $p \in P_0$, $p = \begin{pmatrix} x & & & - & * \\ & y & & & | \\ & & k & & \\ | & & & \overline{y}^{-1} & \\ 0 & - & & & \overline{x}^{-1} \end{pmatrix}$, $x, y \in E^*$, $k \in E^1$, $* \in E$. Let $\mathcal{O}$ denote the ring of integers of E. $|\ |\colon E^* \to F^*$ is unramified, hence $(x, y) \mapsto |\ x\ |^{\alpha_1}|\ y\ |^{\alpha_2}$ for $x, y \in E^*$ is trivial on $E^1 \times E^1 \cong \mathcal{O}^* \times \mathcal{O}^*$. Let $K := U(\mathcal{O})$; this is a maximal compact subgroup of G. We have $\mathrm{G} = KP_0$. Let $f \in V_{0,0}$. There exists a unique extension of $f_{|K} : \mathrm{G} \to V$ to a function $h \in V_{\alpha_1,\alpha_2}$, so $f_{|K} = h_{|K}$. This induces an isomorphism $T_{\alpha_1,\alpha_2} : V_{0,0} \xrightarrow{\sim} V_{\alpha_1,\alpha_2}$. Via the composition with T_{α_1,α_2} we consider all representations π_{α_1,α_2} in $V_{0,0}$.

Let $w \in W$ be the longest element of the Weyl group. Let

$$A(w, \lambda) : |\ |^{\alpha_1}\ 1 \times |\ |^{\alpha_2}\ 1 \rtimes \lambda' \to |\ |^{-\alpha_1}\ 1 \times |\ |^{-\alpha_2}\ 1 \rtimes \lambda'$$

be the standard long intertwining operator.

On $V_{0,0}$ we define a set of non-degenerate hermitian forms $\langle\ ,\ \rangle_{\alpha_1,\alpha_2}$ by

$$\langle f, h\rangle_{\alpha_1,\alpha_2} = \int\limits_{U(\mathcal{O})} A(w, \lambda) f(k) \overline{h(k)} dk,\ f, h \in V_{0,0},$$

such that $\langle\ ,\ \rangle_{\alpha_1,\alpha_2}$ is invariant by $T^{-1}_{\alpha_1,\alpha_2}\pi_{\alpha_1,\alpha_2}T_{\alpha_1,\alpha_2}$.

Fix α_1 and α_2 such that $\alpha_1 + \alpha_2 = 1$. Let $\pi_t = \pi_{t\alpha_1,t\alpha_2}$, for $t \in [0, 1]$, denote a continuous one-parameter family of hermitian representations. Let V_t be the vector space of π_t. Via the isomorphism $T_t : V_0 \xrightarrow{\sim} V_t$, we consider all representations π_t in V_0, as before.

Choose a real polynomial $p(t)$, such that $A(t) = p(t)A(w, \lambda)$ is holomorphic and non-zero for $t \in [0, 1]$. So for the one-parameter family of representations π_t one obtains, on the same space V_0, a set of non-degenerate hermitian forms $\langle\ ,\ \rangle_t$ given by

$$\langle f, h\rangle_t = \int\limits_{U(\mathcal{O})} A(t) f(k) \overline{h(k)} dk,\ f, h \in V_0,$$

such that $\langle\ ,\ \rangle_t$ is invariant under $T_t^{-1}\pi_t T_t$.

$\langle\ ,\ \rangle_0$ is positive definite, hence by Remark 3.1 $\langle\ ,\ \rangle_t$ is positive definite until $|\ |^{t\alpha_1}\ 1 \times |\ |^{t\alpha_2} \rtimes \lambda'$ reduces for the first time, for $t = 1$. By [14], for $t = 1$, the irreducible subquotients of $|\ |^{\alpha_1}\ 1 \times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$ are unitary. Hence for $0 < \alpha_2 \leqslant \alpha_1 \leqslant 1$, $\alpha_1 + \alpha_2 \leqslant 1$, the Langlands quotients $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ are unitary.

(2) and 2.1. $|\ |^2\ 1 \times |\ |\ 1 \rtimes \lambda'$ is reducible.$\lambda'(\det)\,\mathrm{St}_{U(3)}$ is the unique irreducible square-integrable subquotient of $|\ |\ 1 \rtimes \lambda'$ [10]. By [2, p.915] the subquotient $|\ |^{3/2}\ 1_{\mathrm{GL}_2} \rtimes \lambda'$ is reducible and has the subquotients $1_{U(5)} = \mathrm{Lg}(|\ |^2\ 1; |\ |\ 1; \lambda')$ and $\mathrm{Lg}(|\ |^2\ 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$. $1_{U(5)} = \mathrm{Lg}(|\ |^2\ 1; |\ |\ 1; \lambda')$ is unitary, $\mathrm{Lg}(|\ |^2\ 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary ([2]). For $1/2 < \alpha < 3/2$, representations $|\ |^{\alpha}\ 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible by Theorem 5.5, they form a continuous one-parameter family of irreducible hermitian representations on the space V_α. Like before we identify the vector spaces V_α for $1/2 < \alpha < 3/2$. For $\alpha = 3/2$, the irreducible subquotient $\mathrm{Lg}(|\ |^2\ 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ of $|\ |^{3/2}\ 1_{\mathrm{GL}_2} \rtimes \lambda'$ is not unitary [2]. Hence, by [14] and by Remark 3.1,

$|\ |^{\alpha}\ 1_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ is non-unitary for $1/2 < \alpha < 3/2$, that is for $1 < \alpha_1 < 2$, $\alpha_1 - \alpha_2 = 1$.

By [2, p.915] the subquotient $|\ |^2\ 1 \rtimes \lambda'(\det)1_{U(3)}$ of the representation $|\ |^2\ 1\times |\ |\ 1 \rtimes \lambda'$ is reducible. It has the subquotients $1_{U(5)} = \mathrm{Lg}(|\ |^2\ 1; |\ |\ 1; \lambda')$ and $\mathrm{Lg}(|\ |^{3/2}\ \mathrm{St}_{\mathrm{GL}_2}; \lambda')$. $1_{U(5)}$ is unitary, $\mathrm{Lg}(|\ |^{3/2}\ \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is non-unitary [2].

Let $1 < \alpha < 2$. Representations $|\ |^{\alpha}\ 1 \rtimes \lambda'(\det)1_{U(3)}$ are irreducible by Theorem 5.7, they form a continuous one-parameter family of irreducible hermitian representations on the space V_α. Similar as before we identify V_α for $1 < \alpha < 2$. The irreducible subquotient $\mathrm{Lg}(|\ |^{3/2}\ \mathrm{St}_{\mathrm{GL}_2}; \lambda')$ of $|\ |^2\ 1 \rtimes \lambda'(\det)1_{U(3)}$ is non-unitary [2]. Hence by [14] and by Remark 3.1 representations $|\ |^{\alpha}\ 1 \rtimes \lambda'(\det)1_{U(3)} = \mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^1\ 1; \lambda')$ are non-unitary for $1 = \alpha_2 < \alpha_1 = \alpha < 2$.

Let $1 < \alpha_1 < 2$, $0 < \alpha_2 < 1$, $\alpha_1 - \alpha_2 < 1$. $|\ |^{\alpha_1}\ 1\times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$ is irreducible by Theorem 5.2 and equal to its own Langlands quotient $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$. Fix $0 < \alpha_2 < 1$ and let $1 < \alpha_1 \leqslant \alpha_2 + 1$. Let π_{α_1} denote the continuous one-parameter family of hermitian representations $|\ |^{\alpha_1}\ 1\times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$ on the same vector space V. For $\alpha_1 = \alpha_2 + 1$ irreducible subquotients of the representations $|\ |^{\alpha_1}\ 1\times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$ are non-unitary, as seen in the previous paragraph. By [14] and by Remark 3.1 the Langlands quotients $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ are non-unitary. (II in Figure 7.1 on page 130)

Let $\alpha_1 > 2$, $\alpha_1 - \alpha_2 = 1$. $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ is non-unitary by Remark 3.1 and Lemma 3.3. For $\alpha_1 > 2$, $\alpha_2 = 1$, $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ is non-unitary by the same argument.

The same holds for $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$, where $\alpha_1 > 1, 0 < \alpha_2 < 1$, $\alpha_1 - \alpha_2 > 1$, for $1 < \alpha_2 \leqslant \alpha_1$, $\alpha_1 - \alpha_2 < 1$ and for $\alpha_1 > 2$, $\alpha_1 - \alpha_2 > 1$ (III, IV, V in Figure 7.1 on page 130).

Let $\alpha_1 = 1$, $\alpha_2 \in (0, 1]$. We have no proof that $\mathrm{Lg}(|\ |\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ is non-unitary.

