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A TWIST IN THE M_{24} MOONSHINE STORY

ANNE TAORMINA AND KATRIN WENDLAND

Abstract. Prompted by the Mathieu Moonshine observation, we identify a pair of 45-dimensional vector spaces of states that account for the first order term in the massive sector of the elliptic genus of K3 in every \mathbb{Z}_2 -orbifold CFT on K3. These generic states are uniquely characterized by the fact that the action of every geometric symmetry group of a \mathbb{Z}_2 -orbifold CFT yields a well-defined faithful representation on them. Moreover, each such representation is obtained by restriction of the 45-dimensional irreducible representation of the Mathieu group M_{24} constructed by Margolin. Thus we provide a piece of evidence for Mathieu Moonshine explicitly from SCFTs on K3.

The 45-dimensional irreducible representation of M_{24} exhibits a twist, which we prove can be undone in the case of \mathbb{Z}_2 -orbifold CFTs on K3 for all geometric symmetry groups. This twist however cannot be undone for the combined symmetry group $(\mathbb{Z}_2)^4 \rtimes A_8$ that emerges from surfing the moduli space of Kummer K3s. We conjecture that in general, the untwisted representations are exclusively those of geometric symmetry groups in some geometric interpretation of a CFT on K3. In that light, the twist appears as a representation theoretic manifestation of the maximality constraints in Mukai's classification of geometric symmetry groups of K3.

INTRODUCTION

The Mathieu Moonshine observation [8] continues to inspire three years on. It is now proven that the multiplicity spaces of irreducible characters of the $N = 4$ superconformal algebra in the elliptic genus of K3 do indeed correspond to representations of the sporadic group M_{24} [12]. The reason why M_{24} is singled out remains a mystery. From the properties of twining elliptic genera, one may expect a representation of M_{24} on a vertex algebra which governs the elliptic genus of K3, as argued in [16, 12, 17]. However, there are conceptual difficulties in following this lead, particularly in the sector of the elliptic genus corresponding to massless states at leading order.

In a recent paper [28], we suggest a starting point for the construction of a vertex algebra that governs the states occurring at lowest order in the elliptic genus. Our approach uses a subtle interplay between the geometry inherited from the K3 surfaces on which superstrings propagate and the (chiral, chiral) algebra associated with $N = (4, 4)$ superconformal algebras at central charge $c = \bar{c} = 6$ [22]. Since the elliptic genus is an invariant on the moduli space of such superconformal field theories (SCFTs), we are at liberty to choose the special class of \mathbb{Z}_2 -orbifold conformal field theories $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3 to present our arguments¹, which we summarize here.

The elliptic genus counts states (with signs) that appear in the Ramond-Ramond sector of the partition function, after projection onto Ramond ground states in the antiholomorphic sector of the theory. Although the expected vertex algebra $\widehat{\mathcal{X}}$ cannot arise in the Ramond-Ramond sector of such SCFTs, one may, by choosing appropriate holomorphic and antiholomorphic $U(1)$ -currents within the relevant $N = (4, 4)$ superconformal algebra, spectral flow the states into the Neveu-Schwarz sector of the theory where (prior to all projections and truncations) they yield a closed vertex algebra $\widehat{\mathcal{X}}$. The Ramond-Ramond ground states, in particular, flow to (chiral, chiral) states. In fact, the (chiral, chiral) algebra \mathcal{X} of [22], which accounts for the contributions to the lowest order terms of the elliptic genus, is obtained from

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¹Throughout our work, $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ refers to the standard \mathbb{Z}_2 -orbifold construction induced by the Kummer construction for K3 surfaces.

$\widehat{\mathcal{X}}$ by truncation. In any theory $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3, the vector space \mathcal{X} is generated by the 24 fields

$$\xi_1\xi_2\xi_3\xi_4, \quad \xi_i\xi_j \ (1 \leq i < j \leq 4), \quad \mathbb{1}; \quad \widetilde{T}_{\vec{a}} \ (\vec{a} \in \mathbb{F}_2^4), \quad (0.1)$$

where ξ_i are holomorphic-antiholomorphic combinations of the Dirac fermions in the theory, $\mathbb{1}$ denotes the vacuum field, and the $\widetilde{T}_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$, are the sixteen fields spectral-flowed from the RR twist fields.

Interestingly, the truncation of the OPE to the (chiral, chiral) algebra \mathcal{X} leaves the latter independent of all moduli. This may seem desirable, since the elliptic genus does not depend on the moduli. Mathieu Moonshine indicates that one should consider symmetries of some underlying vertex algebra, while it is far from clear from inspecting the fields in (0.1) which linear maps are symmetries of the whole theory $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$. In [28], we motivate why in our setting, we restrict our attention to symmetry groups that are induced geometrically in some geometric interpretation of \mathcal{C} stemming from \mathcal{T} . In other words, all such symmetry groups are subgroups of M_{24} by Mukai's seminal result [24]. Imposing that the superconformal algebra of \mathcal{C} be pointwise fixed also requires that a four-dimensional subspace of (0.1) is fixed under symmetries. Therefore, this condition rules out the possibility for \mathcal{X} to carry a representation of M_{24} , as also argued in [16] albeit from a different perspective. Hence a vertex algebra which governs the leading order terms of the elliptic genus and which at the same time carries the expected representation of M_{24} must be related to \mathcal{X} by some nontrivial map. The Niemeier markings and the overarching maps which were constructed in [27] should be viewed as a first approach towards constructing such a map.

This indicates interesting geometric avenues to explore while searching for a vertex algebra governing the leading order terms of the elliptic genus. In this paper, we take a closer look at the leading order in the *massive* sector of the elliptic genus. After all, the Mathieu Moonshine observation [8] originally refers to the massive sector. We use again the framework of \mathbb{Z}_2 -orbifold CFTs on K3, we analyze the massive states at leading order and show that they populate two complex 45-dimensional representation spaces of the respective symmetry groups, which we call V_{45}^{CFT} and \overline{V}_{45}^{CFT} . This has long been anticipated from mere Moonshine numerology: the massive character with the lowest conformal weight appears with the coefficient 90 in the elliptic genus, and M_{24} has two complex conjugate irreducible 45-dimensional representations. Nevertheless, it is remarkable that one obtains well-defined representations of the symmetry groups from the *net contributions* to the elliptic genus in such a natural fashion. We prove that the representations on V_{45}^{CFT} and \overline{V}_{45}^{CFT} can be induced from the two irreducible, complex conjugate 45-dimensional representations of M_{24} . By working within specific orbifold theories $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$, we gain much deeper insights into the nature of these massive states, and crucially, appreciate how the symmetries of interest act on them. In particular, we show that the space $V_{45}^{CFT} \oplus \overline{V}_{45}^{CFT}$ is uniquely characterized by the quantum numbers of states yielding leading order massive contributions to the elliptic genus together with the requirement that it carries a faithful representation of the *generic* geometric symmetry group $(\mathbb{Z}_2)^4$ of \mathbb{Z}_2 -orbifold CFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$.

An indispensable ingredient is the work of Margolin [23], who constructs an irreducible 45-dimensional representation of M_{24} on the 45-dimensional space V_{45} . There, an action of $(\mathbb{Z}_2)^4 \rtimes A_8 \subset M_{24}$ on V_{45} is exhibited as an important stepping stone in the construction of the full M_{24} action. This maximal subgroup of M_{24} is particularly relevant to us, as we have emphasized its role as combined symmetry group of all holomorphic symplectic automorphism groups of Kummer surfaces in [28]. There, we encrypt the action of $(\mathbb{Z}_2)^4 \rtimes A_8$ unequivocally on the Niemeier lattice with root lattice A_1^{24} , which carries a natural representation of M_{24} . In that setting, the very fact that the Niemeier lattice has definite signature while the full

integral homology lattice $H_*(X, \mathbb{Z})$ of K3 surfaces X has indefinite signature is a geometric obstruction against an action of $(\mathbb{Z}_2)^4 \rtimes A_8$ and a fortiori of M_{24} in terms of geometric symmetries on *any* K3 surface, a fact already appreciated by Mukai [24] and Kondo [21].

In the present work, we discover a new potential obstruction from representation theory to the action of M_{24} as a geometric symmetry group. Indeed, we prove that the group action on V_{45} simplifies when one restricts from the group $(\mathbb{Z}_2)^4 \rtimes A_8$ to the subgroups which happen to be symmetry groups of Kummer surfaces with dual Kähler class induced from the underlying torus. More precisely, we deduce from [23] that V_{45} possesses the structure of a tensor product $V_{45} = \mathcal{V} \otimes \mathcal{B}$, where the factorization however is not respected by the action of $(\mathbb{Z}_2)^4 \rtimes A_8$ on V_{45} . Here, \mathcal{V} is a 3-dimensional complex vector space, and the 15-dimensional space \mathcal{B} is referred to as the BASE of V_{45} . In fact, $(\mathbb{Z}_2)^4 \rtimes A_8$ permutes the 3-dimensional “fibers” $\mathcal{V} \otimes \text{span}_{\mathbb{C}}\{B\}$, $B \in \mathcal{B}$, introducing a TWIST on each fiber (see Def. A.1). Such a twist is necessary for the construction of the 45-dimensional representation of that group, since $(\mathbb{Z}_2)^4 \rtimes A_8$ does not possess any nontrivial 3-dimensional representations that \mathcal{V} could carry. The above-mentioned simplification under restriction to geometric symmetry groups amounts to the fact that these groups act without a twist, as we prove. On the CFT side this is expected from the very structure of the states in $V_{45}^{CFT} \oplus \overline{V}_{45}^{CFT}$, which forbids geometric symmetry group actions induced from \mathcal{T} to exhibit a twist.

Nevertheless, in view of our search for an explanation to Mathieu Moonshine, we interpret the representation space V_{45} as a medium which combines the actions of symmetry groups at distinct points of the moduli space of $N = (4, 4)$ SCFTs on K3 to representations of larger groups, similarly to the ideas we present in [27, 28]. Indeed, we generate the action of the entire group $(\mathbb{Z}_2)^4 \rtimes A_8$ on V_{45} by combining the actions of the maximal symmetry groups of Kummer K3s. Moreover, we prove that this action of $(\mathbb{Z}_2)^4 \rtimes A_8$ is obtained from Margolin’s irreducible representation of M_{24} by restriction to the maximal subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$. Thus we obtain a first piece of evidence for an action of M_{24} on a selection of states generic to all \mathbb{Z}_2 -orbifold CFTs on K3.

We start in Section 1 by a detailed account of the massive states described above. This exercise leads to Proposition 1.1, which provides the mathematical structure organising 90 twisted massive states into two 45-dimensional spaces. More precisely, we find $V_{45}^{CFT} = \mathbf{3} \otimes \mathbf{15}$ and $\overline{V}_{45}^{CFT} = \overline{\mathbf{3}} \otimes \mathbf{15}$, with $\mathbf{3}$ and $\overline{\mathbf{3}}$ being complex representation spaces of $SO(3)$, and $\mathbf{15}$ being a representation space of $\text{Aff}(\mathbb{F}_2^4)$ hosting twisted ground states. In Section 2, we focus on the action of the maximal subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$ of M_{24} on the base $\mathbf{15}$ of the space of states V_{45}^{CFT} . In Proposition 2.1 we show that the representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ on the space $\mathbf{15}$ of twisted ground states is equivalent to the representation of the same group constructed by Margolin [23] on the 15-dimensional base \mathcal{B} of $V_{45} = \mathcal{V} \otimes \mathcal{B}$. Section 3 analyzes the properties of the space V_{45}^{CFT} in order to substantiate the expectation that for every \mathbb{Z}_2 -orbifold CFT, this space carries a representation of a geometric symmetry group $G \subset M_{24}$ which is induced from Margolin’s representation of M_{24} on V_{45} . A first step in proving this is to show that none of the symmetry groups of maximally symmetric Kummer surfaces acts with a twist in Margolin’s representation on V_{45} . This is the purpose of Propositions 3.1, 3.2 and 3.3. A second step is to prove that the representation of each of the three maximal symmetry groups of Kummer surfaces on V_{45}^{CFT} is equivalent to the representation of that same group viewed as a subgroup of $(\mathbb{Z}_2)^4 \rtimes A_8$ acting on V_{45} . This is done in Proposition 3.4. Our main result, Theorem 3.5, generalizes these findings to arbitrary symmetry groups of \mathbb{Z}_2 -orbifold CFTs which are induced by geometric symmetries of the underlying toroidal theories. We then briefly discuss the role and limitations of a lattice of rank 20 that accommodates the combined action of the three maximal symmetry

groups of Kummer surfaces in our efforts to understand the role of M_{24} on the CFT side. In the Appendix we collect the details of Margolin's construction that are relevant for our work.

1. COUNTING STATES IN \mathbb{Z}_2 -ORBIFOLD CFTs ON K3

In this section, we use the conformal field theoretic elliptic genus of K3 to determine a 45-dimensional vector space V_{45}^{CFT} of states, which exists in all \mathbb{Z}_2 -orbifold conformal field theories on K3 and which is expected to be related to a representation of the Mathieu group M_{24} by the Mathieu Moonshine phenomenon [8].

We use the notion of CFTs on K3 which can be found, for example, in [2, 25], along with many further relevant references to the topic. One may also consult the more recent publications [30, 15, 31]. In the present work, however, we solely address \mathbb{Z}_2 -orbifold CFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ with \mathcal{T} a toroidal SCFT at central charge $c = 6 = \bar{c}$ and \mathcal{C} the standard \mathbb{Z}_2 -orbifold of this theory, cf. footnote 1. This ensures the mathematical foundations of the present work: Recall that a definition of the underlying toroidal theories \mathcal{T} has been given in [19], generalizing Kac's lattice algebras [18] to non-holomorphic CFTs. For all cyclic groups G , thus including the case $G = \mathbb{Z}_2$ that is relevant to our work, orbifold techniques for toroidal theories have been put on a solid mathematical foundation in a series of papers by Fröhlich, Fuchs, Runkel and Schweigert, culminating in [11]. That $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ as above obeys all defining properties of a SCFT on K3, in particular that it enjoys $N = (4, 4)$ supersymmetry and that its conformal field theoretic elliptic genus agrees with the geometric elliptic genus of K3 surfaces, has been shown in [9], see also [2, 25, 29, 31].