(3) $1 \times 1 \times \lambda'$ is irreducible by Theorem 5.1 and unitary. For $0 < \alpha < 1$, $|\ |^{\alpha}\ 1 \times 1 \rtimes \lambda'$ is irreducible by Theorem 5.4 and equal to its own Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\ 1; 1 \rtimes \lambda')$. By Remark 3.1 these representations are unitary. For $\alpha = 1$, $|\ |\ 1 \times 1 \rtimes \lambda'$ reduces for the first time, hence $\mathrm{Lg}(|\ |\ 1; 1 \rtimes \lambda')$ is unitary [14].

(4) For $\alpha > 1$, $|\ |^{\alpha}\ 1 \times 1 \rtimes \lambda' = \mathrm{Lg}(|\ |^{\alpha}\ 1; 1 \rtimes \lambda')$ is irreducible by Theorem 5.4 and by Remark 3.1 and Lemma 3.3 non-unitary. □

7.1.4. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$, $0 < \alpha_2 \leqslant \alpha_1$, $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$, $\alpha > 0$, $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$.

Theorem 7.5. — (1) *Let* $0 < \alpha_2 \leqslant \alpha_1 \leqslant 1/2$. *Then* $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ *is unitary.*

(2) *Let* $\alpha_1 > 1/2$, $\alpha_2 \leqslant \alpha_1$, $(\alpha_1, \alpha_2) \neq (3/2, 1/2)$. *If* $0 < \alpha_2 < 1/2$, *then let* $\alpha_1 \notin (1/2, 1 - \alpha_2]$. *Then* $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ *is non-unitary.*

(3) *Let* $0 < \alpha \leqslant 1/2$. *Then* $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ *is unitary.*

(4) *Let* $\alpha > 1$. *Then* $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ *is non-unitary.*

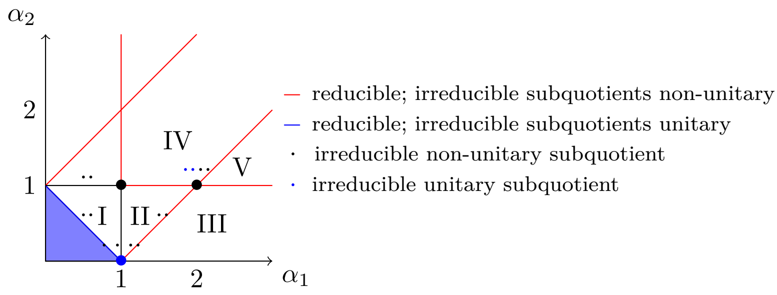

Figure 1

FIGURE 7.1. Let $\alpha_1, \alpha_2 \geqslant 0$, Figure 1 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ 1 \times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$ and the unitary dual. Let $0 < \alpha_2 \leqslant \alpha_1$. $\mathrm{Lg}(|\ |^{\alpha_1}\ 1; |\ |^{\alpha_2}\ 1; \lambda')$ is unitary for $0 < \alpha_2 \leqslant \alpha_1 < 1$, $\alpha_1 + \alpha_2 \leqslant 1$ and for $\alpha_1 = 2$, $\alpha_2 = 1$. Except for $\alpha_1 = 1$, $0 < \alpha_2 \leqslant 1$, it is non-unitary. $\mathrm{Lg}(|\ |^{\alpha}\ 1; 1 \rtimes \lambda')$ is unitary for $0 < \alpha \leqslant 1$ and non-unitary for $\alpha > 1$.

Proof. —

(1) $\chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. For $0 < \alpha_2 \leqslant \alpha_1 < 1/2$, $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.2 and equal to its Langlands quotient $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$. Let $\alpha_1 = 1/2$ and fix $0 < \alpha_2 \leqslant 1/2$. For $t \in [0, 1]$, let $\pi_{(t1/2, t\alpha_2)} =: \pi_t$ denote the continuous one-parameter family of hermitian representations $|\ |^{t1/2}\ \chi_{\omega_{E/F}} \times |\ |^{t\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. For $t \in [0, 1)$ these representations are equal to their own Langlands quotient $\mathrm{Lg}(|\ |^{t1/2}\ \chi_{\omega_{E/F}}; |\ |^{t\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ and by Remark 3.1 unitary. For $t = 1$ the representations π_t reduce for the first time. By [14] $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ is unitary.

(2) Let $1 < \alpha_1 < 3/2$ and let $\alpha_1 - \alpha_2 = 1$. Then $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ is reducible by Theorem 5.2. The subquotients $|\ |^{\frac{\alpha_1+\alpha_2}{2}}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible by Theorem 5.5. They form a continuous 1-parameter family of irreducible hermitian representations, that similar as before, we realize on the same vector space V.

Let $\alpha_1 = 3/2$, $\alpha_2 = 1/2$. Then

$$|\ |^{\frac{3/2+1/2}{2}}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda' = |\ |\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$$

reduces by Theorem 6.4. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique square-integrable subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [10]. By Theorem 1.1 and Remark 4.7 in [7] the irreducible subquotient $\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ of $|\ |\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ is non-unitary. By [14] and Remark 3.1 the representations $|\ |^{\frac{\alpha_1+\alpha_2}{2}}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$, for $1 < \alpha_1 < 3/2$ and $\alpha_1 - \alpha_2 = 1$, that are equal to $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$, for $1 < \alpha_1 < 3/2$ and $\alpha_1 - \alpha_2 = 1$, are non-unitary.

Let $1/2 < \alpha_1 < 3/2$, $\alpha_2 = 1/2$. By Theorem 5.2 the representations $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ are reducible. Let $\pi_{2,\chi_{\omega_{E/F}}}$ be the unique irreducible non-tempered subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ [10]. For $1/2 < \alpha_1 < 3/2$, the representations $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ are irreducible by Theorem 5.9 and equal to the Langlands quotient $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$. They form a 1-parameter family of irreducible hermitian representations, that we realise on the same vector space V. For $\alpha_1 = 3/2$, $|\ |^{3/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{2,\chi_{\omega_{E/F}}}$ reduces by Theorem 6.4, and by Theorem 1.1 and Remark 4.7 in [7] its irreducible subquotient $\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2}; \lambda')$ is non-unitary. By [14] and Remark 3.1

$$|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \rtimes \pi_2 = \mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{1/2}\ \chi_{\omega_{E/F}}; \lambda')$$

is non-unitary for $1/2 < \alpha_1 < 3/2$.

Representations $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ in II,III,IV,V of Figure 7.2 on page 132 are irreducible by Theorem 5.2 and equal to their own Langlands quotient $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$. The Langlands quotients $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ in II are non-unitary by [14] and Remark 3.1. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ in III, IV and V are non-unitary by Remark 3.1 and Lemma 3.3.

(3) $\chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.1 and unitary. For $0 < \alpha < 1/2$, $|\ |^{\alpha}\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.4 and equal to its Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$. By Remark 3.1 these Langlands quotients are unitary.

For $\alpha = 1/2$, $|\ |^{1/2}\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ reduces for the first time (5.4). By [14] $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ is unitary.

(4) For $\alpha > 1$, $|\ |^{\alpha}\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ is irreducible by Theorem 5.4 and equal to its Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}}; \lambda')$. By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary. ☐

Remark 7.6. — Unfortunately we do not have a proof that the representation $\mathrm{Lg}(|\ |^{3/2}\ \chi_{\omega_{E/F}} \times |\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda')$ is unitary. It is the Aubert dual of a square-integrable representation and is expected to be unitary, see Theorem 6.4. See [6], where the proof is given for orthogonal and symplectic groups.

Remark 7.7. — In the Grothendieck group of the category of admissible representations of finite length one has

$$|\ |\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda' = |\ |^{1/2}\ \chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' + |\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'.$$

If we assume that $|\ |^{1/2}\ \chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ and $|\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible (see Remark 6.5), we are able to prove that $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$ is non-unitary for $1/2 < \alpha_1 < 1$, $\alpha_2 \leqslant 1 - \alpha_1$, and that $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}}; \lambda')$ is non-unitary for $1/2 < \alpha \leqslant 1$.

Let $1/2 < \alpha_1 < 1$, $\alpha_2 = 1 - \alpha_1$. The representations $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$ are reducible by Theorem 5.2, and the subquotient $|\ |^{\frac{\alpha_1 - \alpha_2}{2}}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible by Theorem 5.5. It is equal to $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F}}; \lambda')$. By assumption $|\ |^{1/2}\ \chi_{\omega_{E/F}} 1_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible, it is equal to $\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes$

λ'). Hence we can extend the argument 2 in the proof of Theorem 7.5: $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{\omega_{E/F}}; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$ is non-unitary for $1/2 < \alpha_1 < 1$ and $\alpha_2 = 1 - \alpha_1$ and for $1 < \alpha_1 < 3/2$, $\alpha_2 = \alpha_1 - 1$, and $\mathrm{Lg}(|\ | \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ is non-unitary.

Let $1/2 < \alpha_1 < 1$, $\alpha_2 < 1 - \alpha_1$. By Theorem 5.2 the representations $|\ |^{\alpha_1} \chi_{\omega_{E/F}} \times |\ |^{\alpha_2} \chi_{\omega_{E/F}} \rtimes \lambda'$ are irreducible, they are equal to their Langlands quotient $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{\omega_{E/F}}; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$ (I in Figure 7.2, page 132). By [14] and by Remark 3.1 these Langlands quotients are non-unitary.

Let $1/2 < \alpha < 1$. By Theorem 5.4 representations $|\ |^{\alpha} \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ are irreducible, they are equal to their Langlands quotient $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$. $|\ |\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}} \rtimes \lambda'$ is reducible by Theorem 5.4, $\mathrm{Lg}(|\ |\ \chi_{\omega_{E/F}} \times \chi_{\omega_{E/F}}; \lambda')$ is non-unitary by the foregoing argument. By [14] and by Remark 3.1 $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ is non-unitary for $1/2 < \alpha < 1$.

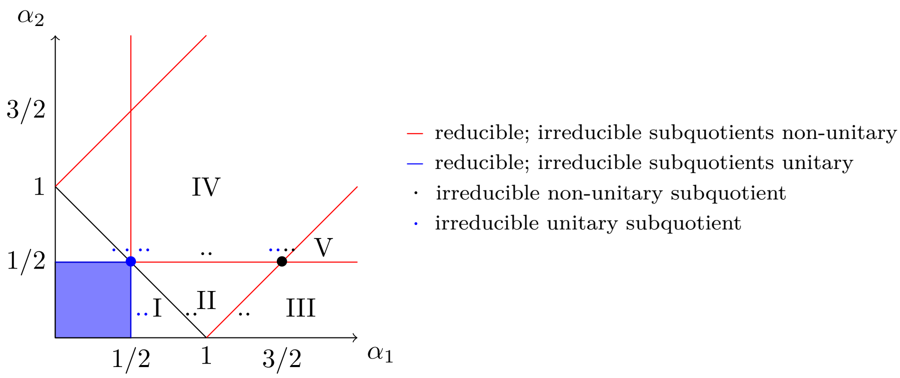

Figure 2

Figure 7.2. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Figure 2 shows lines and points of reducibility of the representation $|\ |^{\alpha_1} \chi_{\omega_{E/F}} \times |\ |^{\alpha_2} \chi_{\omega_{E/F}} \rtimes \lambda'$ and the unitary dual. Let $0 < \alpha_2 \leqslant \alpha_1$. $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{\omega_{E/F}}; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$ is unitary for $0 < \alpha_2 \leqslant \alpha_1 \leqslant 1/2$. Except for $1/2 < \alpha_1 < 1$, $0 < \alpha_2 \leqslant 1 - \alpha_1$ and for $\alpha_1 = 3/2$, $\alpha_2 = 1/2$, it is non-unitary. $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ is unitary for $0 < \alpha \leqslant 1/2$. For $\alpha > 1$ it is non-unitary.

7.1.5. $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{1_{F^*}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$, $0 < \alpha_2 \leqslant \alpha_1$.

Theorem 7.8. — *Let* $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

(1) *Let* $0 < \alpha_2 \leqslant \alpha_1$, $\alpha_1 + \alpha_2 > 1$. *Then* $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{1_{F^*}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$ *is non-unitary.*

(2) *Let* $0 < \alpha_2 \leqslant \alpha_1$, $\alpha_1 + \alpha_2 = 1$. *Then* $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{1_{F^*}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$ *is unitary.*

Proof. — The proof is similar to the proof of Theorem 7.4. See Figure 7.3 on page 133. □

Remark 7.9. — Let $0 < \alpha_2 \leqslant \alpha_1$, $\alpha_1 + \alpha_2 < 1$. We do not have a proof that $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{1_{F^*}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$ is non-unitary.

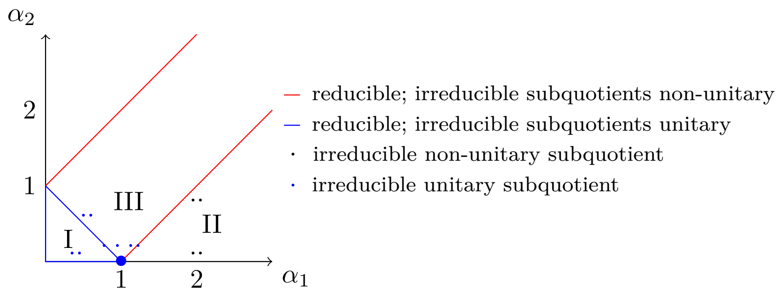

Figure 3

FIGURE 7.3. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi 1_{F^*} \in X_{1_{F^*}}$. Figure 3 shows lines and points of reducibility of the representation $|\ |^{\alpha_1} \chi 1_{F^*} \times |\ |^{\alpha_2} \chi 1_{F^*} \rtimes \lambda'$ and the unitary dual. Let $0 < \alpha_2 \leqslant \alpha_1$. Then $\mathrm{Lg}(|\ |^{\alpha_1} \chi 1_{F^*}; |\ |^{\alpha_2} \chi 1_{F^*}; \lambda')$ is unitary for $\alpha_1 + \alpha_2 = 1$. It is non-unitary for $\alpha_1 + \alpha_2 > 1$.

In the following Theorems 7.10, 7.11 and 7.12, when speaking of the Langlands quotient, we will exceptionally allow that $\alpha_1 < \alpha_2$ for ease of notation.

7.1.6. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$, $\alpha_1, \alpha_2 > 0$, $\mathrm{Lg}(|\ |^{\alpha} 1; \chi_{\omega_{E/F}} \rtimes \lambda')$, $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; 1 \rtimes \lambda')$, $\alpha > 0$, $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ denote the unique square-integrable subquotient and let $\pi_{2,\chi_{\omega_{E/F}}}$ denote the unique non-tempered subquotient of $|\ |^{1/2} \chi_{\omega_{E/F}} \rtimes \lambda'$.

THEOREM 7.10. — *Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$.*

(1) *Let $0 < \alpha_1 \leqslant 1$, $0 < \alpha_2 \leqslant 1/2$. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$ is unitary.*
(2) *Let $\alpha_1 > 1$, $\alpha_2 > 0$, or let $0 < \alpha_1 \leqslant 1$, $\alpha_2 > 1/2$. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$ is non-unitary.*
(3) *Let $0 < \alpha \leqslant 1$. $\mathrm{Lg}(|\ |^{\alpha} 1; \chi_{\omega_{E/F}} \rtimes \lambda')$ is unitary.*
(4) *Let $\alpha > 1$. $\mathrm{Lg}(|\ |^{\alpha} 1; \chi_{\omega_{E/F}} \rtimes \lambda')$ is non-unitary.*
(5) *Let $\alpha \leqslant 1/2$. $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; 1 \rtimes \lambda')$ is unitary.*
(6) *Let $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; 1 \rtimes \lambda')$ is non-unitary.*

Proof. — The proof is similar to the proof of Theorem 7.4. See Figure 7.4 on page 134. □

7.1.7. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi 1_{F^*}; \lambda')$, $\alpha_1, \alpha_2 > 0$, $\mathrm{Lg}(|\ |^{\alpha} \chi 1_{F^*}; 1 \rtimes \lambda')$, $\alpha > 0$, $\chi 1_{F^*} \in X_{1_{F^*}}$. Let $\chi 1_{F^*} \in X_{1_{F^*}}$,

THEOREM 7.11. — (1) *Let $\alpha_1, \alpha_2 > 0$. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi 1_{F^*}; \lambda')$ is non-unitary.*
(2) *Let $\alpha > 0$. $\mathrm{Lg}(|\ |^{\alpha} \chi 1_{F^*}; 1 \rtimes \lambda')$ is non-unitary.*

Proof. — The proof is similar to the proof of Theorem 7.4. See Figure 7.5 on page 134. □

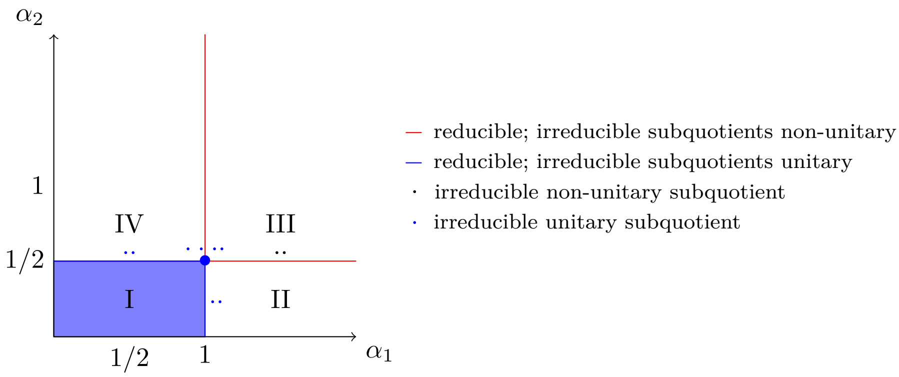

Figure 4

FIGURE 7.4. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Figure 4 shows lines and points of reducibility of the representation $|\ |^{\alpha_1} 1 \times |\ |^{\alpha_2} \chi_{\omega_{E/F}} \rtimes \lambda'$ and the unitary dual. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi_{\omega_{E/F}}; \lambda')$ is unitary for $0 < \alpha_1 \leqslant 1$, $0 < \alpha_2 \leqslant 1/2$. Otherwise it is non-unitary. $\mathrm{Lg}(|\ |^{\alpha} 1; \chi_{\omega_{E/F}} \rtimes \lambda')$ is unitary for $0 < \alpha \leqslant 1$, $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F}}; 1 \rtimes \lambda')$ is unitary for $0 < \alpha \leqslant 1/2$. Otherwise these Langlands-quotients are non-unitary.