1.1. Evidence for the $45 \oplus \overline{45}$ of M_{24} from the elliptic genus of K3. The conformal field theoretic elliptic genus of $N = (2, 2)$ superconformal field theories (SCFTs) [1, 33] yields an invariant on every connected component of the moduli space of $N = (2, 2)$ SCFTs. For SCFTs on K3, one has central charges $c = \bar{c} = 6$ and $N = (4, 4)$ supersymmetry, and the conformal field theoretic elliptic genus $Z_{K3}(\tau, z)$ may be defined for $\tau, z \in \mathbb{C}, \Im(\tau) > 0$ as

$$Z_{K3}(\tau, z) := \text{tr}_{\mathcal{H}^R} \left((-1)^F y^{J_0} q^{L_0 - \frac{1}{4}} \bar{q}^{\bar{L}_0 - \frac{1}{4}} \right), \quad q := e^{2\pi i \tau}, \quad y := e^{2\pi i z}.$$

Here, the trace is over the states belonging to the Ramond-Ramond sector \mathcal{H}^R of the theory, F is the fermion number operator, J_0 is the zero mode of a choice of $U(1)$ -current in the holomorphic $N = 4$ superconformal algebra, while L_0 (resp. \bar{L}_0) is the zero-mode of the holomorphic (resp. antiholomorphic) Virasoro field. In fact, the above definition implies that the K3 elliptic genus is obtained from the partition function of any $N = (4, 4)$ SCFT on K3 by

$$Z_{K3}(\tau, z) = Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z} = 0),$$

where $Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z})$ is the Ramond-Ramond ($\tilde{R}\tilde{R}$) partition function with fermion number insertion in both the holomorphic and antiholomorphic sectors. By standard cohomological arguments, the insertion of $\bar{z} = 0$ in the $\tilde{R}\tilde{R}$ partition function suppresses the dependence of the resulting function on $\bar{\tau}$. One therefore expects a decomposition of Z_{K3} in terms of $N = 4$ characters stemming from the holomorphic sector of the partition function. Such a decomposition was achieved in [9], where the $N = (4, 4)$ SCFTs chosen for calculation were Gepner models at $c = \bar{c} = 6$. The very form in which the Mathieu Moonshine phenomenon was observed appears in [26, 29]. Indeed, in terms of characters of irreducible representations of the $N = 4$ superconformal algebra one may write

$$Z_{K3}(\tau, z) = -2 \text{ch}_{1, \frac{1}{2}}^{\tilde{R}}(\tau, z) + 20 \text{ch}_{1, 0}^{\tilde{R}}(\tau, z) + e(\tau) \text{ch}^{\tilde{R}}(\tau, z). \quad (1.1)$$

Here, the $N = 4$ massless characters $\text{ch}_{1,0}^{\widetilde{R}}(\tau, z)$ and $\text{ch}_{1,\frac{1}{2}}^{\widetilde{R}}(\tau, z)$ may be obtained from the Ramond sector characters derived in [10] through the shift $z \mapsto z + \frac{1}{2}$, and with $h \in \mathbb{R}$, $h > 0$, the massive $N = 4$ characters in this sector are of the form

$$q^h \widetilde{\text{ch}}^{\widetilde{R}}(\tau, z) = q^{h-\frac{1}{8}} \frac{\vartheta_1^2(\tau, z)}{\eta^3(\tau)} = q^h (2-y-y^{-1}) (1 + q(1-2y-2y^{-1}) + \dots). \quad (1.2)$$

The function $e(\tau)$ in (1.1) is closely related to a weakly holomorphic mock modular form of weight $\frac{1}{2}$ on $SL(2, \mathbb{Z})$ [6], and its q -expansion starts with

$$e(\tau) = 90q + 462q^2 + \dots. \quad (1.3)$$

The root of the Mathieu Moonshine phenomenon lies in the coefficients of the series (1.3): it was observed in [8] that these coefficients appeared to be twice the dimensions of some representations of the sporadic Mathieu group M_{24} . A proof of this fact, along with its highly non-trivial generalizations to twining genera, was given recently in [12], which builds on the works [4, 14, 13, 7]. The field theoretic reason for M_{24} to act on the states in the massive sector that Z_{K3} accounts for has remained a mystery so far. In order to unveil some of this M_{24} Moonshine Mystery, it seems natural to track the states of one's favourite $N = (4, 4)$ SCFT on K3 and determine explicitly which ones contribute to the elliptic genus in the form of representations of M_{24} or its subgroups. In this work we shall do so for the leading order term of (1.3) in the case of any \mathbb{Z}_2 -orbifold conformal field theory on K3, which we denote $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$, where \mathcal{T} is the underlying toroidal CFT in four dimensions. The current section is devoted to determining a 45-dimensional space V_{45}^{CFT} of states which is generic to all such theories, such that $V_{45}^{CFT} \oplus \overline{V}_{45}^{CFT}$ accounts for the leading order coefficient 90 of $e(\tau)$ in (1.3).

1.2. Counting massive states in \mathbb{Z}_2 -orbifold conformal field theories. Every toroidal conformal field theory \mathcal{T} possesses two free Dirac fermions on the holomorphic side, which we denote by $\chi_+^1(z)$, $\chi_+^2(z)$. The fields of the complex conjugates are denoted $\chi_-^1(z)$, $\chi_-^2(z)$, such that

$$\chi_+^k(z) \chi_-^\ell(w) \sim \frac{\delta_{k\ell}}{z-w}, \quad k, \ell \in \{1, 2\}, \quad (1.4)$$

while the antiholomorphic counterparts are denoted $\overline{\chi}_\pm^1(\overline{z})$, $\overline{\chi}_\pm^2(\overline{z})$. The superpartners of the two Dirac fermions $\chi_+^1(z)$, $\chi_+^2(z)$ are given by

$$j_+^1(z) := \frac{1}{\sqrt{2}}(j^1(z) + i j^2(z)) \quad \text{and} \quad j_+^2(z) := \frac{1}{\sqrt{2}}(j^3(z) + i j^4(z)), \quad (1.5)$$

with $j^K(z)$, $K \in \{1, \dots, 4\}$, four real holomorphic $U(1)$ -currents. We remark that the introduction of the fields $\chi_\pm^k(z)$, $\overline{\chi}_\pm^k(\overline{z})$ and their superpartners amounts to a choice of basis for fields with appropriate quantum numbers, which is tantamount to a choice of geometric interpretation (see [25, 29] for extensive discussions of this issue). Indeed, the fields $j_\pm^k(z)$, $k \in \{1, 2\}$, are identified with the holomorphic coordinate vector fields $\frac{\partial}{\partial z_k}$, $k \in \{1, 2\}$, in such a geometric interpretation. As was argued in the introduction, in this work we are only interested in geometric symmetries of our respective CFTs. For this notion to make sense, the choice of a geometric interpretation is inevitable. As usual, in \mathcal{T} we have the mode expansions

$$j^K(z) = \sum_{n \in \mathbb{Z}} a_n^K z^{n-1}, \quad \chi_\pm^k(z) = \sum_{n \in \mathbb{Z}+r} (\chi_\pm^k)_n z^{n-1/2}, \quad (1.6)$$

where $r = \frac{1}{2}$ in the Neveu-Schwarz sector and $r = 0$ in the Ramond sector. The charges with respect to $(j^1, \dots, j^4; \overline{j}^1, \dots, \overline{j}^4)$ are denoted $p := (p_L; p_R) \in \Gamma \subset \mathbb{R}^{4,4}$, where Γ is a self-dual even integral lattice with signature $(4, 4)$.

The \mathbb{Z}_2 -orbifold action on the Dirac fermions is given by $\chi_\pm^k(z) \mapsto -\chi_\pm^k(z)$, and on the $U(1)$ -currents by $j^K(z) \mapsto -j^K(z)$. One may construct \mathbb{Z}_2 -invariant

generators of the $N = 4$ superconformal algebra from these free fields, namely the $U(1)$ -current

$$J^3 = \frac{1}{2}(:\chi_+^1\chi_-^1: + :\chi_+^2\chi_-^2:),$$

the energy-momentum tensor

$$T = :j_+^1j_-^1: + :j_+^2j_-^2: + \frac{1}{2}(:\partial\chi_+^1\chi_-^1: + :\partial\chi_-^1\chi_+^1: + :\partial\chi_+^2\chi_-^2: + :\partial\chi_-^2\chi_+^2:),$$

and the remaining $SU(2)$ -currents and $N = 4$ supercurrents

$$J^\pm = \pm:\chi_\pm^1\chi_\pm^2:, \quad G^\pm = \sqrt{2}(:\chi_\pm^1j_\mp^1: + :\chi_\pm^2j_\mp^2:), \quad G'^\pm = \sqrt{2}(:\chi_\mp^1j_\mp^2: - :\chi_\mp^2j_\mp^1:). \quad (1.7)$$

Similar expressions are obtained in the antiholomorphic sector. We denote by Q and h (resp. \bar{Q} and \bar{h}) the eigenvalues of J_0^3 and L_0 (resp. \bar{J}_0^3 and \bar{L}_0).

We label the four RR ground states that correspond to the two Dirac fermions by $\sigma_i^{\pm\pm}$, $i \in \{1, 2\}$. These states are odd under the \mathbb{Z}_2 -action. We summarize the RR ground state content of the untwisted sector of the \mathbb{Z}_2 -orbifold conformal field theory $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3 in Table 1.1. Note that each of the four charged RR ground

charged ground state	$(h, Q; \bar{h}, \bar{Q})$	uncharged ground state	$(h, Q; \bar{h}, \bar{Q})$
$\sigma_1^{++}\sigma_2^{++}$	$(\frac{1}{4}, 1; \frac{1}{4}, 1)$	$\sigma_1^{++}\sigma_2^{--}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$
$\sigma_1^{+-}\sigma_2^{+-}$	$(\frac{1}{4}, 1; \frac{1}{4}, -1)$	$\sigma_1^{--}\sigma_2^{++}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$
$\sigma_1^{-+}\sigma_2^{-+}$	$(\frac{1}{4}, -1; \frac{1}{4}, 1)$	$\sigma_1^{-+}\sigma_2^{+-}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$
$\sigma_1^{--}\sigma_2^{--}$	$(\frac{1}{4}, -1; \frac{1}{4}, -1)$	$\sigma_1^{+-}\sigma_2^{-+}$	$(\frac{1}{4}, 0; \frac{1}{4}, 0)$

Table 1.1: RR ground states in the untwisted sector of \mathcal{T}/\mathbb{Z}_2 .

states can be obtained from, say, $\sigma := \sigma_1^{--}\sigma_2^{--}$ by application of the zero-modes J_0^+ , \bar{J}_0^+ of the $SU(2)$ -currents listed in (1.7); these states comprise the ground states of the vacuum representation of the $N = (4, 4)$ superconformal algebra in the Ramond-Ramond sector. On the other hand, each of the uncharged RR ground states is the ground state of a massless matter representation.

Moreover, there is a 16-dimensional space of twisted ground states in the Ramond-Ramond sector with orthonormal basis $T_{\vec{a}}$, where each state has quantum numbers $(h, Q; \bar{h}, \bar{Q}) = (\frac{1}{4}, 0; \frac{1}{4}, 0)$. In any geometric interpretation of \mathcal{C} on a Kummer surface with underlying torus $T = \mathbb{R}^4/\Lambda$, the label $\vec{a} \in \mathbb{F}_2^4 \cong \frac{1}{2}\Lambda/\Lambda$ refers to the fixed point of \mathbb{Z}_2 at which the respective field is localized.

We now identify the states in our theory $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ which are expected to form a $\mathbf{45} \oplus \overline{\mathbf{45}}$ representation of M_{24} because they generically contribute to the leading coefficient 90 in (1.3). In the next two sections, we show how a maximal subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$ of M_{24} acts on these states, and we establish a link with the geometric picture we have developed in [27, 28].

The \widetilde{RR} partition function of \mathcal{T}/\mathbb{Z}_2 may be read off [9, 29] after suitable spectral flow. We write

$$Z^{\widetilde{R}} := Z_{\text{untwisted}}^{\widetilde{R}} + Z_{\text{twisted}}^{\widetilde{R}},$$

with

$$Z_{\text{untwisted}}^{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z}) = \frac{1}{2|\eta(\tau)|^8} \left(1 + \sum_{\substack{(p_L; p_R) \in \Gamma, \\ (p_L; p_R) \neq (0; 0)}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} \right) \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^4 + 8 \left| \frac{\vartheta_2(\tau, z)}{\vartheta_2(\tau)} \right|^4, \quad (1.8)$$

$$Z_{\text{twisted}}^{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z}) = 8 \left| \frac{\vartheta_3(\tau, z)}{\vartheta_3(\tau)} \right|^4 + 8 \left| \frac{\vartheta_4(\tau, z)}{\vartheta_4(\tau)} \right|^4, \quad (1.9)$$

with Γ the charge lattice. From (1.1) and (1.2) we deduce that the massive states which contribute to the leading order term of $e(\tau)$ have weights $(h; \bar{h}) = (\frac{5}{4}; \frac{1}{4})$. Moreover, it suffices to focus on the states with quantum numbers $(h, Q; \bar{h}, \bar{Q}) = (\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$.

The untwisted sector. We first identify all states with these quantum numbers which generically come from the untwisted sector (1.8). Since each factor $1/\eta(\tau)$ counts states created from the vacuum by the bosonic oscillators a_n^K , $n \in \mathbb{N}$, for fixed K , and since each factor $\vartheta_1(\tau, z)/\eta(\tau)$ counts bosonic and fermionic states (with signs) created from the vacuum by the fermionic modes $(\chi_{\pm}^k)_n$, $n \in \mathbb{N}$, for a fixed value of k , one can read from the expansion

$$\frac{1}{\eta^4(\tau)} \left(\frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right)^2 = -y^{-1}(1 - 2y + y^2)(1 + q(4 - 2y - 2y^{-1}) + \dots)$$

that the factor y^{-1} accounts for a Ramond ground state with $U(1)$ -charge $Q = -1$, the term $(-2y)$ in $(1 - 2y + y^2)$ accounts for the two fermionic zero modes $(\chi_{\pm}^k)_0$, $k \in \{1, 2\}$, while the term y^2 accounts for the bilinear $(\chi_{\pm}^1)_0(\chi_{\pm}^2)_0$. On the other hand, the term $4q$ in the factor $(1 + q(4 - 2y - 2y^{-1}) + \dots)$ accounts for the four bosonic oscillators a_1^K , while $(-2yq)$ and $(-2y^{-1}q)$ account for $(\chi_{\pm}^k)_1$ and $(\chi_{\pm}^k)_1$, $k \in \{1, 2\}$, respectively. The states with quantum numbers $(\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ in

$$\frac{1}{|\eta(\tau)|^8} \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^4$$

that are \mathbb{Z}_2 -invariant are thus encoded in the terms

$$(y^{-1}\bar{y}^{-1}) ((y^2)(4q)(-2\bar{y}) + (-2y)(-2qy)(1 + \bar{y}^2)). \quad (1.10)$$

Since the charge lattice Γ depends on the moduli of \mathcal{T} , and since the term

$$\left| 2 \frac{\vartheta_2(\tau, z)}{\vartheta_2(\tau)} \right|^4$$

in (1.9) implements the projection onto the \mathbb{Z}_2 -invariant states, (1.10) accounts for all those \mathbb{Z}_2 -invariant untwisted states with quantum numbers $(\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ which exist in every \mathbb{Z}_2 -orbifold conformal field theory $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$. Hence, the generic contribution from the untwisted sector of the theory to this class of states amounts to eight fermions and eight bosons. Indeed, the eight fermions are (with signs) given by²

$$(4q)(y^2)(-2\bar{y})(y^{-1}\bar{y}^{-1}): \quad a_1^K \chi_0^1 \chi_0^2 \bar{\chi}_0^\ell \sigma, \quad K \in \{1, \dots, 4\}, \ell \in \{1, 2\}, [\bar{Q} = 0], \quad (1.11)$$

and the eight bosons are given by

$$\begin{aligned} (2qy)(2y)(1)(y^{-1}\bar{y}^{-1}): \quad & \chi_1^k \chi_0^\ell \sigma \quad [\bar{Q} = -1], \\ (2qy)(2y)(\bar{y}^2)(y^{-1}\bar{y}^{-1}): \quad & \chi_1^k \chi_0^\ell \bar{\chi}_0^1 \bar{\chi}_0^2 \sigma \quad [\bar{Q} = 1], \end{aligned} \quad k, \ell \in \{1, 2\}, \quad (1.12)$$

where $\sigma := \sigma_1^- \sigma_2^-$ is the ground state with charges $(Q; \bar{Q}) = (-1; -1)$ from Table 1.1, such that the $\chi_0^k \bar{\chi}_0^\ell \sigma$ with $k, \ell \in \{1, 2\}$ yield the four uncharged states from Table 1.1, while $\bar{\chi}_0^1 \bar{\chi}_0^2 \sigma$ yields the one with charges $(Q; \bar{Q}) = (-1; 1)$. Actually,

²From now on, we set $\chi^\ell := \chi_+^\ell$ and $\bar{\chi}^\ell := \bar{\chi}_+^\ell$, $\ell \in \{1, 2\}$, as the fields χ_-^ℓ and $\bar{\chi}_-^\ell$ do not appear in the expressions of the states we are considering.

the eight fermionic states in (1.11) are massless, since they are the images of the massless matter states $\chi_0^k \bar{\chi}_0^\ell \sigma$, $k, \ell \in \{1, 2\}$, under the modes $G_1^+, G_1'^-$ of the $N = 4$ supercurrents listed in (1.7), respectively. Each set of four bosons in (1.12) consists of one massless boson $L_1 J_0^+ \sigma$ or $L_1 J_0^+ \bar{\chi}_0^1 \bar{\chi}_0^2 \sigma$, and three massive ones. The occurrence of six massive contributions in total can also readily be checked by rewriting the $\widetilde{R}\widetilde{R}$ partition function of the untwisted sector of $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ in terms of $N = 4$ characters. Indeed, using

$$\begin{aligned} \widetilde{\text{ch}}_{1, \frac{1}{2}}^{\widetilde{R}}(\tau, z) &= -(y + y^{-1}) + q(2 - y - y^{-1}) + \dots, & \text{vacuum, massless} \\ \widetilde{\text{ch}}_{1,0}^{\widetilde{R}}(\tau, z) &= 1 + q(2 - 2y - 2y^{-1} + y^2 + y^{-2}) + \dots, & \text{massless matter} \\ \widetilde{\text{ch}}^{\widetilde{R}}(\tau, z) &= 2 - y - y^{-1} + q(6 - 5y - 5y^{-1} + 2y^2 + 2y^{-2}) + \dots, & \text{massive} \end{aligned}$$

one obtains

$$\begin{aligned} Z_{\text{untwisted}}^{\widetilde{R}}(\tau, z; \bar{\tau}, \bar{z}) &= \widetilde{\text{ch}}_{1, \frac{1}{2}}^{\widetilde{R}}(\tau, z) \overline{\widetilde{\text{ch}}_{1, \frac{1}{2}}^{\widetilde{R}}(\bar{\tau}, \bar{z})} + 4 \widetilde{\text{ch}}_{1,0}^{\widetilde{R}}(\tau, z) \overline{\widetilde{\text{ch}}_{1,0}^{\widetilde{R}}(\bar{\tau}, \bar{z})} \\ &+ 3q \widetilde{\text{ch}}^{\widetilde{R}}(\tau, z) \overline{\widetilde{\text{ch}}^{\widetilde{R}}(\bar{\tau}, \bar{z})} + \dots, \end{aligned}$$

which reflects accurately the order q (and \bar{q}^0) contribution. In summary, the 16 generic Ramond-Ramond states with quantum numbers $(\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ in the untwisted sector of $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ are

- one massless boson with $\bar{Q} = 1$ and one massless boson with $\bar{Q} = -1$,
- eight massless fermions with $\bar{Q} = 0$,
- three massive bosons with $\bar{Q} = 1$ and three massive bosons with $\bar{Q} = -1$.

These massive bosons contribute a term $-6q$ to the function $e(\tau)$ in (1.3).

The twisted sector. On the other hand, the expansion of $Z_{\text{twisted}}^{\widetilde{R}}(\tau, z; \bar{\tau}, \bar{z})$ encodes the contribution to the states with quantum numbers $(\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ in the term $16(-8qy)$, where the factor 16 accounts for the number of twisted ground states $T_{\bar{a}}$, $\bar{a} \in \mathbb{F}_2^4$. By a similar analysis as above, the 128 states of interest are fermionic of the form³ $a_{\frac{1}{2}}^K \chi_{\frac{1}{2}}^\ell T_{\bar{a}}$, $K \in \{1, \dots, 4\}$, $\ell \in \{1, 2\}$, with $\bar{Q} = 0$. Since one can write

$$Z_{\text{twisted}}^{\widetilde{R}}(\tau, z; \bar{\tau}, \bar{z}) = 16 \left\{ \widetilde{\text{ch}}_{1,0}^{\widetilde{R}}(\tau, z) \overline{\widetilde{\text{ch}}_{1,0}^{\widetilde{R}}(\bar{\tau}, \bar{z})} + 6q \widetilde{\text{ch}}^{\widetilde{R}}(\tau, z) \overline{\widetilde{\text{ch}}^{\widetilde{R}}(\bar{\tau}, \bar{z})} + \dots \right\},$$

it follows that 32 of these states are massless, while the remaining 96 are massive, and contribute $+96q$ to (1.3). Since our goal is to study these massive states, it is imperative to determine which combinations of the $a_{\frac{1}{2}}^K \chi_{\frac{1}{2}}^\ell T_{\bar{a}}$ are massless. These are the 32 states which are created from the massless ground states $T_{\bar{a}}$, $\bar{a} \in \mathbb{F}_2^4$, by the modes $G_1^+, G_1'^-$ of the $N = 4$ supercurrents listed in (1.7). We therefore introduce the modes of the complex currents (1.5),

$$\begin{aligned} (j_+^1)_n &:= \frac{1}{\sqrt{2}}(a_n^1 + ia_n^2), & j_-^1 &:= (j_+^1)^*, \\ (j_+^2)_n &:= \frac{1}{\sqrt{2}}(a_n^3 + ia_n^4), & j_-^2 &:= (j_+^2)^*, \end{aligned}$$

and from (1.7) we find that the 32 massless states may be written as⁴

$$(\chi^1 j_-^1 + \chi^2 j_-^2) T_{\bar{a}}, \quad (\chi^1 j_+^2 - \chi^2 j_+^1) T_{\bar{a}}, \quad \bar{a} \in \mathbb{F}_2^4. \quad (1.13)$$

³Recall that the mode expansion (1.6) of the $U(1)$ -currents and free fermions in the twisted RR sector has modes a_n^K, χ_n^ℓ with $n \in \mathbb{Z} + \frac{1}{2}$.

⁴We suppress the modes for ease of reading.

The massive states are perpendicular to these massless ones with respect to the standard metric induced by the Zamolodchikov metric. Therefore we conveniently set

$$3 := \{\chi^1 j_+^2 + \chi^2 j_+^1, \chi^1 j_+^1, \chi^2 j_+^2\}, \quad \bar{3} := \{\chi^1 j_-^1 - \chi^2 j_-^2, \chi^1 j_-^2, \chi^2 j_-^1\}, \quad (1.14)$$

such that the 96-dimensional vector space of massive twisted states with quantum numbers $(\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ has the basis

$$\{WT_{\vec{a}} \mid W \in 3 \cup \bar{3}, \vec{a} \in \mathbb{F}_2^4\}. \quad (1.15)$$

Note that all these states are fermionic.

1.3. A generic space V_{45}^{CFT} in \mathbb{Z}_2 -orbifold CFTs on K3. In the previous subsection, we have determined all massive states with quantum numbers $(\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$ which exist generically in \mathbb{Z}_2 -orbifold conformal field theories on K3. Indeed, we have recovered a 6-dimensional space of untwisted bosonic states, along with a 96-dimensional space of twisted fermionic states, correctly accounting for a net contribution of $-90qy$ to the elliptic genus. As explained in Section 1.1, our interpretation of the Mathieu Moonshine observations predicts that a pair $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ of 45-dimensional representation spaces of the Mathieu group M_{24} should arise from these states. While the emergence of the group M_{24} remains mysterious, we expect to observe, on the space V_{45}^{CFT} , the representation of subgroups of M_{24} which occur as geometric symmetry groups of \mathbb{Z}_2 -orbifold limits of K3 surfaces, in accordance with ideas already promoted in our previous works [27, 28].

Indeed, we focus on symmetry groups of SCFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ which are induced by geometric symmetries of the underlying toroidal conformal field theories.⁵ We emphasize that this notion only makes sense *after* the *choice* of a geometric interpretation for the theory \mathcal{T} on some torus \mathbb{R}^4/Λ . As is explained in detail in [27, 28], we even have to make a choice of generators for the lattice Λ , and this means that in fact we are working on a cover of the moduli space of SCFTs on K3. These choices in particular induce an identification $\frac{1}{2}\Lambda/\Lambda \cong \mathbb{F}_2^4$, such that every geometric symmetry group G acts on the twisted ground states $T_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$, as permutation group by means of affine linear maps on the space of labels \mathbb{F}_2^4 . In other words, we have a natural representation

$$R_G: G \longrightarrow \text{Aff}(\mathbb{F}_2^4),$$

once the very choices listed above have been made; see [28] and Section 2.1 for details. Furthermore, G acts linearly as subgroup of $SO(3)$ on the $U(1)$ -currents j_+^1, j_+^2 of (1.5). More precisely, j_+^1, j_+^2 form a doublet $\mathbf{2}$ of $SU(2)$, as do their fermionic superpartners χ_+^1, χ_+^2 of (1.4), while j_-^1, j_-^2 carry a $\bar{\mathbf{2}}$. By a direct calculation one checks that the states (1.13) are invariant under the resulting action of $SU(2)$ and that the action respects the decomposition (1.14). In fact, we have $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$, $\mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{1} \oplus \bar{\mathbf{3}}$. Since $-\mathbf{1} \in SU(2)$ acts trivially on $\mathbf{2} \otimes \bar{\mathbf{2}}$, we have an action of $SO(3) = SU(2)/\{\pm \mathbf{1}\}$ on $\mathbf{3}$ and $\bar{\mathbf{3}}$. This action thus descends to a representation on the 96-dimensional space of massive twisted states with basis (1.15), as does the action of $\text{Aff}(\mathbb{F}_2^4)$ on the indices of the twisted ground states. We formally denote the resulting representation of $SO(3)$ by S , where

$$\widehat{V} := \text{span}_{\mathbb{C}} \{WT_{\vec{a}} \mid W \in 3 \cup \bar{3}, \vec{a} \in \mathbb{F}_2^4\}, \quad S: SO(3) \longrightarrow \text{End}_{\mathbb{C}}(\widehat{V}). \quad (1.16)$$

Every symmetry group G of $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ which is induced by geometric symmetries of \mathcal{T} has the form $G = (\mathbb{Z}_2)^4 \rtimes G_T$ with $G_T \subset SO(3)$, see for example [27, 28] for

⁵This includes the symmetries which are induced by shifts by half lattice vectors on the underlying toroidal theory.

an exposition. Then the representation $R_G^{CFT} : G \rightarrow \text{End}_{\mathbb{C}}(\widehat{V})$ obtained from the symmetries of \mathcal{C} is given by

$$\begin{aligned} \forall g = (\vec{c}, g_T) \in G = (\mathbb{Z}_2)^4 \rtimes G_T, \quad W \in 3 \cup \bar{3}, \quad \vec{a} \in \mathbb{F}_2^4: \\ R_G^{CFT}(g)(WT_{\vec{a}}) = S(g_T)(W)T_{R_G(g)(\vec{a})}. \end{aligned} \quad (1.17)$$

It is important to note that \widehat{V} thereby is simply a tensor product of the representation spaces $\mathbf{3} \oplus \bar{\mathbf{3}}$ of $SO(3)$ by a 16-dimensional representation space of $\text{Aff}(\mathbb{F}_2^4)$, a fact that will be crucial later on, when we discuss group actions on this space of states.

In fact, we immediately obtain a natural decomposition of \widehat{V} according to

$$\mathbf{96} = (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes \mathbf{1} \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes \mathbf{15}$$

as follows: we decompose the 16-dimensional space of twisted ground states into a one-dimensional space generated by $N_{0000} := \frac{1}{4} \sum_{\vec{a} \in \mathbb{F}_2^4} T_{\vec{a}}$, and its orthogonal complement \mathcal{A} . Since N_{0000} is invariant under the action of $\text{Aff}(\mathbb{F}_2^4)$, this action descends to a representation on \mathcal{A} . We now obtain the desired 45-dimensional vector space V_{45}^{CFT} as the space which is generated by the states WA with $W \in \{\chi^1 j_+^2 + \chi^2 j_+^1, \chi^1 j_+^1, \chi^2 j_+^2\}$ and $A \in \mathcal{A}$. Then \bar{V}_{45}^{CFT} is defined analogously by using the $\bar{\mathbf{3}}$ from (1.14) instead of the $\mathbf{3}$ as above. The restrictions of the representations R_G and S to V_{45}^{CFT} and \bar{V}_{45}^{CFT} are denoted by R_G and S as well. While at this point the choice of $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ in \widehat{V} is only justified by the fact that it is natural and compatible with a restriction of the representations R_G and S , in Theorem 3.5 we prove that these spaces are in fact uniquely determined.

In summary, we have obtained the result that the generic field content of \mathbb{Z}_2 -orbifold conformal field theories on K3 ensures the existence of a space of states which naturally accounts for the massive net contributions to the elliptic genus in leading order:

PROPOSITION 1.1. — *Consider the orthogonal complement \mathcal{A} of*

$$N_{0000} := \frac{1}{4} \sum_{\vec{a} \in \mathbb{F}_2^4} T_{\vec{a}}$$

in the space of twisted ground states of an arbitrary \mathbb{Z}_2 -orbifold conformal field theory on K3. Then the space

$$V_{45}^{CFT} := \text{span}_{\mathbb{C}} \{WA \mid W \in \{\chi^1 j_+^2 + \chi^2 j_+^1, \chi^1 j_+^1, \chi^2 j_+^2\}, A \in \mathcal{A}\}$$

is a 45-dimensional vector space of massive states which together with \bar{V}_{45}^{CFT} accounts for the leading order contribution to the function $e(\tau)$ that governs the elliptic genus of K3 according to (1.1). In terms of representations of symmetry groups, V_{45}^{CFT} is a tensor product $\mathcal{W} \otimes \mathcal{A}$, where \mathcal{W} is the three-dimensional representation space $\mathbf{3}$ of $SO(3)$, while \mathcal{A} is a 15-dimensional representation space of $\text{Aff}(\mathbb{F}_2^4)$. Similarly, $\bar{V}_{45}^{CFT} = \bar{\mathcal{W}} \otimes \mathcal{A} = \bar{\mathbf{3}} \otimes \mathbf{15}$.