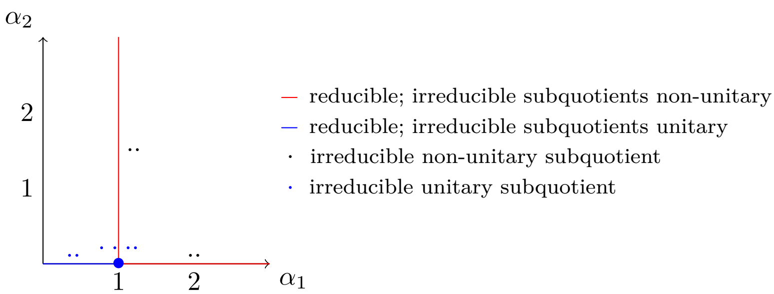

Figure 5

FIGURE 7.5. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Figure 5 shows lines and points of reducibility of the representation $|\ |^{\alpha_1} 1 \times |\ |^{\alpha_2} \chi_{1_{F^*}} \rtimes \lambda'$ and the unitary dual. $\mathrm{Lg}(|\ |^{\alpha_1} 1; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$ is non-unitary for all $\alpha_1, \alpha_2 > 0$. $\mathrm{Lg}(|\ |^{\alpha} \chi_{1_{F^*}}; 1 \rtimes \lambda')$ is non-unitary for all $\alpha > 0$.

7.1.8. $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{\omega_{E/F}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$, $\alpha_1, \alpha_2 > 0$, $\mathrm{Lg}(|\ |^{\alpha} \chi_{1_{F^*}}; \chi_{\omega_{E/F}} \rtimes \lambda')$, $\alpha > 0$, $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$, $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$, let $\chi_{1_{F^*}} \in X_{1_{F^*}}$.

THEOREM 7.12. — (1) *Let $\alpha_1, \alpha_2 > 0$. $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{\omega_{E/F}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$ is non-unitary.*

(2) *Let $\alpha > 0$. $\mathrm{Lg}(|\ |^{\alpha} \chi_{1_{F^*}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ is non-unitary.*

Proof. — The proof is similar to the proof of Theorem 7.4. See Figure 7.6 on page 135. □

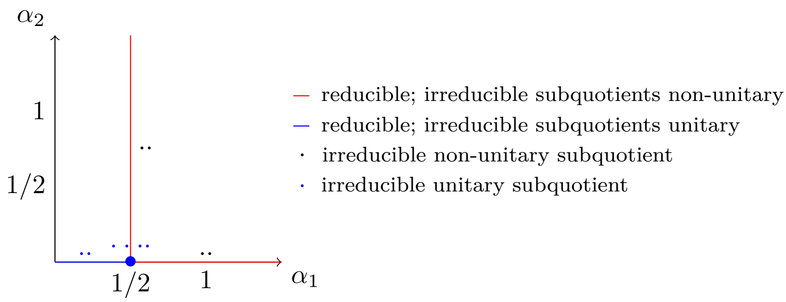

Figure 6

FIGURE 7.6. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}, \chi_{1_{F^*}} \in X_{1_{F^*}}$. Figure 6 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi_{1_{F^*}} \rtimes \lambda'$ and the unitary dual. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F}}; |\ |^{\alpha_2} \chi_{1_{F^*}}; \lambda')$ is non-unitary for all $\alpha_1, \alpha_2 > 0$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}; \chi_{\omega_{E/F}} \rtimes \lambda')$ is non-unitary for all $\alpha > 0$.

7.1.9. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F},1}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F},2}; \lambda')$, $0 < \alpha_2 \leqslant \alpha_1$, $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F},1}; \chi_{\omega_{E/F},2} \rtimes \lambda')$, $\alpha > 0$, $\chi_{\omega_{E/F},1}, \chi_{\omega_{E/F},2} \in X_{\omega_{E/F}}$, $\chi_{\omega_{E/F},1} \ncong \chi_{\omega_{E/F},2}$.

THEOREM 7.13. — *Let $\chi_{\omega_{E/F},1}, \chi_{\omega_{E/F},2} \in X_{\omega_{E/F}}$, such that $\chi_{\omega_{E/F},1} \ncong \chi_{\omega_{E/F},2}$.*

1. *Let $0 < \alpha_2 \leqslant \alpha_1 \leqslant 1/2$. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F},1}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F},2}; \lambda')$ is unitary.*
2. *Let $\alpha_1 > 1/2$, $0 < \alpha_2 \leqslant \alpha_1$. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{\omega_{E/F},1}; |\ |^{\alpha_2}\ \chi_{\omega_{E/F},2} \rtimes \lambda')$ is non-unitary.*
3. *Let $0 < \alpha \leqslant 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F},1}; \chi_{\omega_{E/F},2} \rtimes \lambda')$ is unitary.*
4. *Let $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F},1}; \chi_{\omega_{E/F},2} \rtimes \lambda')$ is non-unitary.*

Proof. — The proof is similar to the proof of Theorem 7.4. See Figure 7.7 on page 136. □

7.1.10. $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{1_{F^*},1}; |\ |^{\alpha_2}\ \chi_{1_{F^*},2}; \lambda')$, $0 < \alpha_2 \leqslant \alpha_1$. Let $\chi_{1_{F^*},1}, \chi_{1_{F^*},2} \in X_{1_{F^*}}$, such that $\chi_{1_{F^*}} \ncong \chi_{2_{F^*}}$.

THEOREM 7.14. — *Let $\alpha_1, \alpha_2 > 0$. Then $\mathrm{Lg}(|\ |^{\alpha_1}\ \chi_{1_{F^*},1}; |\ |^{\alpha_2}\ \chi_{1_{F^*},2}; \lambda')$ is non-unitary.*

Proof. — The proof is similar to the proof of Theorem 7.4. See Figure 7.8 on page 136. □

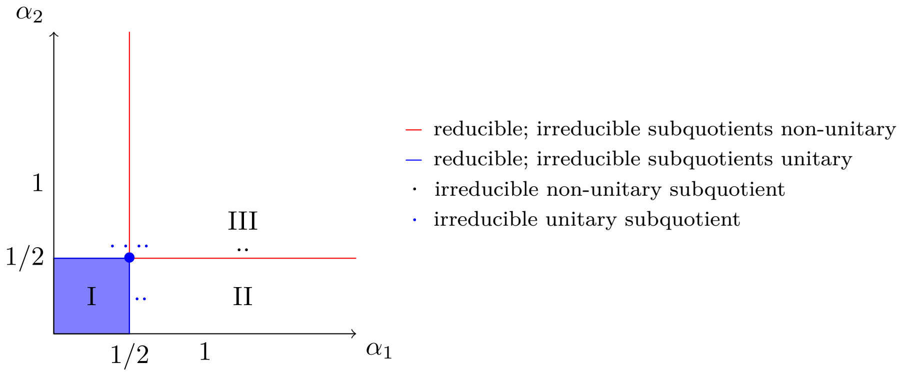

Figure 7

Figure 7.7. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F},1}, \chi_{\omega_{E/F},2} \in X_{\omega_{E/F}}$, be such that $\chi_{\omega_{E/F},1} \ncong \chi_{\omega_{E/F},2}$. Figure 7 shows lines and points of reducibility of the representation $|\ |^{\alpha_1} \chi_{\omega_{E/F},1} \times |\ |^{\alpha_2} \chi_{\omega_{E/F},2} \rtimes \lambda'$ and the unitary dual. $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{\omega_{E/F},1}; |\ |^{\alpha_2} \chi_{\omega_{E/F},2}; \lambda')$ is unitary for $0 < \alpha_2 \leqslant \alpha_1 \leqslant 1/2$. Otherwise it is non-unitary. $\mathrm{Lg}(|\ |^{\alpha} \chi_{\omega_{E/F},1}; \chi_{\omega_{E/F},2} \rtimes \lambda')$ is unitary for $0 < \alpha \leqslant 1/2$. Otherwise it is non-unitary.