From the above proposition it follows that the vector space underlying V_{45}^{CFT} serves as a medium to collect the actions of the geometric symmetry groups when symmetry-surfing the moduli space of \mathbb{Z}_2 -orbifold conformal field theories $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3. In the remaining sections of this paper, we show that the combined action of these symmetry groups generates an action of $\text{Aff}(\mathbb{F}_2^4)$ which can be induced from a 45-dimensional irreducible representation of M_{24} by restriction to $\text{Aff}(\mathbb{F}_2^4) \cong (\mathbb{Z}_2)^4 \rtimes A_8$.

To clear notations, we make use of the fact that N_{0000} is invariant under both the action of $SO(3)$ and of $\text{Aff}(\mathbb{F}_2^4)$. We let

$$\mathcal{W}_0 := \{WN_{0000} \mid W \in \{\chi^1 j_+^2 + \chi^2 j_+^1, \chi^1 j_+^1, \chi^2 j_+^2\}\} \cong \mathcal{W}$$

and frequently view S as a representation on \mathcal{W}_0 ,

$$S: SO(3) \longrightarrow \text{End}_{\mathbb{C}}(\mathcal{W}_0),$$

the MODEL FIBER of V_{45}^{CFT} .

2. THE ACTION OF $(\mathbb{Z}_2)^4 \rtimes \mathbf{A}_8$ ON TWISTED GROUND STATES

In Prop. 1.1, we have determined a 45-dimensional space V_{45}^{CFT} of states which in every theory $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ yields a representation of the group of symmetries induced from geometric symmetries of the underlying toroidal theory. This representation depends on a choice of geometric interpretation for the theory \mathcal{T} on some torus \mathbb{R}^4/Λ together with a choice of generators for Λ , thus lifting our construction onto a cover of the moduli space of SCFTs. As explained in Section 1.1, Mathieu Moonshine predicts that this space is in fact related to a representation of the Mathieu group M_{24} . Indeed, M_{24} possesses an irreducible representation of dimension 45 on the space V_{45} which has been constructed by Margolin [23], see Appendix A for a summary. Eventually we would like to understand how our space of states V_{45}^{CFT} can be identified with the space V_{45} as a representation of M_{24} . In the present work we focus on the action of the maximal subgroup $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$ of M_{24} , which is obtained by combining all groups of symmetries of SCFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ induced by geometric symmetry groups of the underlying toroidal theory.

The structure of the space V_{45}^{CFT} of fields obtained in the previous section is that of a tensor product $\mathcal{W} \otimes \mathcal{A}$, where \mathcal{A} is a fifteen-dimensional space of twisted ground states in our \mathbb{Z}_2 -orbifold CFT, and \mathcal{W} is three-dimensional and furnishes a triplet $\mathbf{3}$ of $SO(3)$. By choice of an appropriate orthonormal basis $\{N_X \mid X \in \{A, B, \dots, N, O\}\}$ of \mathcal{A} one can thus write this space in a form which is very similar to the form of V_{45} given in (A.3),

$$V_{45}^{CFT} = \mathcal{W}_A \oplus \mathcal{W}_B \oplus \dots \mathcal{W}_N \oplus \mathcal{W}_O, \quad \mathcal{W}_X := \text{span}_{\mathbb{C}}\{N_X\} \otimes \mathcal{W},$$

for all $X \in \{A, B, \dots, N, O\}$. By construction, see Appendix A, Margolin's representation M of $(\mathbb{Z}_2)^4 \rtimes A_8$ induces a well-defined action on the fifteen-dimensional vector space which we call the base \mathcal{B} of V_{45} , and which is generated by the counterparts P_X (see the discussion of (A.3)) of the 'CFT' orthonormal basis $\{N_X \mid X \in \{A, \dots, O\}\}$ of \mathcal{A} . On the other hand, in Section 1.3 we mentioned that $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$ acts naturally on \mathcal{A} by affine linear maps on the indices of the twisted ground states $T_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$. Hence we expect that the latter representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ on \mathcal{A} is equivalent to the representation M of this group on \mathcal{B} described in Appendix A. In the current section, we prove that this expectation holds true.

To make the claim precise, let us describe the space \mathcal{A} in more detail. The space of twisted ground states in our CFT $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ has a natural orthonormal basis $\{T_{\vec{a}} \mid \vec{a} \in \mathbb{F}_2^4\}$, where $\vec{a} \in \mathbb{F}_2^4$ labels the sixteen resolved singular points in any geometric interpretation on an orbifold limit of K3. As is explained in our previous work [27, 28], the group $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$ therefore acts naturally on these states by affine linear transformations on the indices $\vec{a} \in \mathbb{F}_2^4$. In Prop. 1.1 the space \mathcal{A} is obtained as the orthogonal complement of the state $N_{0000} := \frac{1}{4} \sum_{\vec{a} \in \mathbb{F}_2^4} T_{\vec{a}}$, which is invariant under the action of $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$ by construction. The space \mathcal{A} hence indeed carries an action of $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$.

PROPOSITION 2.1. — *Consider the orthogonal complement \mathcal{A} of the state*

$$N_{0000} = \frac{1}{4} \sum_{\vec{a} \in \mathbb{F}_2^4} T_{\vec{a}}$$

in the space of twisted ground states in a \mathbb{Z}_2 -orbifold conformal field theory on K3. The natural representation of $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$ on \mathcal{A} through affine linear transformations of the indices $\vec{a} \in \mathbb{F}_2^4$ of the twisted ground states $T_{\vec{a}}$ is equivalent

to the representation M of $(\mathbb{Z}_2)^4 \rtimes A_8$ constructed by Margolin on the base \mathcal{B} of V_{45} .

We postpone the proof of Prop. 2.1 to Section 2.2, since as a preparation and for later convenience we first recall some of the constructions and notations of [27, 28].

2.1. The action of $(\mathbb{Z}_2)^4 \rtimes A_8$ as combined symmetry group. In [27, 28] we show that the group $(\mathbb{Z}_2)^4 \rtimes A_8 \cong \text{Aff}(\mathbb{F}_2^4)$ can be obtained by combining the symmetry groups of the three maximally symmetric Kummer surfaces⁶. More precisely, the images $R_{G_k}(G_k)$, $k \in \{0, 1, 2\}$, of these three groups under their natural representations on \mathbb{F}_2^4 generate the entire group $\text{Aff}(\mathbb{F}_2^4)$. The three maximally symmetric Kummer surfaces are the square Kummer surface X_0 with symmetry group $G_0 := (\mathbb{Z}_2)^4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, the tetrahedral Kummer surface X_1 with symmetry group $G_1 := (\mathbb{Z}_2)^4 \rtimes A_4$, and the triangular Kummer surface X_2 with symmetry group $G_2 := (\mathbb{Z}_2)^4 \rtimes S_3$. Let us denote by Λ_k , $k \in \{0, 1, 2\}$, the defining lattices of the complex tori underlying the Kummer surfaces X_k . Then each group G_k acts on the twisted ground states $T_{\vec{a}}$ through the permutations induced on $\mathbb{F}_2^4 \cong \frac{1}{2}\Lambda_k/\Lambda_k$ by the geometric action on Λ_k . This defines the representations $R_{G_k} : G_k \rightarrow \text{Aff}(\mathbb{F}_2^4)$.

Let us fix some additional notations. The translational subgroup $(\mathbb{Z}_2)^4$ is common to all symmetry groups of Kummer K3s, and its elements $\iota_{\vec{c}}$ with $\vec{c} \in \mathbb{F}_2^4$ act by

$$\iota_{\vec{c}} : T_{\vec{a}} \mapsto T_{\vec{a}+\vec{c}} \quad \forall \vec{a} \in \mathbb{F}_2^4 \quad (2.1)$$

on the twisted ground states. To realize the action of the non-translational part of each symmetry group G_k , we first fix convenient generators for each of the lattices Λ_k and for the groups G_k , $k \in \{0, 1, 2\}$, see [28, (1.5)-(1.9)] for our particular choices. In the case of the square Kummer surface X_0 with non-translational symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$, we introduce two generators α_1, α_2 in [27, (4.23)] whose action on the $T_{\vec{a}}, \vec{a} \in \mathbb{F}_2^4$, is⁷

$$\begin{aligned} R_{G_0}(\alpha_1) : & \begin{cases} T_{1000} \longleftrightarrow T_{0100}, & T_{0010} \longleftrightarrow T_{0001}, & T_{1010} \longleftrightarrow T_{0101}, \\ T_{1001} \longleftrightarrow T_{0110}, & T_{1110} \longleftrightarrow T_{1101}, & T_{1011} \longleftrightarrow T_{0111}, \end{cases} \\ R_{G_0}(\alpha_2) : & \begin{cases} T_{1000} \longleftrightarrow T_{0010}, & T_{0100} \longleftrightarrow T_{0001}, & T_{1100} \longleftrightarrow T_{0011}, \\ T_{1001} \longleftrightarrow T_{0110}, & T_{1110} \longleftrightarrow T_{1011}, & T_{1101} \longleftrightarrow T_{0111}. \end{cases} \end{aligned} \quad (2.2)$$

Similarly, for the non-translational symmetry group A_4 of the tetrahedral Kummer surface X_1 , we introduce three generators $\gamma_1, \gamma_2, \gamma_3$ in [27, (4.23)], where

$$\begin{aligned} R_{G_1}(\gamma_1) : & \begin{cases} T_{1000} \longleftrightarrow T_{0100}, & T_{0010} \longleftrightarrow T_{1110}, & T_{0001} \longleftrightarrow T_{0111}, \\ T_{1001} \longleftrightarrow T_{0011}, & T_{0101} \longleftrightarrow T_{1111}, & T_{1011} \longleftrightarrow T_{1101}, \end{cases} \\ R_{G_1}(\gamma_2) : & \begin{cases} T_{1000} \longleftrightarrow T_{0010}, & T_{0100} \longleftrightarrow T_{1110}, & T_{0001} \longleftrightarrow T_{1101}, \\ T_{1001} \longleftrightarrow T_{1111}, & T_{0101} \longleftrightarrow T_{0011}, & T_{1011} \longleftrightarrow T_{0111}, \end{cases} \quad (2.3) \\ R_{G_1}(\gamma_3) : & \begin{cases} T_{1000} \mapsto T_{0101} \mapsto T_{1101} \mapsto T_{1000}, \\ T_{0100} \mapsto T_{0011} \mapsto T_{0111} \mapsto T_{0100}, \\ T_{0010} \mapsto T_{1001} \mapsto T_{1011} \mapsto T_{0010}, \\ T_{0001} \mapsto T_{1110} \mapsto T_{1111} \mapsto T_{0001}, \\ T_{1100} \mapsto T_{0110} \mapsto T_{1010} \mapsto T_{1100}. \end{cases} \end{aligned}$$

Finally, generators β_1 and β_2 for the triangular Kummer surface with non-translational symmetry group S_3 are given in [28, (1.9)]. Note that $\beta_2 = \alpha_2$ acts as in

⁶On a K3 surface X , we call a biholomorphic map $f : X \rightarrow X$ a SYMMETRY, if its induced action on cohomology fixes the holomorphic volume form and the dual Kähler class of X . As is explained in [27, 28], this implies that all symmetry groups of K3 surfaces are finite. We require all our Kummer surfaces to be equipped with the dual Kähler class which is induced from the standard Euclidean metric on the underlying torus.

⁷Here and in the following we only list the action on those $T_{\vec{a}}$ which are not invariant under the respective symmetry.

(2.2), $R_{G_2}(\beta_2) = R_{G_0}(\alpha_2)$, while

$$R_{G_2}(\beta_1): \begin{cases} T_{1000} \mapsto T_{0100} \mapsto T_{1100} \mapsto T_{1000}, \\ T_{0010} \mapsto T_{0011} \mapsto T_{0001} \mapsto T_{0010}, \\ T_{1010} \mapsto T_{0111} \mapsto T_{1101} \mapsto T_{1010}, \\ T_{0110} \mapsto T_{1111} \mapsto T_{1001} \mapsto T_{0110}, \\ T_{0101} \mapsto T_{1110} \mapsto T_{1011} \mapsto T_{0101}. \end{cases} \quad (2.4)$$

The permutations $R_{G_0}(\alpha_1)$, $R_{G_0}(\alpha_2)$, $R_{G_1}(\gamma_1)$, $R_{G_1}(\gamma_2)$, $R_{G_1}(\gamma_3)$, $R_{G_2}(\beta_1)$ generate the action of $A_8 \cong \mathrm{GL}_4(\mathbb{F}_2)$ on the indices $\vec{a} \in \mathbb{F}_2^4$ of the twisted ground states $T_{\vec{a}}$ in a form which is convenient for us [28], but of course they do not furnish a minimal set of generators.

At this point, it is important to keep in mind that we view any of our Kummer surfaces as coming equipped with a preferred choice of generators for the lattice Λ which defines the underlying torus. The indexing of the twisted ground states by $\vec{a} \in \mathbb{F}_2^4$ is directly correlated to this choice. As we explain in [27, 28], this induces a choice of common marking for all our Kummer surfaces, that is, an isometry between the lattice $H_*(X, \mathbb{Z})$ of integral homology on K3 with a standard lattice of signature $(4, 20)$. This marking ensures that we can view the location of the Kummer lattice⁸ Π in $H_*(X, \mathbb{Z})$ as fixed among all Kummer surfaces.

Another lattice that plays an important role in our previous work is the Niemeier lattice N with root lattice A_1^{24} . We denote by f_n , $n \in \mathcal{I}$, with $\mathcal{I} = \{1, \dots, 24\}$ a choice of 24 pairwise perpendicular roots in N . The Mathieu group M_{24} acts faithfully by lattice automorphisms on the Niemeier lattice N , permuting the roots $\{f_1, \dots, f_{24}\}$. The Niemeier lattice⁹ $N(-1)$ contains a lattice $\tilde{\Pi}(-1)$ which is isometric to the Kummer lattice [27, Prop. 2.3.3]. In [27, Prop. 2.3.4] we construct an explicit isometry $\iota: \Pi \rightarrow \tilde{\Pi}(-1)$, where we define the lattice $\tilde{\Pi}(-1)$ by choosing a reference octad $\mathcal{O}_9 := \{3, 5, 6, 9, 15, 19, 23, 24\}$ from the Golay code,

$$\tilde{\Pi}(-1) := \{v \in N \mid \langle v, f_n \rangle = 0 \ \forall n \in \mathcal{O}_9\}. \quad (2.5)$$

The lattice $H_*(X, \mathbb{Z})$ is central to our description of symmetries, since every symmetry of a K3 surface X induces a lattice automorphism on $H_*(X, \mathbb{Z})$ which by the Torelli theorem for K3 surfaces determines the symmetry uniquely. The discussion of symmetries hence reduces to a discussion of lattice automorphisms. In particular, for every symmetry group G of a K3 surface X , the action on the orthogonal complement L_G of the invariant lattice, $L_G := (H_*(X, \mathbb{Z})^G)^\perp \cap H_*(X, \mathbb{Z})$, uniquely determines the action of G . If X is a Kummer surface with induced dual Kähler class, then G also induces a lattice automorphism on the Kummer lattice Π , which in turn uniquely determines the action of G on X . Since L_G can never contain Π , and Π in general does not contain L_G , in [27] we propose to consider the lattice M_G which is generated by L_G , Π , and the vector $v_0 - v$, where $v_0 \in H_0(X, \mathbb{Z})$ and $v \in H_4(X, \mathbb{Z})$ with¹⁰ $\langle v_0, v \rangle = 1$.