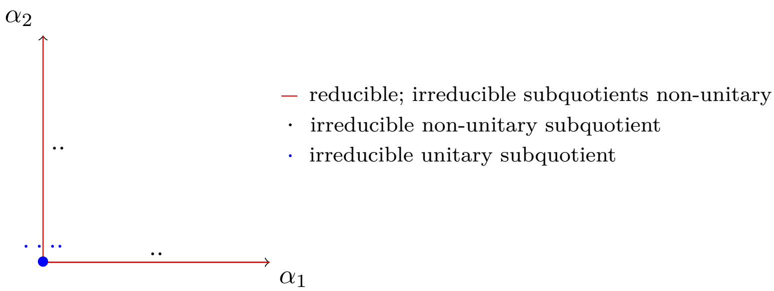

Figure 8

Figure 7.8. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{1_{F^*},1}, \chi_{1_{F^*},2} \in X_{1_{F^*}}$ be such that $\chi_{1_{F^*},1} \ncong \chi_{1_{F^*},2}$. Figure 8 shows lines and points of reducibility of the representation $|\ |^{\alpha_1} \chi_{1_{F^*},1} \times |\ |^{\alpha_2} \chi_{1_{F^*},2} \rtimes \lambda'$ and the unitary dual. $\mathrm{Lg}(|\ |^{\alpha_1} \chi_{1_{F^*},1}; |\ |^{\alpha_2} \chi_{1_{F^*},2}; \lambda')$ is non-unitary for $0 < \alpha_2 \leqslant \alpha_1$.

7.2. Representations induced from M_1, with cuspidal support in M_0, not fully-induced.

7.2.1. *Irreducible subquotients of* $\chi \, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$. Let χ be a unitary character of E^*.

As $\chi \, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is tempered unitary, all irreducible subquotients of $\chi \, \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ are tempered and unitary.

Remark 7.15. — By Proposition 5.6 $\chi\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is reducible if and only if $\chi =: \chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. $\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ has two tempered subquotients.

7.2.2. $\mathrm{Lg}(|\ |^{\alpha}\,\chi\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$, $\alpha > 0$.

Theorem 7.16. — *Let χ be a unitary character of E^* such that $\chi \notin X_{N_{E/F}(E^*)}$. $\mathrm{Lg}(|\ |^{\alpha}\,\chi\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary for all $\alpha > 0$.*

Proof. — If $\chi \notin X_{N_{E/F}(E^*)}$, the representations $\mathrm{Lg}(|\ |^{\alpha}\,\chi\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ are not hermitian. □

Theorem 7.17. — (1) *Let $0 < \alpha \leqslant 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is unitary.*
(2) *Let $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary.*

Proof. —

(1) Let $0 < \alpha < 1/2$. The representations $|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible by Theorem 5.5. They form a continuous one-parameter family of irreducible hermitian representations, that, similar as in Theorem 7.4, we realize on the same vector space V. $\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible by Proposition 5.6 and tempered, hence unitary. By Remark 3.1 the representations $|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda' = \mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ are unitary for $0 < \alpha < 1/2$. By Theorem 5.5 $|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is reducible. By [14] $\mathrm{Lg}(|\ |^{1/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is unitary.
(2) Let $1/2 < \alpha < 3/2$. The representations $|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ are irreducible by Theorem 5.5 and equal to their Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$. They form a continuous 1-parameter family of irreducible hermitian representations, that we realize on the same vector space V. For $\alpha = 3/2$, $\mathrm{Lg}(|\ |^{3/2}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is a subquotient of the representation $|\ |^{2}\,1 \times |\ |^{1}\,1 \rtimes \lambda'$ (Theorem 6.2). By results of Casselmann [2], page 915, it is non-unitary. By [14] and Remark 3.1 $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is not unitary for $1/2 < \alpha < 3/2$.
The representations $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$, $\alpha > 3/2$, form a continuous 1-parameter family of irreducible hermitian representations. If there existed $\alpha > 3/2$ such that $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ was unitary, then by Remark 3.1 $\mathrm{Lg}(|\ |^{\alpha}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ would be unitary for all $\alpha > 3/2$, in contradiction to Lemma 3.3. (Figure 7.9, page 138). □

Theorem 7.18. — *Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary.*

Proof. — Let $1/2 < \alpha < 1$. $|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible by Theorem 5.5 and equal to its own Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$. For $\alpha = 1$, by Theorem 6.4 $|\ |\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is reducible, by Remark 4.7 in [7], $\mathrm{Lg}(|\ |\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary. By [14] and Remark 3.1 $\mathrm{Lg}(|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ is not unitary for $1/2 < \alpha < 1$.

Let $\alpha > 1$. $|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible by Theorem 5.5 and equal to its own Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$. If there existed $\alpha > 1$ such that $\mathrm{Lg}(|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ was unitary, then by Remark 3.1 $\mathrm{Lg}(|\ |^{\alpha}\,\chi_{\omega_{E/F}}\,\mathrm{St}_{\mathrm{GL}_2};\lambda')$ would be unitary for all $\alpha > 1$, in contradiction to Lemma 3.3. (Figure 7.10, page 138) □

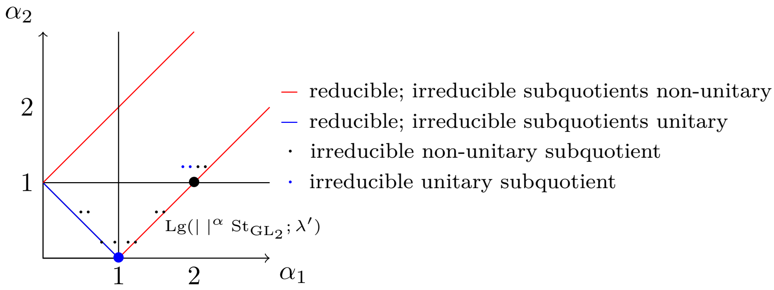

Figure 9

Figure 7.9. Let $\alpha_1, \alpha_2 \geqslant 0$. Figure 9 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ 1 \times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$. For $0 < \alpha \leqslant 1/2$, $\mathrm{Lg}(|\ |^{\alpha}\ \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda')$ is unitary, for $\alpha > 1/2$ it is non-unitary.

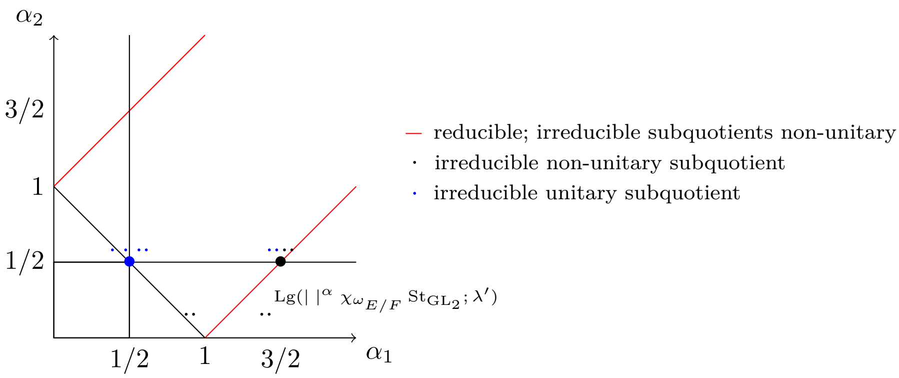

Figure 10

Figure 7.10. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Figure 10 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2} \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary.

Remark 7.19. — Let $0 < \alpha < 1/2$. Then $|\ |^{\alpha}\ \chi_{\omega_{E/F}}\ \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible by Theorem 5.5 and equal to its own Langlands quotient $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$. If we assume that $|\ |^{1/2}\ \chi_{\omega_{E/F}}\ \mathrm{St}_{\mathrm{GL}_2} \rtimes \lambda'$ is irreducible and equal to $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$, see Remark 6.5, then we can extend the argument that $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary to $\alpha > 0$.

Theorem 7.20. — *Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$.*
Let $0 < \alpha \leqslant 1/2$. Then $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$ is unitary.
Let $\alpha > 1/2$. Then $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$ is non-unitary.

Proof. — The proof is similar to the proof of Theorem 7.17. See Figure 7.11 on page 139. □

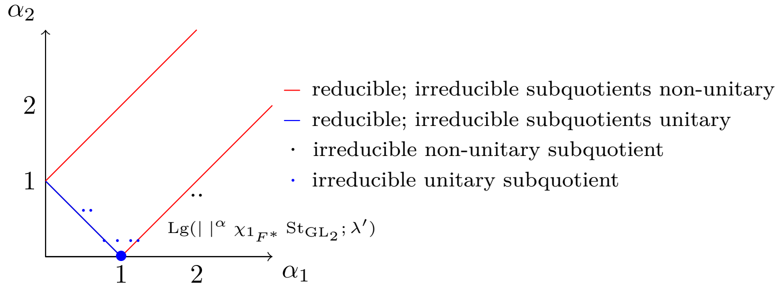

Figure 11

FIGURE 7.11. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Figure 11 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ \chi_{1_{F^*}} \times |\ |^{\alpha_2}\chi_{1_{F^*}} \rtimes \lambda'$. For $0 < \alpha \leqslant 1/2$, $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}\ \mathrm{St}_{\mathrm{GL}_2};\lambda')$ is unitary, for $\alpha > 1/2$ it is non-unitary.