A key result is [27, Theorem 3.3.7], which in the case of Kummer surfaces with induced dual Kähler class generalizes and improves techniques introduced by Kondo [21]. It states that for each such Kummer surface with symmetry group G , the lattice M_G mentioned above can be primitively embedded in $N(-1)$ in such a way that on the Kummer lattice Π , the embedding induces the isometry $\iota: \Pi \rightarrow \tilde{\Pi}(-1)$ of [27, Prop. 2.3.4]. We call such an embedding $\iota_G: M_G \hookrightarrow N(-1)$ a NIEMEIER MARKING. The Niemeier marking allows us to represent the group G as a group of lattice automorphisms on the Niemeier lattice $N(-1)$, where the action on the

⁸For a Kummer surface X obtained by blowing up the 16 singularities of T/\mathbb{Z}_2 for some complex torus T , the KUMMER LATTICE Π is the smallest primitive sublattice of $H_*(X, \mathbb{Z})$ which contains the classes of the sixteen rational curves obtained from the blow-up.

⁹For a lattice Γ , by $\Gamma(n)$, $n \in \mathbb{Z}$, we denote the \mathbb{Z} -module Γ with quadratic form rescaled by the factor n .

¹⁰On $H_*(X, \mathbb{Z})$, we use the standard quadratic form induced by the intersection form.

image of M_G is prescribed by enforcing the Niemeier marking to be G -equivariant, while on the orthogonal complement of $\iota_G(M_G)$, the group G acts trivially. This allows us to elegantly realize G as a subgroup of the Mathieu group M_{24} . Its action on the Niemeier lattice $N(-1)$ is uniquely determined by its action on the sublattice $\tilde{\Pi}(-1)$. Moreover, this construction allows us to combine symmetry groups of distinct Kummer surfaces by means of their action on $N(-1)$.

As was mentioned above, in [28] we show that the combined action of all symmetry groups of Kummer K3s on $N(-1)$ – and by the above, equivalently, on \mathbb{F}_2^4 – yields the group $(\mathbb{Z}_2)^4 \rtimes A_8$. Here, the normal subgroup $(\mathbb{Z}_2)^4$ is the common translational subgroup of all symmetry groups of Kummer surfaces, which on the labels $\vec{a} \in \mathbb{F}_2^4$ acts by translation as in (2.1). This naturally fixes the action on the sublattice $\tilde{\Pi}(-1)$ of the Niemeier lattice $N(-1)$. The translational group $(\mathbb{Z}_2)^4$ acts trivially on the orthogonal complement of $\tilde{\Pi}(-1)$ in $N(-1)$.

The non-translational group $A_8 \cong \mathrm{GL}_4(\mathbb{F}_2)$ acts on the labels $\vec{a} \in \mathbb{F}_2^4$ as the linear group $\mathrm{GL}_4(\mathbb{F}_2)$. In terms of our favourite generators, this is encoded in (2.2), (2.3) and (2.4), and this naturally determines the action on the sublattice $\tilde{\Pi}(-1)$ of the Niemeier lattice $N(-1)$. On the orthogonal complement of $\tilde{\Pi}(-1)$ in $N(-1)$, the action is obtained from this by means of the isomorphism $A_8 \cong \mathrm{GL}_4(\mathbb{F}_2)$. The result is most conveniently described in terms of the induced permutation of the roots in this lattice, which by (2.5) are labelled by our reference octad \mathcal{O}_9 . The permutations of the eight points of this octad that are induced by our symmetries $\gamma_1, \gamma_2, \gamma_3, \alpha_1, \alpha_2, \beta_1$ are [28, (3.1),(3.2),(3.9)]

$$\begin{aligned} \check{\gamma}_1 &= (9, 24)(15, 19), & \check{\gamma}_2 &= (9, 19)(15, 24), & \check{\gamma}_3 &= (9, 19, 24), \\ \check{\alpha}_1 &= (6, 19)(23, 24), & \check{\alpha}_2 &= (3, 9)(23, 24), & \check{\beta}_1 &= (5, 24, 23). \end{aligned} \quad (2.6)$$

2.2. The proof of Proposition 2.1. To prove Prop. 2.1, let us first consider the translational subgroup $(\mathbb{Z}_2)^4 \subset \mathrm{Aff}(\mathbb{F}_2^4)$, which acts by (2.1) on the twisted ground states. In Margolin's representation $M: \mathrm{Aff}(\mathbb{F}_2^4) \rightarrow \mathrm{End}_{\mathbb{C}}(V_{45})$, this group is simultaneously diagonalised by the basis $\{P_X \mid X \in \{A, B, \dots, N, O\}\}$ which yields the decomposition (A.3). Hence we need to use the common eigenbasis of the translational group $(\mathbb{Z}_2)^4$ on \mathcal{A} , which is given by

$$N_{\vec{a}}^{CFT} := \frac{1}{4} \sum_{\vec{b} \in \mathbb{F}_2^4} (-1)^{\langle \vec{a}, \vec{b} \rangle} T_{\vec{b}}, \quad \text{such that } \iota_{\vec{c}}(N_{\vec{a}}^{CFT}) = (-1)^{\langle \vec{c}, \vec{a} \rangle} N_{\vec{a}}^{CFT} \quad \forall \vec{c} \in \mathbb{F}_2^4, \quad (2.7)$$

for all $\vec{a} \in \mathbb{F}_2^4$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{F}_2^4 . Hence an isomorphism of representations of $(\mathbb{Z}_2)^4$ between \mathcal{A} and the base \mathcal{B} of V_{45} is induced by identifying the translations ι_1, \dots, ι_4 by the four standard basis vectors of \mathbb{F}_2^4 with any set of four generators of $(\mathbb{Z}_2)^4 = \{\mathbb{1}, A', B', \dots, N', O'\}$ according to Table A.5 in the Appendix. We choose

$$\iota_1 = A', \quad \iota_2 = B', \quad \iota_3 = D', \quad \iota_4 = F'.$$

Then from Table A.5 we read, for example,

$$\iota_1(N_A) = N_A, \quad \iota_2(N_A) = N_A, \quad \iota_3(N_A) = -N_A, \quad \iota_4(N_A) = -N_A$$

and hence (2.7) implies $N_A = N_{0011}^{CFT}$. Altogether we have

$$\begin{aligned} N_A &= N_{0011}^{CFT}, & N_B &= N_{0001}^{CFT}, & N_C &= N_{0010}^{CFT}, & N_D &= N_{1001}^{CFT}, & N_E &= N_{1010}^{CFT}, \\ N_F &= N_{1110}^{CFT}, & N_G &= N_{1101}^{CFT}, & N_H &= N_{1000}^{CFT}, & N_I &= N_{1011}^{CFT}, & N_J &= N_{0111}^{CFT}, \\ N_K &= N_{0100}^{CFT}, & N_L &= N_{0110}^{CFT}, & N_M &= N_{0101}^{CFT}, & N_N &= N_{1111}^{CFT}, & N_O &= N_{1100}^{CFT}. \end{aligned} \quad (2.8)$$

To complete the proof of Prop. 2.1, it remains to check that the induced action of $A_8 \cong \mathrm{GL}_4(\mathbb{F}_2)$ on the orthonormal basis $\{N_X \mid X \in \{A, B, \dots, N, O\}\}$ of \mathcal{A} indeed yields the representation on \mathcal{A} equivalent to the one on the base \mathcal{B} of V_{45}

described in Appendix A, by means of the isomorphism induced by $N_X \mapsto P_X$ for all $X \in \{A, B, \dots, N, O\}$.

To do so, one first calculates the permutation of $\{A, B, \dots, N, O\}$ induced by (2.2), (2.3), (2.4). For example, γ_1 interchanges N_A and N_C , N_D and N_J, \dots, N_I and N_K , and we write

$$M(\gamma_1): (A, C)(D, J)(E, M)(G, O)(H, L)(I, K).$$

Next, using the array $\mathcal{A}_{\text{even}}$ of Table A.1 in the Appendix one checks that $s = (0, 2)(1, 5)$ is the unique even permutation of the eight points $\{\infty, 0, \dots, 6\}$, such that conjugation by s induces the permutation

$$(A, C)(D, J)(E, M)(G, O)(H, L)(I, K)$$

of the rows of $\mathcal{A}_{\text{even}}$. We denote this by

$$(A, C)(D, J)(E, M)(G, O)(H, L)(I, K) = \rho_{(0,2)(1,5)},$$

and we proceed analogously for the other generators listed in (2.2), (2.3), (2.4). Altogether we obtain

$$\begin{aligned} M(\alpha_1): & (B, C)(D, L)(E, M)(F, G)(H, K)(I, J) = \rho_{(\infty,2)(1,6)}, \\ M(\alpha_2): & (A, O)(B, K)(C, H)(D, L)(F, I)(G, J) = \rho_{(\infty,2)(0,4)}, \\ M(\gamma_1): & (A, C)(D, J)(E, M)(G, O)(H, L)(I, K) = \rho_{(0,2)(1,5)}, \\ M(\gamma_2): & (A, G)(C, O)(D, L)(E, I)(H, J)(K, M) = \rho_{(0,1)(2,5)}, \\ M(\gamma_3): & (A, H, I)(B, N, F)(C, J, M)(D, K, G)(E, O, L) = \rho_{(0,1,2)}, \\ M(\beta_1): & (A, C, B)(D, N, L)(E, G, J)(F, M, I)(H, O, K) = \rho_{(\infty,3,2)}. \end{aligned} \tag{2.9}$$

We can now confirm that (2.8) under $N_X \mapsto P_X$ for all $X \in \{A, \dots, O\}$ furnishes an isomorphism between the space \mathcal{A} of twisted ground states, on the one hand, and the base \mathcal{B} of V_{45} , on the other hand, as representations of $(\mathbb{Z}_2)^4 \rtimes A_8$. Indeed, by construction, this correctly identifies the action of the translational subgroup $(\mathbb{Z}_2)^4$. As explained in Section 2.1, the action of the non-translational subgroup A_8 as permutation group is most efficiently determined by the action (2.6) of A_8 on the reference octad \mathcal{O}_9 . Finally, the following bijection $\mathcal{O}_9 \rightarrow \{\infty, 0, \dots, 6\}$ induces $\check{\gamma}_1 \mapsto \rho_{(0,2)(1,5)}, \dots, \check{\beta}_1 \mapsto \rho_{(\infty,3,2)}$ and thus proves that $N_X \mapsto P_X$, $X \in \{A, \dots, O\}$, gives an isomorphism between representations of $(\mathbb{Z}_2)^4 \rtimes A_8$:

$$3 \mapsto 4, \quad 5 \mapsto 3, \quad 6 \mapsto 6, \quad 9 \mapsto 0, \quad 15 \mapsto 5, \quad 19 \mapsto 1, \quad 23 \mapsto \infty, \quad 24 \mapsto 2. \quad \square$$

3. A TWIST IN THE $(\mathbb{Z}_2)^4 \rtimes A_8$ ACTION

In the previous sections, we have constructed a $(45 + \overline{45})$ -dimensional space of states $V_{45}^{CFT} \oplus \overline{V}_{45}^{CFT}$, which is generic to all \mathbb{Z}_2 -orbifold conformal field theories on K3, and which accounts for the leading order term $90q$ in the function $\epsilon(\tau)$ of (1.3) that governs the massive contributions to the elliptic genus (1.1). According to Prop. 1.1, this space decomposes as $V_{45}^{CFT} = \mathcal{W} \otimes \mathcal{A}$, where \mathcal{W} is a complex 3-dimensional vector space, while \mathcal{A} is a 15-dimensional space of twisted ground states which carries a faithful action of $(\mathbb{Z}_2)^4 \rtimes A_8$. According to Prop. 2.1, this representation is equivalent to the representation M of $(\mathbb{Z}_2)^4 \rtimes A_8$ on the base \mathcal{B} of the space $V_{45} = \mathcal{V} \otimes \mathcal{B}$ of (A.3) which was constructed by Margolin [23].

We are now ready to explain how the properties of V_{45}^{CFT} give evidence in favour of our surfing ideas, whose ultimate goal is to unravel the role of the full group M_{24} in the context of Mathieu Moonshine, and so far provide a mathematical framework for the action of the maximal subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$. Namely, the representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ generated by the action of geometric symmetry groups on V_{45}^{CFT} can be identified in a natural way with the 45-dimensional irreducible representation M of this maximal subgroup of the Mathieu group M_{24} described by Margolin on

V_{45} . Recall however from Appendix A that the representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ on V_{45} does not respect the tensor product structure $V_{45} = \mathcal{V} \otimes \mathcal{B}$ in a simple way. More precisely, we have the orthogonal direct decomposition

$$V_{45} = \mathcal{V}_A \oplus \mathcal{V}_B \oplus \dots \oplus \mathcal{V}_N \oplus \mathcal{V}_O$$

according to (A.3), where every $M(g)$ with $g \in (\mathbb{Z}_2)^4 \rtimes A_8$ permutes the FIBERS \mathcal{V}_X of V_{45} , and the induced maps $\mathcal{V}_X \rightarrow \mathcal{V}_{M(g)(X)}$ depend non-trivially on g and on $X \in \{A, B, \dots, N, O\}$. Indeed, such a ‘‘twist’’ (see Def. A.1) is necessary, since there exists no nontrivial three-dimensional representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ that \mathcal{V} could carry¹¹.

This may appear counter-intuitive at first sight, as we have not observed a twist in our space of states $V_{45}^{CFT} = \mathcal{W} \otimes \mathcal{A}$. Indeed, according to Prop. 1.1, the three-dimensional space \mathcal{W} is identified with the representation $\mathbf{3}$ of $SO(3)$ under $S: SO(3) \rightarrow \text{End}_{\mathbb{C}}(V_{45}^{CFT})$: as was explained in Section 1.3, every symmetry group of a \mathbb{Z}_2 -orbifold CFT on K3 which is induced from the underlying toroidal theory by geometric symmetry groups acts as a subgroup of $SO(3)$ on \mathcal{W} by means of the representation $\mathbf{3}$ of $SO(3)$.

The key to this puzzle lies in the very groups that can occur as such symmetry groups. As we recalled in Section 2.1, the maximal groups in our setting are the symmetry groups $G_0 = (\mathbb{Z}_2)^4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ of the square Kummer K3, $G_1 = (\mathbb{Z}_2)^4 \rtimes A_4$ of the tetrahedral Kummer K3, and $G_2 = (\mathbb{Z}_2)^4 \rtimes S_3$ of the triangular Kummer K3, where the common translational subgroup $(\mathbb{Z}_2)^4$ acts trivially on \mathcal{W} . In other words, only the finite subgroups $G_T = \mathbb{Z}_2 \times \mathbb{Z}_2$, A_4 and S_3 of $SO(3)$ are of relevance here, all of which have standard nontrivial 3-dimensional representations on \mathcal{W} , induced by $G_T \subset SO(3)$, $S: SO(3) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{W})$.

In order to understand how V_{45}^{CFT} can be identified with Margolin’s V_{45} in a natural way, we first need to prove that these three groups act on V_{45} *without a twist* (see Def. A.1). We may view the twist in Margolin’s representation on V_{45} as yet another obstruction for any known (orbifold) CFT on K3 to enjoy a larger geometric symmetry.

Let us briefly comment on the possibility of an action of a subgroup of $(\mathbb{Z}_2)^4 \rtimes A_8$ on $V_{45} = \mathcal{V} \otimes \mathcal{B}$ without a twist. Since the translational group $(\mathbb{Z}_2)^4$ acts trivially on \mathcal{V} , we can restrict our attention to subgroups of A_8 , in accord with Def. A.1. Recall from [23] or from Appendix A that the action of $\tau \in A_8$ between any two fibers \mathcal{V}_X and \mathcal{V}_Y with $Y = M(\tau)(X)$ of V_{45} is given in terms of a permutation $m_\tau^{(X,Y)}$ on the seven points $\{0, \dots, 6\}$ of the Fano plane $\mathbb{P}(\mathbb{F}_2^3)$. Such a permutation encodes a linear map $\mathcal{V}_X \rightarrow \mathcal{V}_Y$, because a preferred set of generators of the vector spaces \mathcal{V}_X and \mathcal{V}_Y , namely the root vectors of the lattice Λ_3^{b7} , is conveniently encoded in terms of lines with marked points in $\mathbb{P}(\mathbb{F}_2^3)$. The precise labelling by $\{0, \dots, 6\}$ of the A_8 permutations of cycle shape 2^4 in the rows X and Y of the array $\mathcal{A}_{\text{even}}$ of Table A.1 thus corresponds to a specific choice of generators for the vector spaces \mathcal{V}_X and \mathcal{V}_Y . This in particular means that a relabelling of a row X simply amounts to a change of basis in \mathcal{V}_X , as long as the relabelling respects the projective linear structure of $\mathbb{P}(\mathbb{F}_2^3)$. It follows that for every $\tau \in A_8$, there exists a labelling of the array $\mathcal{A}_{\text{even}}$ such that τ acts without a twist according to Def. A.1, i.e. such that the permutations $m_\tau^{(X, M(\tau)(X))}$ agree for all $X \in \{A, B, \dots, N, O\}$. That the subgroups $G_T = \mathbb{Z}_2 \times \mathbb{Z}_2$, A_4 and S_3 of A_8 which are relevant to our construction can act without a twist is a nontrivial claim which we need to prove:

PROPOSITION 3.1. — *Consider the square Kummer surface X_0 with symmetry group $G_0 = (\mathbb{Z}_2)^4 \rtimes (G_T)_0$, where $(G_T)_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators α_1, α_2 as in [27, (4.23)], whose action on the base \mathcal{B} of V_{45} is given in (2.9). Then the group $(G_T)_0$ acts without a twist on V_{45} .*

¹¹The minimal dimension of a nontrivial linear representation of the alternating group A_8 is seven [3].

Proof. — We claim that the labelling of the array \mathcal{A}_{even} of Table A.1 exhibits no element of $(G_T)_0$ with a twist. The proof is a straightforward calculation, where we check that for none of the generators α_1, α_2 , there is a twist.

From (2.9) we read that the generator α_1 of $(G_T)_0$ acts by the conjugation $\rho_{(\infty,2)(1,6)}$ on the rows of the array \mathcal{A}_{even} , and it induces the permutation $\tau_1 := (B, C)(D, L)(E, M)(F, G)(H, K)(I, J)$ of the rows. One then checks for every pair $(X, \tau_1(X))$ with $X \in \{A, B, \dots, N, O\}$ that the induced permutation $m_{\tau_1}^{(X, \tau_1(X))}$ of Fano plane labels in the rows is $(0, 3)(4, 5)$, independently of X . In other words, α_1 acts without a twist. For example, τ_1 maps row A into itself, where conjugation by $(\infty, 2)(1, 6)$ interchanges the first entry $(\infty, 0)(1, 5)(2, 3)(4, 6)$, labelled 0, with the entry $(\infty, 3)(0, 2)(1, 4)(5, 6)$, labelled 3.

Similarly, from (2.9) we read that the generator α_2 of $(G_T)_0$ acts by the conjugation $\rho_{(\infty,2)(0,4)}$ on the rows of the array \mathcal{A}_{even} , that is by

$$\tau_2 := (A, O)(B, K)(C, H)(D, L)(F, I)(G, J).$$

One then checks for every pair $(X, \tau_2(X))$ with $X \in \{A, B, \dots, N, O\}$ that the induced permutation $m_{\tau_2}^{(X, \tau_2(X))}$ of Fano plane labels in the rows is $(0, 3)(1, 6)$, independently of X . In other words, α_2 acts without a twist. \square

Since the square Kummer surface has maximal symmetry, we cannot expect the labelling of \mathcal{A}_{even} given in Table A.1 to exhibit an action without twist for the other maximal symmetry groups G_1, G_2 of Kummer surfaces as well. Nevertheless, we have

PROPOSITION 3.2. — *Consider the tetrahedral Kummer surface X_1 with symmetry group $G_1 = (\mathbb{Z}_2)^4 \rtimes (G_T)_1$, where $(G_T)_1 = A_4$ with generators $\gamma_1, \gamma_2, \gamma_3$ as in [27, (4.10), (4.11)], whose action on the base \mathcal{B} of V_{45} is given in (2.9). Then the group $(G_T)_1$ acts without a twist on V_{45} .*

Proof. — We claim that the following relabelling yields the action of $(G_T)_1$ without a twist, where for each row of Table A.1, we list the seven labels from left to right:

A :	0	1	2	3	4	5	6	I :	0	1	2	5	4	6	3
B :	2	1	3	0	6	4	5	J :	2	1	6	5	3	4	0
C :	3	5	0	2	1	4	6	K :	5	3	2	6	0	4	1
D :	6	5	2	0	1	4	3	L :	6	4	2	5	1	0	3
E :	0	2	3	1	6	4	5	M :	4	3	2	6	0	1	5
F :	0	1	5	6	4	2	3	N :	0	2	6	5	3	4	1
G :	0	1	4	3	5	2	6	O :	5	6	1	3	0	4	2
H :	0	1	2	6	4	3	5								

One checks that this relabelling respects the linear structure of each $\mathbb{P}_X(\mathbb{F}_2^3)$ with $X \in \{A, B, \dots, N, O\}$. The rest of the proof is analogous to the proof of Prop. 3.1: From (2.9) one reads the action of the generators $\gamma_1, \gamma_2, \gamma_3$ of $(G_T)_1$ on the rows of the array \mathcal{A}_{even} and checks that the induced permutation $m_{\gamma_1}^{(X, M(\gamma_1)(X))}$ of Fano plane labels in the rows is $(0, 2)(1, 4)$, independently of X , while $m_{\gamma_2}^{(X, M(\gamma_2)(X))}$ yields $(0, 1)(2, 4)$, independently of X , and $m_{\gamma_3}^{(X, M(\gamma_3)(X))}$ yields $(0, 1, 2)(3, 5, 6)$, independently of X . In other words, $(G_T)_1$ acts without a twist. \square

The final case that we need to study works analogously:

PROPOSITION 3.3. — *Consider the triangular Kummer surface X_2 with symmetry group $G_2 = (\mathbb{Z}_2)^4 \rtimes (G_T)_2$, where $(G_T)_2 = S_3$ with generators $\beta_1, \beta_2 = \alpha_2$ as in [28, (1.9)], whose action on the base \mathcal{B} of V_{45} is given in (2.9). Then the group $(G_T)_2$ acts without a twist on V_{45} .*

Proof. — We work analogously to the proof of Prop. 3.2 and claim that the following relabelling yields the action of $(G_T)_2$ without a twist:

A : 0 1 2 3 4 5 6	I : 1 6 5 2 4 3 0
B : 2 6 3 0 1 5 4	J : 6 1 2 3 0 5 4
C : 3 4 0 2 6 5 1	K : 4 1 2 3 6 5 0
D : 4 2 3 6 1 0 5	L : 5 2 0 3 1 4 6
E : 6 1 2 4 3 5 0	M : 0 6 2 5 1 4 3
F : 2 0 3 6 1 4 5	N : 1 2 4 0 6 3 5
G : 1 6 5 3 2 4 0	O : 4 1 6 3 0 5 2
H : 1 4 2 5 0 6 3	

One checks that this relabelling respects the linear structure of each $\mathbb{P}_X(\mathbb{F}_2^3)$ with $X \in \{A, B, \dots, N, O\}$. From (2.9) one reads the action of the generators $\beta_1, \beta_2 = \alpha_2$ of $(G_T)_2$ on the rows of the array \mathcal{A}_{even} and checks that the induced permutation $m_{\beta_1}^{(X, M(\beta_1)(X))}$ of Fano plane labels in the rows is $(0, 2, 3)(4, 5, 6)$, independently of X , while $m_{\beta_2}^{(X, M(\beta_2)(X))}$ yields $(0, 3)(4, 5)$, independently of X . In other words, $(G_T)_2$ acts without a twist. \square

In summary, for each of the three maximal symmetry groups $G_k = (\mathbb{Z}_2)^4 \rtimes (G_T)_k$ of Kummer surfaces, $k \in \{0, 1, 2\}$, there exists a consistent labelling of the array \mathcal{A}_{even} in Table A.1, such that no twist is exhibited for the action of the group $(G_T)_k$ on V_{45} . We furthermore find

PROPOSITION 3.4. — *For $k \in \{0, 1, 2\}$ consider the three maximal symmetry groups $G_k = (\mathbb{Z}_2)^4 \rtimes (G_T)_k$ of Kummer surfaces whose generators are given in [27, (4.10), (4.11), (4.23)] and [28, (1.9)]. For each group G_k , the natural action on V_{45}^{CFT} , which is induced by the respective symmetries of the states of \mathbb{Z}_2 -orbifold conformal field theories according to (1.17), is equivalent to the action of G_k viewed as a subgroup of $(\mathbb{Z}_2)^4 \rtimes A_8$ in Margolin's representation $M: G_k \rightarrow \text{End}_{\mathbb{C}}(V_{45})$.*

Proof. — Prop. 2.1 implies that Margolin's representation M of G_k on the base \mathcal{B} of $V_{45} = \mathcal{V} \otimes \mathcal{B}$ is equivalent to the representation R_{G_k} of G_k on the base \mathcal{A} of $V_{45}^{CFT} = \mathcal{W} \otimes \mathcal{A}$. Moreover, the translational subgroup $(\mathbb{Z}_2)^4$ of $(\mathbb{Z}_2)^4 \rtimes A_8$ acts trivially both on \mathcal{W} and \mathcal{V} in the tensor products $V_{45}^{CFT} = \mathcal{W} \otimes \mathcal{A}$ and $V_{45} = \mathcal{V} \otimes \mathcal{B}$. Finally, by Props. 3.1, 3.2 and 3.3 along with Prop. 1.1, G_k acts without a twist both on V_{45} and on V_{45}^{CFT} for each $k \in \{0, 1, 2\}$. Hence for each G_k , $k \in \{0, 1, 2\}$, it remains to be shown that the representation M of the subgroup $(G_T)_k$ on the fibers of V_{45}^{CFT} is equivalent to the representation $S: (G_T)_k \rightarrow \text{End}_{\mathbb{C}}(\mathcal{W}_0)$ which is induced by $(G_T)_k \subset SO(3)$ on the fibers $\mathcal{W} \cong \mathcal{W}_0$ of V_{45} .

To do so, let us fix some notations first. Let

$$w_1 := i(\chi^1 j_+^2 + \chi^2 j_+^1) N_{0000}, \quad w_2 := \chi^1 j_+^1 N_{0000}, \quad w_3 := \chi^2 j_+^2 N_{0000} \quad (3.1)$$

denote a basis of $\mathcal{W}_0 = \text{span}_{\mathbb{C}}\{N_{0000}\} \otimes \mathcal{W}$ according to Prop. 1.1. For the generators α_1, α_2 of $(G_T)_0$, $\gamma_1, \gamma_2, \gamma_3$ of $(G_T)_1$ and $\beta_1, \beta_2 = \alpha_2$ of $(G_T)_2$ we determine the induced actions on the respective fields of the \mathbb{Z}_2 -orbifold CFT on the square, the tetrahedral and the triangular Kummer K3, according to [28, (1.5), (1.7), (1.9)]. Here, in accord with [28] the complex currents j_+^k , $k \in \{1, 2\}$, whose real and imaginary parts are the four left-handed $U(1)$ -currents generating a $U(1)^4$ -symmetry in \mathcal{T} , are identified with the holomorphic coordinate vector fields $\frac{\partial}{\partial z_k}$, $k \in \{1, 2\}$.