7.3. **Representations induced from M_2, with cuspidal support in M_0, not fully-induced.**

7.3.1. *Irreducible subquotients of $\chi \rtimes \tau$, τ tempered non-cuspidal of $U(3)$, not fully-induced.* Recall that $\lambda'(\det)\,\mathrm{St}_{U(3)}$ is the unique square-integrable subquotient of $|\ |\ 1 \rtimes \lambda'$ [10]. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ denote the unique square-integrable irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. We have $\chi_{1_{F^*}} \rtimes \lambda' = \sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$, where $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are irreducible tempered [10].

$\lambda'(\det)\,\mathrm{St}_{U(3)}$, $\pi_{1,\chi_{\omega_{E/F}}}$, $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are all non-cuspidal tempered representations of $U(3)$ that are not fully induced [10].

Let χ be a unitary character of E^*. The representations $\chi \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $\chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $\chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are tempered, hence unitary. Hence all their irreducible subquotients are tempered, hence unitary.

Remark 7.21. — By Proposition 5.8 $\chi \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ is reducible if and only if $\chi = 1$ or $\chi \in X_{1_{F^*}}$. By Proposition 5.10 $\chi \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ is reducible if and only if $\chi \in X_{1_{F^*}}$. By Theorem 5.1 $\chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are reducible if and only if $\chi \in X_{1_{F^*}}$ but $\chi \ncong \chi_{1_{F^*}}$.

Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. $1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $\chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $\chi_{1_{F^*}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $\chi \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi \rtimes \sigma_{2,\chi_{1_{F^*}}}$, where $\chi \in X_{1_{F^*}}$ but $\chi \ncong \chi_{1_{F^*}}$, have two tempered subquotients (Propositions 5.8, 5.10 and Theorem 5.1).

7.3.2. $\mathrm{Lg}(|\ |^{\alpha}\ \chi;\tau)$, *$\alpha > 0$, τ tempered non-cuspidal of $U(3)$, not fully-induced.* Recall that $\lambda'(\det)\,\mathrm{St}_{U(3)}$ is the unique square-integrable subquotient of $|\ |\ 1 \rtimes \lambda'$ [10]. Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ denote the unique square-integrable irreducible subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. We have $\chi_{1_{F^*}} \rtimes \lambda' = \sigma_{1,\chi_{1_{F^*}}} \oplus \sigma_{2,\chi_{1_{F^*}}}$, where $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are irreducible tempered [10].

$\lambda'(\det)\,\mathrm{St}_{U(3)}$, $\pi_{1,\chi_{\omega_{E/F}}}$, $\sigma_{1,\chi_{1_{F^*}}}$ and $\sigma_{2,\chi_{1_{F^*}}}$ are all non-cuspidal tempered representations of $U(3)$ that are not fully induced [10].

Theorem 7.22. — *Let $\chi \notin X_{N_{E/F}}(E^*)$. Let $\alpha > 0$.* $\mathrm{Lg}(|\ |^\alpha \chi; \lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha \chi, \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha \chi; \sigma_{1,\chi_{1_{F^*}}})$ *and* $\mathrm{Lg}(|\ |^\alpha \chi; \sigma_{2,\chi_{1_{F^*}}})$ *are non-unitary.*

Proof. — The representations $\mathrm{Lg}(|\ |^\alpha \chi\ ;\ \lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha \chi\ ;\ \pi_{1,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha \chi; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha \chi; \sigma_{2,\chi_{1_{F^*}}})$ are not hermitian, hence not unitary. $\square$

Let $\chi \in X_{N_{E/F}}(E^*) = 1 \cup X_{\omega_{E/F}} \cup X_{1_{F^*}}$.

Let $\chi_{\omega_{E/F}}$, $\chi'_{\omega_{E/F}} \in X_{\omega_{E/F}}$, such that $\chi'_{\omega_{E/F}} \neq \chi_{\omega_{E/F}}$, and let $\chi_{1_{F^*}}$, $\chi'_{1_{F^*}} \in X_{1_{F^*}}$, such that $\chi'_{1_{F^*}} \neq \chi_{1_{F^*}}$.

Theorem 7.23. — (1) *Let $\alpha > 1$.* $\mathrm{Lg}(|\ |^\alpha 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ *is non-unitary.*

(2) *Let $0 < \alpha \leqslant 1$.* $\mathrm{Lg}(|\ |^\alpha 1; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha 1; \sigma_{1,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha 1; \sigma_{2,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}})$ *and* $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$ *are unitary.*

(3) *Let $\alpha > 1$.* $\mathrm{Lg}(|\ |^\alpha 1; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha 1; \sigma_{1,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha 1; \sigma_{2,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}})$ *and* $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$ *are non-unitary.*

(4) *Let $0 < \alpha \leqslant 1/2$.* $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \sigma_{1,\chi_{1_{F^*}}})$ *and* $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \sigma_{2,\chi_{1_{F^*}}})$ *are unitary.*

(5) *Let $\alpha > 1/2$. Then* $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha \chi'_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \sigma_{1,\chi_{1_{F^*}}})$ *and* $\mathrm{Lg}(|\ |^\alpha \chi_{\omega_{E/F}}; \sigma_{2,\chi_{1_{F^*}}})$ *are non-unitary.*

(6) *Let $\alpha > 0$.* $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha \chi'_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}})$, *and* $\mathrm{Lg}(|\ |^\alpha \chi'_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$ *are non-unitary.*

Proof. —

(1) Let $1 < \alpha < 2$. By Theorem 5.7 $|\ |^\alpha 1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ is irreducible and equal to its Langlands quotient $\mathrm{Lg}(|\ |^\alpha 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$. By [2] the representation $|\ |^2 1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ is reducible. By the same author $\mathrm{Lg}(|\ |^2 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary. By [14] and Remark 3.1 $\mathrm{Lg}(|\ |^\alpha 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary for $1 < \alpha < 2$.

Let $\alpha > 2$. $|\ |^\alpha 1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$ is irreducible by Theorem 5.7 and equal to its Langlands quotient $\mathrm{Lg}(|\ |^\alpha 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$. By Remark 3.1 and Lemma 3.3, $\mathrm{Lg}(|\ |^\alpha 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary for $\alpha > 2$.

Let $0 < \alpha \leqslant 1$. We have no proof that $\mathrm{Lg}(|\ |^\alpha 1; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary. (Figure 7.12, page 142).

(2) The representations $1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ are irreducible by Proposition 5.10, representations $1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$, $1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$, $\chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible by Theorem 5.1. All representations are unitary. For $0 < \alpha < 1$, representations $|\ |^\alpha 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}$ are irreducible by Theorem 5.9, representations $|\ |^\alpha 1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$, $|\ |^\alpha 1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$, $|\ |^\alpha \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible by Theorem 5.11. The representations are equal to their own Langlands quotients $\mathrm{Lg}(|\ |^\alpha 1; \pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha 1; \sigma_{1,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha 1; \sigma_{2,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha \chi_{1_{F^*}}; \sigma_{2,\chi_{1_{F^*}}})$, respectively. By Remark 3.1 these Langlands quotients are unitary. For $\alpha = 1$, $|\ | 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ | 1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$, $|\ | 1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$, $|\ | \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ | \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ reduce for the first time (Theorems 5.9 and 5.11).

By [14] $\mathrm{Lg}(|\ |\ 1;\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |\ 1;\sigma_{1,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |\ 1;\sigma_{2,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{1_{F^*}};\sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha\ \chi_{1_{F^*}};\sigma_{2,\chi_{1_{F^*}}})$ are unitary.

(3) For $\alpha > 1$, representations $|\ |^\alpha\ 1 \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^\alpha\ 1 \rtimes \sigma_{1,\chi_{1_{F^*}}}$, $|\ |^\alpha\ 1 \rtimes \sigma_{2,\chi_{1_{F^*}}}$, $|\ |^\alpha\ \chi_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha\ \chi_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible (Theorems 5.9 and 5.11) and equal to their Langlands quotients $\mathrm{Lg}(|\ |^\alpha\ 1;\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha\ 1;\sigma_{1,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha\ 1;\sigma_{2,\chi_{1_{F^*}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{1_{F^*}};\sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha\ \chi_{1_{F^*}},\sigma_{2,\chi_{1_{F^*}}})$, respectively. By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary.

(4) Representations $\chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $\chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $\chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $\chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $\chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible (Propositions 5.8, 5.10 and Theorem 5.1) and unitary. For $0 < \alpha < 1/2$, representations $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^\alpha\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible (Theorems 5.7, 5.9 and 5.11) and equal to their Langlands quotients $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi'_{\omega_{E/F}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\sigma_{2,\chi_{1_{F^*}}})$, respectively. By Remark 3.1 these Langlands quotients are unitary. For $\alpha = 1/2$, $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^{1/2}\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ reduce for the first time (Theorems 5.7, 5.9 and 5.11). $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}};\lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^{1/2}\ \chi'_{\omega_{E/F}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}};\sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{1/2}\ \chi_{\omega_{E/F}};\sigma_{2,\chi_{1_{F^*}}})$ are unitary by [14].