Hence the induced symmetries on the free fields in \mathcal{T} are given by

$$\alpha_1: \begin{cases} \chi^1 \mapsto i\chi^1 \\ \chi^2 \mapsto -i\chi^2 \\ j_+^1 \mapsto ij_+^1 \\ j_+^2 \mapsto -ij_+^2 \end{cases}, \quad \alpha_2: \begin{cases} \chi^1 \mapsto \chi^2 \\ \chi^2 \mapsto -\chi^1 \\ j_+^1 \mapsto j_+^2 \\ j_+^2 \mapsto -j_+^1 \end{cases}, \quad \beta_1: \begin{cases} \chi^1 \mapsto \zeta\chi^1 \\ \chi^2 \mapsto \zeta^{-1}\chi^2 \\ j_+^1 \mapsto \zeta j_+^1 \\ j_+^2 \mapsto \zeta^{-1}j_+^2 \end{cases},$$

$$\gamma_1: \begin{cases} \chi^1 \mapsto i\chi^1 \\ \chi^2 \mapsto -i\chi^2 \\ j_+^1 \mapsto ij_+^1 \\ j_+^2 \mapsto -ij_+^2 \end{cases}, \quad \gamma_2: \begin{cases} \chi^1 \mapsto \chi^2 \\ \chi^2 \mapsto -\chi^1 \\ j_+^1 \mapsto j_+^2 \\ j_+^2 \mapsto -j_+^1 \end{cases}, \quad \gamma_3: \begin{cases} \chi^1 \mapsto \frac{i+1}{2}(i\chi^1 - \chi^2) \\ \chi^2 \mapsto -\frac{i+1}{2}(i\chi^1 + \chi^2) \\ j_+^1 \mapsto \frac{i+1}{2}(ij_+^1 - j_+^2) \\ j_+^2 \mapsto -\frac{i+1}{2}(ij_+^1 + j_+^2) \end{cases},$$

where as before $\zeta = e^{2\pi i/3}$. Hence we have

$$S(\alpha_1): \begin{cases} w_1 \mapsto w_1 \\ w_2 \mapsto -w_2 \\ w_3 \mapsto -w_3 \end{cases}, \quad S(\alpha_2): \begin{cases} w_1 \mapsto -w_1 \\ w_2 \mapsto w_3 \\ w_3 \mapsto w_2 \end{cases}, \quad S(\beta_1): \begin{cases} w_1 \mapsto w_1 \\ w_2 \mapsto \zeta^{-1}w_2 \\ w_3 \mapsto \zeta w_3 \end{cases}. \quad (3.2)$$

$$S(\gamma_1): \begin{cases} w_1 \mapsto w_1 \\ w_2 \mapsto -w_2 \\ w_3 \mapsto -w_3 \end{cases}, \quad S(\gamma_2): \begin{cases} w_1 \mapsto -w_1 \\ w_2 \mapsto w_3 \\ w_3 \mapsto w_2 \end{cases}, \quad S(\gamma_3): \begin{cases} w_1 \mapsto -w_2 - w_3 \\ w_2 \mapsto \frac{i}{2}(-w_1 - w_2 + w_3) \\ w_3 \mapsto \frac{i}{2}(w_1 - w_2 + w_3) \end{cases}.$$

Note that $\gamma_2 = \gamma_1^2 \gamma_3 \gamma_1 \gamma_3^{-1}$, so henceforth we will not continue to consider the generator γ_2 . We now argue that the respective representations are equivalent for the three relevant cases.

(1) **The square Kummer surface \mathbf{X}_0 with $(\mathbf{G}_T)_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$**

For $(G_T)_0$, we have a nontrivial action for each nontrivial element of this group both on \mathcal{W}_0 and on \mathcal{V} . Moreover, in both spaces, every $g \in (G_T)_0$ is represented by a unitary involution with determinant 1. Hence there is a common eigenbasis of \mathcal{V} for the entire group $M((G_T)_0)$, such that for each nontrivial $g \in (G_T)_0$, $M(g)$ has a two-fold eigenvalue -1 and a simple eigenvalue 1, as in (3.2). Thus there exists an orthonormal basis $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$ of \mathcal{V} such that

$$M(\alpha_1): (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \mapsto (\tilde{w}_1, -\tilde{w}_2, -\tilde{w}_3),$$

$$M(\alpha_2): (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \mapsto (-\tilde{w}_1, \tilde{w}_3, \tilde{w}_2).$$

Then $w_k \mapsto \tilde{w}_k$ for $k \in \{1, 2, 3\}$ induces an equivalence of representations of $(G_T)_0$.

(2) **The tetrahedral Kummer surface \mathbf{X}_1 with $(\mathbf{G}_T)_1 = A_4$**

For $(G_T)_1 = A_4$, we know from [3] that every linear 3-dimensional representation of the alternating group A_4 is either irreducible, or it decomposes into the direct sum of three one-dimensional representations. Moreover, there is only one equivalence class of irreducible linear 3-dimensional representations of A_4 .

By (3.2), there exists no common eigenvector of $S(\gamma_1)$ and $S(\gamma_3)$, hence \mathcal{W}_0 cannot be the sum of three one-dimensional representations of $S((G_T)_1)$, and thus it carries the 3-dimensional irreducible representation of A_4 . It suffices to show the same for \mathcal{V} .

From the proof of Prop. 3.2 we know that $M(\gamma_1)$ acts on the seven points $\{0, \dots, 6\}$ of the Fano plane by means of the permutation $m_{\gamma_1} := (0, 2)(1, 4)$, while γ_3 is represented by $(0, 1, 2)(3, 5, 6)$. Hence $M(\gamma_1)$ maps the lines with marked points $(0, 023)$, $(0, 501)$, $(0, 460)$ to $(2, 023)$, $(2, 245)$, $(2, 612)$, implying that the basis $\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}$ of $\mathcal{V} \cong \mathbb{C}^3$ is mapped to $\{\pm(0, b\bar{7}, b\bar{7}), \pm(-\bar{b}\bar{7}, 1, -1), \pm(\bar{b}\bar{7}, 1, -1)\}$. The signs are

uniquely determined by the fact that $M(\gamma_1)$ is represented on \mathcal{V} by a linear map with determinant one, and m_{γ_1} fixes the point labelled 3 in the Fano plane, such that $M(\gamma_1)$ permutes the three pairs of root vectors which belong to the point frame 3 in Table A.2. From this and by similar arguments for γ_3 one obtains the following matrix representations for the generators of $(G_T)_1$ with respect to the standard basis of $\mathbb{C}^3 \cong \mathcal{V}$:

$$M(\gamma_1) = \frac{1}{2} \begin{pmatrix} 0 & \bar{b7} & \bar{b7} \\ b7 & -1 & 1 \\ b7 & 1 & -1 \end{pmatrix}, \quad M(\gamma_3) = \frac{1}{2} \begin{pmatrix} b7 & 1 & -1 \\ 0 & \bar{b7} & \bar{b7} \\ b7 & -1 & 1 \end{pmatrix}.$$

Calculating the eigenvectors of $M(\gamma_1)$, one finds a one-dimensional eigenspace with eigenvalue $+1$ and the corresponding eigenvector $\tilde{v}_1 = (\bar{b7}, 1, 1)^T$, which as one immediately checks is not an eigenvector of $M(\gamma_3)$. Hence $M(\gamma_1)$ and $M(\gamma_3)$ do not have a common eigenbasis. It follows that \mathcal{V} cannot be the sum of three one-dimensional representations of A_4 . Hence it agrees with the irreducible 3-dimensional representation of A_4 .

(3) **The triangular Kummer surface X_2 with $(G_T)_2 = S_3$**

From the proof of Prop. 3.3 we know that $M(\beta_1)$ acts on the seven points $\{0, \dots, 6\}$ of the Fano plane by means of the permutation $(0, 2, 3)(4, 5, 6)$, while β_2 is represented by $(0, 3)(4, 5)$. By a calculation similar to the one performed for the tetrahedral Kummer surface, one obtains the following matrix representations for the generators of $(G_T)_2$ with respect to the standard basis of $\mathbb{C}^3 \cong \mathcal{V}$:

$$M(\beta_1) = \frac{1}{2} \begin{pmatrix} 0 & -\bar{b7} & \bar{b7} \\ b7 & -1 & -1 \\ b7 & 1 & 1 \end{pmatrix}, \quad M(\alpha_2) = M(\beta_2) = \frac{1}{2} \begin{pmatrix} 0 & \bar{b7} & -\bar{b7} \\ b7 & -1 & -1 \\ -b7 & -1 & -1 \end{pmatrix}.$$

Calculating the respective eigenvectors one finds that with the eigenbasis

$$\tilde{w}_1 = \begin{pmatrix} 1 \\ 0 \\ b7 \end{pmatrix}, \quad \tilde{w}_2 = \begin{pmatrix} \bar{b7} \\ i\sqrt{3} \\ -1 \end{pmatrix}, \quad \tilde{w}_3 = \zeta^{-1} \begin{pmatrix} -\bar{b7} \\ i\sqrt{3} \\ 1 \end{pmatrix}$$

of $M(\beta_1)$, $\zeta = e^{2\pi i/3}$ as above, the isomorphism induced by $w_k \mapsto \tilde{w}_k$ for $k \in \{1, 2, 3\}$ induces an equivalence of representations of $(G_T)_2$. \square

From Prop. 3.4 we now infer how to identify the space of states V_{45}^{CFT} in an arbitrary \mathbb{Z}_2 -orbifold CFT $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ with the representation space V_{45} of M_{24} in a fashion which is compatible with the relevant group \tilde{G} of symmetries of \mathcal{C} . The idea is similar to the surfing procedure described in [28]. As always, we assume that all symmetries in \tilde{G} are induced from geometric symmetries of \mathcal{T} in a fixed geometric interpretation on some torus $T = \mathbb{R}^4/\tilde{\Lambda}$. As detailed in [28, Section 4], for at least one $k \in \{0, 1, 2\}$, we find $\tilde{G} \subset G_k$ along with a smooth deformation of Λ_k into $\tilde{\Lambda}$, call it Λ^t with $t \in [0, 1]$ and $\Lambda^0 = \Lambda_k$, $\Lambda^1 = \tilde{\Lambda}$, such that the linear automorphism group of each Λ^t with $t \neq 0$ is $\tilde{G}'_T \subset SU(2)$ where $\tilde{G}_T = \tilde{G}'_T/\mathbb{Z}_2$. By Prop. 3.4, there exists an isomorphism from V_{45}^{CFT} to V_{45} which induces an equivalence of representations of G_k . By construction, this isomorphism yields the desired identification of the space of states $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ of \mathcal{C} . Note that on Margolin's $V_{45} = \mathcal{V} \otimes \mathcal{B}$, the translational subgroup $(\mathbb{Z}_2)^4$ which is *common* to all geometric symmetry groups of Kummer K3s acts faithfully on \mathcal{B} and trivially on \mathcal{V} . Hence its fixed point set is $\{0\}$. This implies that our selection of the 90-dimensional subspace $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ in \hat{V} (see (1.16)) is in fact unique. Indeed, by construction, the only fixed state of the translational $(\mathbb{Z}_2)^4$ in $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ is 0, while the group acts trivially on the six-dimensional orthogonal complement of this space in \hat{V} . The 90-dimensional subspace $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ is thus uniquely

characterized by the requirement that it carries a faithful representation of the translational subgroup $(\mathbb{Z}_2)^4$. In summary, we have shown

THEOREM 3.5. — *Consider a \mathbb{Z}_2 -orbifold CFT \mathcal{C} on K3, and let $\tilde{G} \subset \text{Aff}(\mathbb{F}_2^4) = (\mathbb{Z}_2)^4 \rtimes A_8$ denote the group of those symmetries of \mathcal{C} which are induced from the geometric symmetries of the underlying toroidal theory in a fixed geometric interpretation. Then the natural representation of \tilde{G} in terms of symmetries of \mathcal{C} on the space V_{45}^{CFT} of massive states of Prop. 1.1 is equivalent to the representation of this group on V_{45} which is obtained by restricting Margolin’s representation $M : M_{24} \rightarrow \text{End}_{\mathbb{C}}(V_{45})$ to \tilde{G} . In other words, the representation of \tilde{G} on V_{45}^{CFT} can be viewed as a representation which is induced by Margolin’s representation $M : M_{24} \rightarrow \text{End}_{\mathbb{C}}(V_{45})$.*

Moreover, within the 96-dimensional space \hat{V} of generic states with the appropriate quantum numbers in \mathbb{Z}_2 -orbifold CFTs on K3, the subspace singled out as $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ is uniquely determined by the property that the action of any geometric symmetry group of a \mathbb{Z}_2 -orbifold conformal field theory is equivalent to the one induced by M .

This theorem encompasses the main result of the present work. Indeed, we have shown that on a large component of the moduli space, there is a 45-dimensional subspace of the space of states, which exists generically and which accounts for the expected *net contributions* to the elliptic genus. We have also shown that these states are actually uniquely characterized by the action of the symmetry groups.

Furthermore, our surfing procedure predicts that the combined action of the symmetry groups at distinct points of the moduli space generates the action of a subgroup of M_{24} ; that this should be the case is by no means clear a priori. Not only do we confirm this part of our prediction, but the group that we generate is a maximal subgroup of M_{24} , which is not a subgroup of M_{23} , and it acts in precisely the predicted way. This is the first piece of evidence in the literature whatsoever for a trace of M_{24} that is intrinsic to CFTs on K3.

As we recall in Section 2.1, according to our previous work [28] the images of the maximal symmetry groups G_0, G_1, G_2 of Kummer surfaces under the respective representations $R_{G_0}, R_{G_1}, R_{G_2}$ altogether generate the group $\text{Aff}(\mathbb{F}_2^4) \cong (\mathbb{Z}_2)^4 \rtimes A_8$. By construction this implies that the combined action of these groups on V_{45} yields the representation of $\text{Aff}(\mathbb{F}_2^4)$ on V_{45} induced by Margolin’s representation M . By Thm. 3.5, in analogy to our construction of overarching symmetry groups by means of Niemeier markings, this procedure combines symmetry groups that are obtained at distinct points in moduli space. Due to the twisting in the representation space V_{45} it is not clear how to interpret an induced combined action on V_{45}^{CFT} geometrically. Note for example that, according to (3.2), the generators α_1 and γ_1 of this combined group have the same representation $S(\alpha_1) = S(\gamma_1)$ on \mathcal{W} . On the other hand, by (2.9), $M(\alpha_1)$ and $M(\gamma_1)$ both fix the label N , so both induce a linear map on the fiber \mathcal{V}_N of V_{45} . However, one checks that these maps are distinct. Indeed, α_1 permutes the seven points in row N of Table A.1 according to $(0, 3)(4, 5)$ with respect to the square labelling, such that for example the first entry of that row, $\infty 0.13.24.56$, is mapped to $\infty 4.02.15.36$. On the other hand, γ_1 permutes the seven points in row N according to $(0, 2)(1, 4)$ with respect to the tetrahedral labelling given in Prop. 3.2, i.e. the first entry of that row is mapped to $\infty 2.04.16.35$. This implies that there is a nontrivial twist which is induced on \mathcal{W} on transition between distinct points of the moduli space of SCFTs on K3.