(5) For $\alpha > 1/2$, representations $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^\alpha\ \chi'_{\omega_{E/F}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha\ \chi_{\omega_{E/F}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible (Theorems 5.7, 5.9 and 5.11) and equal to their Langlands quotients $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi'_{\omega_{E/F}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha\ \chi_{\omega_{E/F}};\sigma_{2,\chi_{1_{F^*}}})$, respectively. By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary.

(6) Let $\alpha > 0$. The representations $|\ |^\alpha\ \chi_{1_{F^*}} \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)}$, $|\ |^\alpha\ \chi_{1_{F^*}} \rtimes \pi_{1,\chi_{\omega_{E/F}}}$, $|\ |^\alpha\ \chi'_{1_{F^*}} \rtimes \sigma_{1,\chi_{1_{F^*}}}$ and $|\ |^\alpha\ \chi'_{1_{F^*}} \rtimes \sigma_{2,\chi_{1_{F^*}}}$ are irreducible (Theorems 5.7, 5.9 and 5.11) and equal to their Langlands quotients $\mathrm{Lg}(|\ |^\alpha\ \chi_{1_{F^*}};\lambda'(\det)\,\mathrm{St}_{U(3)})$, $\mathrm{Lg}(|\ |^\alpha\ \chi_{1_{F^*}};\pi_{1,\chi_{\omega_{E/F}}})$, $\mathrm{Lg}(|\ |^\alpha\ \chi'_{1_{F^*}};\sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^\alpha\ \chi'_{1_{F^*}};\sigma_{2,\chi_{1_{F^*}}})$. By Remark 3.1 and Lemma 3.3 these Langlands quotients are non-unitary. $\square$

7.4. **Representations with cuspidal support in M_1.** Recall $M_1 \cong \mathrm{GL}(2,E) \times E^1$.

7.4.1. $\mathrm{Lg}(|\ |^\alpha\ \pi;\lambda')$, *$\alpha > 0$, π a cuspidal unitary representation of* $\mathrm{GL}_2(E)$. Let π be a cuspidal unitary representation of $\mathrm{GL}_2(E)$. Let $\alpha > 0$. Assume it exists $g \in \mathrm{GL}(2,E)$ such that $\pi(g) \neq \pi((\overline{g}^t)^{-1})$. Then the induced representations $\pi \rtimes \lambda'$ and $|\ |^\alpha\ \pi \rtimes \lambda'$ are irreducible.$\pi \rtimes \lambda'$ is unitary, $|\ |^\alpha\ \pi \rtimes \lambda'$ is not hermitian for all $\alpha > 0$, hence not unitary.

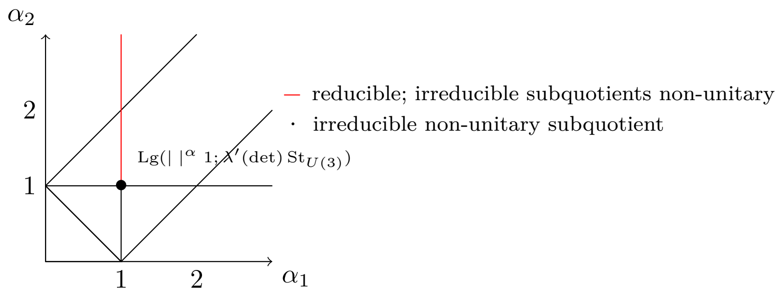

Figure 12

FIGURE 7.12. Let $\alpha_1, \alpha_2 \geqslant 0$. Figure 12 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ 1 \times |\ |^{\alpha_2}\ 1 \rtimes \lambda'$. For $\alpha > 1$, $\mathrm{Lg}(|\ |^{\alpha}\ 1 \rtimes \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary.

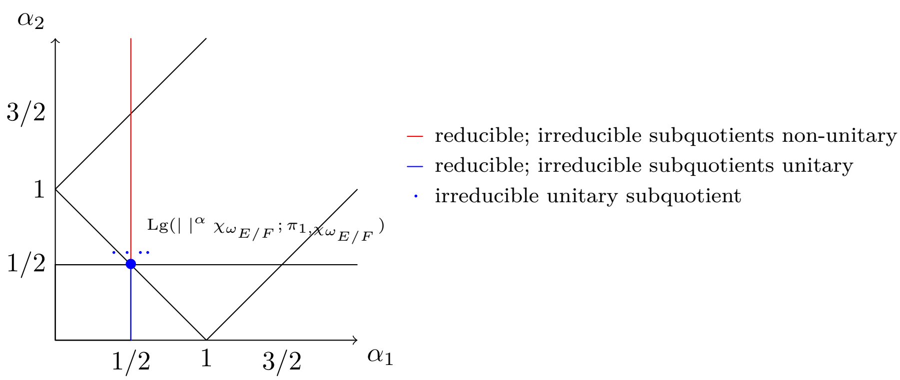

Figure 13

FIGURE 7.13. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Figure 13 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2} \chi_{\omega_{E/F}} \rtimes \lambda'$. Let $\pi_{1,\chi_{\omega_{E/F}}}$ be the unique square-integrable subquotient of $|\ |^{1/2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. Then $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \pi_{1,\chi_{\omega_{E/F}}})$ is unitary for $0 < \alpha \leqslant 1/2$. It is non-unitary for $\alpha > 1/2$.

Assume $\pi(g) = \pi((\overline{g}^t)^{-1})$ for all $g \in \mathrm{GL}_2(E)$. Then π is obtained by base change lift from $U(2)$ to $\mathrm{GL}(2, E)$, that is by endoscopic liftings from endoscopic data of $U(2)$ to data of $\mathrm{GL}_2(E)$ [16].

Let $\mathrm{G} := U(2)$ and $\widetilde{\mathrm{G}} = \mathrm{Res}_{E/F}\,\mathrm{G} = \mathrm{GL}(2, E)$.

Let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Let ${}^L\mathrm{G}$ be the L-group of G. Recall that σ is defined to be the non-trivial element of $\mathrm{Gal}(E, F)$. Let $\widetilde{\sigma}$ denote the F-automorphism of $\widetilde{\mathrm{G}}$ associated to σ by the F-structure of $\widetilde{\mathrm{G}}$. Let Γ denote the absolute Galois group of E, let W_F and W_E denote the Weil groups of F and E, respectively. Let ρ_{G}

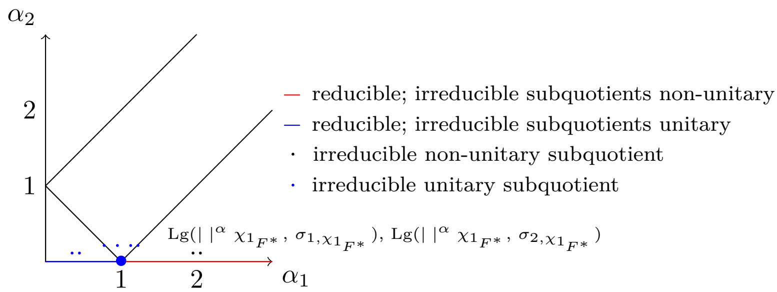

Figure 14

Figure 7.14. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Figure 14 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ \chi_{1_{F^*}} \times |\ |^{\alpha_2}\chi_{1_{F^*}} \rtimes \lambda'$. Let $\alpha > 0$. Then $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}, \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}, \sigma_{2,\chi_{1_{F^*}}})$ are unitary for $0 < \alpha \leqslant 1$ and non-unitary for $\alpha > 1$.

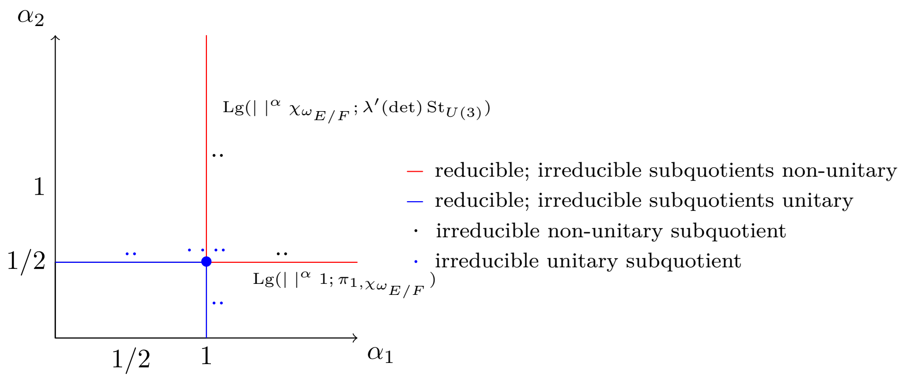

Figure 15

Figure 7.15. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$. Figure 15 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ 1\times |\ |^{\alpha_2}\ \chi_{\omega_{E/F}} \rtimes \lambda'$. $\mathrm{Lg}(|\ |^{\alpha}\ 1; \pi_{1,\chi_{\omega_{E/F}}})$ is unitary for $0 < \alpha \leqslant 1$. It is non-unitary for $\alpha > 1$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is unitary for $0 < \alpha \leqslant 1/2$. It is non-unitary for $\alpha > 1/2$.

denote an L-action of Γ on G and let $\rho_{\widetilde{\mathrm{G}}}$ denote an L-action of Γ on $\widetilde{\mathrm{G}}$. One fixes $\omega_\sigma \in W_F \backslash W_E$.

Lemma 7.24 ([16], 4.7). — *Up to isomorphism, the base change problem for $U(2)$ consists of the endoscopic liftings from endoscopic data $(\mathrm{G}, {}^L\mathrm{G}, 1, \xi)$ and*

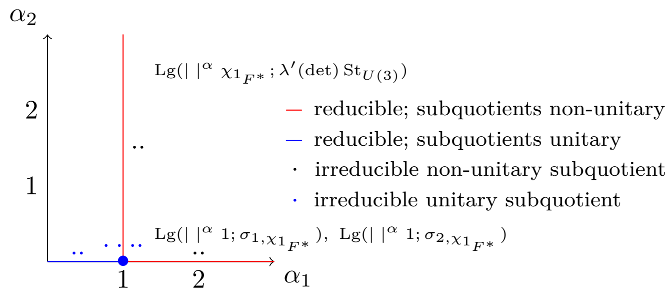

Figure 16

FIGURE 7.16. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Figure 16 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ 1 \times |\ |^{\alpha_2}\ \chi_{1_{F^*}} \rtimes \lambda'$. $\mathrm{Lg}(|\ |^{\alpha}\ 1; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{\alpha}\ 1; \sigma_{2,\chi_{1_{F^*}}})$ are unitary for $0 < \alpha \leqslant 1$ and non-unitary for $\alpha > 1$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}; \lambda'(\det)\,\mathrm{St}_{U(3)})$ is non-unitary $\forall \alpha > 0$.

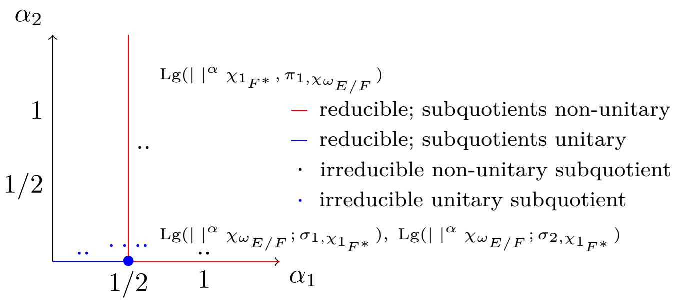

Figure 17

FIGURE 7.17. Let $\alpha_1, \alpha_2 \geqslant 0$, let $\chi_{\omega_{E/F}} \in X_{\omega_{E/F}}$ and $\chi_{1_{F^*}} \in X_{1_{F^*}}$. Figure 17 shows lines and points of reducibility of the representation $|\ |^{\alpha_1}\ \chi_{\omega_{E/F}} \times |\ |^{\alpha_2}\ \chi_{1_{F^*}} \rtimes \lambda'$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \sigma_{1,\chi_{1_{F^*}}})$ and $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{\omega_{E/F}}; \sigma_{2,\chi_{1_{F^*}}})$ are unitary for $0 < \alpha \leqslant 1/2$ and non-unitary for $\alpha > 1/2$. $\mathrm{Lg}(|\ |^{\alpha}\ \chi_{1_{F^*}}, \pi_{1,\chi_{\omega_{E/F}}})$ is non-unitary for all $\alpha > 0$.

$(\mathrm{G}, {}^L\mathrm{G}, 1, \xi_{\chi_{\omega_{E/F}}})$ *for* $(\widetilde{\mathrm{G}}, \widetilde{\sigma}, 1)$ *to* $\widetilde{\mathrm{G}}$. *Here*

$$\xi : {}^L\mathrm{G} \ni g \rtimes_{\rho_{\mathrm{G}}} \omega \mapsto (g, g) \rtimes_{\rho_{\widetilde{\mathrm{G}}}} \omega \in {}^L\widetilde{\mathrm{G}}$$

$$\xi_{\chi_{\omega_{E/F}}} : {}^L\mathrm{G} \ni g \rtimes_{\rho_{\mathrm{G}}} \omega \mapsto \begin{cases} (g\chi_{\omega_{E/F}}(\omega), g\chi_{\omega_{E/F}}(\omega)) \rtimes_{\rho_{\widetilde{\mathrm{G}}}} \omega \in {}^L\widetilde{\mathrm{G}} \text{ if } \omega \in W_E \\ (g, -g) \rtimes_{\rho_{\widetilde{\mathrm{G}}}} \omega_\sigma \in {}^L\widetilde{\mathrm{G}} \text{ if } \omega = \omega_\sigma \end{cases}.$$

ξ is called standard base change and $\xi_{\chi_{\omega_{E/F}}}$ is called twisted base change ([11]).

Let $\Pi_{temp}(\mathrm{G})$ be the set of equivalence classes of irreducible admissible tempered representations of G. Let $\Pi_{temp}(\widetilde{\mathrm{G}})$ be the set of equivalence classes of irreducible admissible tempered representations of $\widetilde{\mathrm{G}}$. Let Π be a tempered L-packet of G, then $\xi(\Pi), \xi_{\chi_{\omega_{E/F}}}(\Pi) \in \Pi_{temp}(\widetilde{\mathrm{G}})$.

As before let π be a cuspidal unitary representation of $\mathrm{GL}(2,E)$. If $\pi(g) = \pi((\overline{g}^t)^{-1})$ for all $g \in \mathrm{GL}(2,E)$, then $\pi = \xi_{\chi_{\omega_{E/F}}}(\Pi)$ or $\pi = \xi(\Pi)$ ([16, 4.2]).

Let π be a cuspidal unitary representation of $\mathrm{GL}_2(E)$.

(1) If $\pi = \xi_{\chi_{\omega_{E/F}}}(\Pi)$, then $\pi \rtimes \lambda'$ is reducible ([11, 4.2]; [4, 6.2]). $\pi \rtimes \lambda' = \tau_1(\pi) + \tau_2(\pi)$, where $\tau_1(\pi)$ and $\tau_2(\pi)$ are irreducible tempered.
   $| \, |^{\alpha} \, \pi \rtimes \lambda'$ is irreducible and never unitarisable for $\alpha > 0$ ([4, 6.3]).

(2) If $\pi = \xi(\Pi)$, then $\pi \rtimes \lambda'$ is irreducible ([11, 4.2]; [4, 6.2]).
   By results of Goldberg ([4, 6.3]) one has:
   (a) $| \, |^{\alpha} \, \pi \rtimes \lambda'$ is irreducible and unitarisable for $0 < \alpha < 1/2$.
   (b) $| \, |^{1/2} \, \pi \rtimes \lambda'$ is reducible. One has $| \, |^{1/2} \, \pi \rtimes \lambda' = \sigma + \mathrm{Lg}(| \, |^{1/2} \, \pi; \lambda')$, where σ is a generic, non-supercuspidal and square-integrable subrepresentation, and $\mathrm{Lg}(| \, |^{1/2} \, \pi; \lambda')$ is unitary.
   (c) $| \, |^{\alpha} \, \pi \rtimes \lambda'$ is irreducible and never unitarisable for $\alpha > 1/2$

We obtain the following.

Theorem 7.25. — *Let* $\pi = \xi_{\chi_{\omega_{E/F}}}(\Pi)$, *let* $\alpha > 0$. *Then*

- $\mathrm{Lg}(| \, |^{\alpha} \, \pi; \lambda')$ *is non-unitary.*
- $\pi \rtimes \lambda' = \tau_1(\pi) + \tau_2(\pi)$, *where* $\tau_1(\pi)$ *and* $\tau_2(\pi)$ *are irreducible tempered.*

Let $\pi = \xi(\Pi)$.

- *Let* $0 < \alpha < 1/2$. $\mathrm{Lg}(| \, |^{\alpha} \, \pi, \lambda')$ *is unitary.*
- *Let* $\alpha = 1/2$. $\mathrm{Lg}(| \, |^{1/2} \, \pi, \lambda')$ *is unitary.*
- *Let* $\alpha > 1/2$. $\mathrm{Lg}(| \, |^{\alpha} \, \pi, \lambda')$ *is non-unitary.*
- $\pi \rtimes \lambda'$ *is irreducible and unitary.*

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Manuscript received October 30, 2017,
revised June 29, 2020,
accepted July 6, 2020.

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