It may be useful to push the analogy to the Niemeier markings of [28] a little bit further. Consider a SCFT $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ as before, where \mathcal{T} has a geometric interpretation on the torus $T = \mathbb{R}^4/\Lambda$. Let $\vec{\mu}_1, \dots, \vec{\mu}_4$ denote generators of the lattice Λ^* which by means of the Euclidean scalar product we identify as a lattice in $\mathbb{R}^4 \cong (\mathbb{R}^4)^*$. With μ_1^l, \dots, μ_4^l denoting the Euclidean coordinates of $\vec{\mu}_1, \dots, \vec{\mu}_4$,

according to [28, (A.2)] we consider the fields

$$J_k(z) := \sum_{l=1}^4 \mu_k^l j^l(z), \quad k \in \{1, \dots, 4\},$$

and their superpartners $\tilde{\Psi}_k(z)$ as building blocks to construct a lattice which within the (chiral, chiral) algebra of \mathcal{C} plays a role analogous to that of the integral homology of K3 within the real K3 homology. For our purposes, a slightly different lattice might be helpful. We set

$$\kappa_{jk} := \frac{1}{2} \left((\tilde{\Psi}_j)_{\frac{1}{2}} (J_k)_{\frac{1}{2}} + (\tilde{\Psi}_k)_{\frac{1}{2}} (J_j)_{\frac{1}{2}} \right) N_{0000}, \quad j, k \in \{1, \dots, 4\} \text{ with } j \leq k.$$

Let \mathcal{K} denote the complex 10-dimensional vector space with basis $\{\kappa_{jk}, j, k \in \{1, \dots, 4\}, j \leq k\}$, and let $K \subset \mathcal{K}$ be the lattice of rank 20 generated over \mathbb{Z} by the κ_{jk} and the $i\kappa_{jk}$. By construction, \mathcal{K} contains the vector space \mathcal{W}_0 which yields the model fiber of V_{45}^{CFT} . We have a natural action of each of our maximal symmetry groups G_0, G_1, G_2 on the lattice K which induces the action of these groups on \mathcal{W}_0 . In light of the fact that the very representations R_{G_k} of our groups $G_k, k \in \{0, 1, 2\}$, on the base \mathcal{A} of V_{45}^{CFT} are encoded by means of their action on $\mathbb{F}_2^4 \cong \frac{1}{2}\Lambda/\Lambda$, in other words by their description in terms of the lattice Λ , this description of the representation is more natural than the one obtained by the representation $S: SO(3) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{W}_0)$ that was mentioned in Prop. 1.1. The combined action yields an infinite group which descends to $\text{GL}_4(\mathbb{F}_2)$ when we project to $\frac{1}{2}K/K$.

Note that compared to the standard descriptions of the moduli space of SCFTs on K3 [2, 25], the space \mathcal{K} plays the role of the real K3 homology, and the lattice K is the analog of the integral K3 homology. Then the space \mathcal{W}_0 plays the role of the positive definite four-plane in K3 homology whose relative position with respect to the integral K3 homology determines the point in moduli space. Indeed, the relative position of the basis vectors w_1, w_2, w_3 of (3.1) with respect to K depends on the moduli of our CFT \mathcal{C} . For example, in the SCFTs associated with the square and the tetrahedral Kummer surfaces, respectively, the basis vector w_3 is given by

$$\kappa_{33} - \kappa_{44} + i\kappa_{34}, \quad \kappa_{33} + \frac{1}{2}\kappa_{34} + \frac{i}{2}(\kappa_{34} + \kappa_{44}).$$

That these two expressions differ is the source of the twist which we observed above when comparing the action of α_1 and γ_1 on the fiber \mathcal{V}_N of V_{45} . The precise meaning and interpretation of this twist clearly needs further investigation.

4. CONCLUSIONS

\mathbb{Z}_2 -orbifold conformal field theories on K3 provide us with a concrete framework to investigate the nature of the CFT states counted by the elliptic genus of K3 surfaces. In this paper, we have focused on the massive states contributing to leading order. We have taken a close look at the symmetries that act on them in an effort to identify signatures of the M_{24} Moonshine phenomenon. Our motivation has been to carry over, in a field theory context, the essence of what we have recently discovered by scrutinizing the geometry of Kummer surfaces, which form a large class of K3 surfaces [27, 28]. In our previous work, we considered the finite symplectic automorphism groups of Kummer surfaces equipped with a dual Kähler class induced from the underlying torus. These groups are subgroups of three maximal symmetry groups $G_0 = (\mathbb{Z}_2)^4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, $G_1 = (\mathbb{Z}_2)^4 \rtimes A_4$ and $G_2 = (\mathbb{Z}_2)^4 \rtimes S_3$, and we showed that they have a combined action on the Niemeier lattice with root lattice A_1^{24} . This yields the action of the combined symmetry group $(\mathbb{Z}_2)^4 \rtimes A_8$ of all Kummer surfaces. In [28] we have shown that this group is $(\mathbb{Z}_2)^4 \rtimes A_8 \subset M_{24}$, and that it is the largest group one can expect to generate on the Niemeier lattice, given the restrictions imposed. The Niemeier lattice may be seen as a device that provides a ‘memory’ of the action of all finite symplectic

automorphism groups of Kummer surfaces, by accommodating the action of a group which is maximal in M_{24} but not contained in M_{23} . Of course, there is a geometric obstruction to any string theory propagating on a Kummer surface enjoying this combined symmetry: the Niemeier lattice and the full integral homology lattice $H_*(X, \mathbb{Z})$ of a Kummer surface X have same rank but different signatures.

In the present work, we impose analogous restrictions on the symmetries of \mathbb{Z}_2 -orbifold conformal field theories on K3. We assume that these theories come with a choice of generators of the $N = (4, 4)$ superconformal algebra, which in particular fixes the $U(1)$ -currents and a preferred $N = (2, 2)$ subalgebra. We furthermore require the symmetries to fix the superconformal algebra pointwise. These restrictions ensure that every symmetry preserves the conformal weights and $U(1)$ -charges of every field. In [28] we motivate why we restrict our attention to symmetries which are compatible with taking a large volume limit. These restrictions imply that the symmetry groups of interest to us are those induced geometrically in the underlying toroidal theory in a fixed geometric interpretation, that is, the subgroups of G_0, G_1 and G_2 .

Our first task has been to show that there are ninety massive states accounting for the net contribution to the leading massive order of the elliptic genus of K3, which organise themselves into two 45-dimensional vector spaces with tensor product structure $V_{45}^{CFT} = \mathcal{W} \otimes \mathcal{A}$ and $\bar{V}_{45}^{CFT} = \bar{\mathcal{W}} \otimes \mathcal{A}$. Here, \mathcal{W} and $\bar{\mathcal{W}}$ are the 3-dimensional representation spaces $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SO(3)$ which accommodate massive fermionic excitations from the twisted sector of the theory, while \mathcal{A} is a 15-dimensional representation space of $\text{Aff}(\mathbb{F}_2^4) = (\mathbb{Z}_2)^4 \rtimes A_8$ accommodating twisted ground states. The next task has been to show how closely these two 45-dimensional spaces are related to the complex 45(45)-dimensional irreducible representations of M_{24} constructed by Margolin [23]. Since the groups G_0, G_1 and G_2 are all subgroups of $(\mathbb{Z}_2)^4 \rtimes A_8$, which is also the combined symmetry group of all Kummer surfaces, it is natural to study its action on the space V_{45}^{CFT} . We found that the representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ on the space \mathcal{A} is equivalent to the representation of $(\mathbb{Z}_2)^4 \rtimes A_8$ constructed by Margolin on the 15-dimensional ‘‘base’’ \mathcal{B} of $V_{45} = \mathcal{V} \otimes \mathcal{B}$, where V_{45} carries an irreducible representation of M_{24} . In particular, symmetry-surfing the moduli space of \mathbb{Z}_2 -orbifold CFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3 one generates the action of the group $\text{Aff}(\mathbb{F}_2^4)$ on V_{45}^{CFT} from the combined actions of G_0, G_1 and G_2 . However, the action of $(\mathbb{Z}_2)^4 \rtimes A_8$ on the space V_{45} does not factorize according to this tensor product structure: a twist is necessary between fibers, as there are no nontrivial 3-dimensional representations of this combined symmetry group that the fiber \mathcal{V} could carry. On the other hand, such a twist is not apparent in the CFT space V_{45}^{CFT} in its natural description in terms of \mathbb{Z}_2 -orbifold CFTs. In fact, the three maximal symmetry groups $G_k, k \in \{0, 1, 2\}$, act without a twist on V_{45} , as we proved. Moreover, we showed that their natural action on V_{45}^{CFT} , which is induced by the respective symmetries of CFT massive states, is equivalent to the action of these groups viewed as subgroups of $(\mathbb{Z}_2)^4 \rtimes A_8$ in Margolin’s representation.

We have been discussing generic states in \mathbb{Z}_2 -orbifold CFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3 that account for the net contribution to the leading massive order of the elliptic genus of K3. There is a 96-dimensional vector space of fermionic such states, canceling a contribution of 6 generic bosonic states. We have shown that this 96-dimensional space contains $V_{45}^{CFT} \oplus \bar{V}_{45}^{CFT}$ as the unique 90-dimensional subspace on which the generic geometric symmetry group $(\mathbb{Z}_2)^4$ of all \mathbb{Z}_2 -orbifold CFTs $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ on K3 acts faithfully.

Since we restrict ourselves to maximal symmetry groups G_k , finding a \mathbb{Z}_2 -orbifold CFT on K3 with more geometric symmetry is obviously impossible. We view the twist in Margolin’s representation as another manifestation of the geometric obstruction to accommodate larger geometric symmetry groups, reinforcing the statement just made in the previous sentence. Vice versa we conjecture that for

any $N = (4, 4)$ SCFT on K3 whose symmetry group \tilde{G} is a subgroup of one of the eleven subgroups of M_{24} , which Mukai identifies as maximal symmetry groups of K3 surfaces, \tilde{G} acts without a twist on Margolin's representation on V_{45} . Moreover, assume that (1) \mathcal{C} is a SCFT on K3 with geometric interpretation on a K3 surface with symmetry group $\tilde{G} \subset M_{24}$, (2) the B-field of \mathcal{C} in this geometric interpretation is invariant under \tilde{G} , such that \tilde{G} acts as a group of geometric symmetries of \mathcal{C} . Then we expect that there is a 45-dimensional space of massive states \tilde{V}_{45}^{CFT} of \mathcal{C} with quantum numbers $(h, Q; \bar{h}, \bar{Q}) = (\frac{5}{4}, 1; \frac{1}{4}, \bar{Q})$, such that \tilde{G} acts on \tilde{V}_{45}^{CFT} by means of symmetries of \mathcal{C} . We also expect that this representation is equivalent to the representation of \tilde{G} on V_{45} which is induced by Margolin's representation $M : M_{24} \rightarrow \text{End}_{\mathbb{C}}(V_{45})$. If true, this provides information about states in SCFTs on K3 which nobody has been able to construct so far.

Finally, it would be illuminating to pin down the analog of the Niemeier markings [27, 28], which were designed to bring the combined group action into light. As mentioned in the introduction, we interpret the representation space V_{45} as a medium which can collect the actions of symmetry groups from distinct points of the moduli space and combine them to representations of larger groups, making its role directly comparable to that of the Niemeier lattice. Moreover, we have identified a 10-dimensional complex vector space \mathcal{K} and a rank 20 lattice $K \subset \mathcal{K}$ that play analogous roles to the real K3 homology and the integral K3 homology that were so crucial in constructing the Niemeier markings. There are however interesting novel features we inherit from the representation theory of M_{24} , which will have to await interpretation in a way that would lift a corner of the veil surrounding Mathieu Moonshine.

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APPENDIX A. A 45-DIMENSIONAL VECTOR SPACE WITH M_{24} ACTION

We review Margolin's construction of a 45-dimensional irreducible representation of M_{24} [23]. We focus on the features of this representation that are crucial for our present work.

A core component in the construction is a 3-dimensional vector space V over \mathbb{F}_2 . The associated projective plane $\mathbb{P}(V) \cong \mathbb{P}(\mathbb{F}_2^3)$ is the so-called FANO PLANE: it contains seven points represented by the seven non-zero vectors of \mathbb{F}_2^3 . There are seven lines, each comprising three distinct points whose representatives in \mathbb{F}_2^3 together with the origin form a hyperplane. Under addition in \mathbb{F}_2^3 , the four points of every hyperplane form a Kleinian 4-group, that is, a group isomorphic to $(\mathbb{Z}_2)^2$. Figure A.1(a) illustrates the structure of the Fano plane. The projective linear transformations of $\mathbb{P}(\mathbb{F}_2^3)$ act by permuting the seven points of the Fano plane, and they form a group isomorphic to the group $L_3(2) := \text{GL}_3(\mathbb{F}_2)$ of linear automorphisms of V . It will be helpful for our purposes to label the non-zero elements of V by integers modulo 7, according to Figure A.1(b), such that the set of points in each projective line in $\mathbb{P}(V)$ has the form $\{i, i+2, i+3\}$, $i \in \mathbb{F}_7$. In addition, the origin of V is labelled ∞ . Note that translation by a fixed vector in V permutes

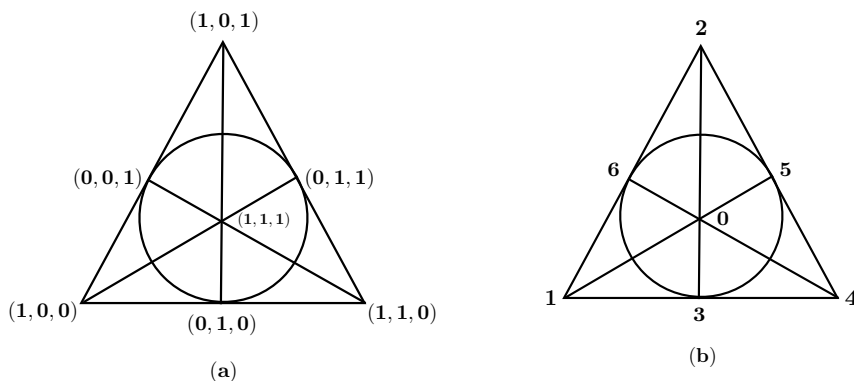


Figure A.1: The Fano plane (a) labelled by points in \mathbb{F}_2^3 ; (b) labelled by points in \mathbb{F}_7 .

the points of V . In the notations of Figure A.1(b), the eight resulting translations are

$$\begin{aligned}
 \text{translation by } (0, 0, 0) \ (\infty) : & \quad \mathbb{1} \text{ (identity),} \\
 \text{translation by } (1, 1, 1) \ (\text{point } 0) : & \quad (0, \infty)(1, 5)(2, 3)(4, 6), \\
 \text{translation by } (1, 0, 0) \ (\text{point } 1) : & \quad (1, \infty)(0, 5)(2, 6)(3, 4), \\
 \text{translation by } (1, 0, 1) \ (\text{point } 2) : & \quad (2, \infty)(0, 3)(1, 6)(4, 5), \\
 \text{translation by } (0, 1, 0) \ (\text{point } 3) : & \quad (3, \infty)(0, 2)(1, 4)(5, 6), \\
 \text{translation by } (1, 1, 0) \ (\text{point } 4) : & \quad (4, \infty)(0, 6)(1, 3)(2, 5), \\
 \text{translation by } (0, 1, 1) \ (\text{point } 5) : & \quad (5, \infty)(0, 1)(2, 4)(3, 6), \\
 \text{translation by } (0, 0, 1) \ (\text{point } 6) : & \quad (6, \infty)(0, 4)(1, 2)(3, 5).
 \end{aligned} \tag{A.1}$$

The Fano plane structure appears in two incarnations in Margolin’s construction, as we shall discuss now.

First consider the 105 permutations of cycle shape 2^4 in the alternating group A_8 , i.e. all permutations of type $(a_1, a_2)(a_3, a_4)(a_5, a_6)(a_7, a_8)$ with $a_i \in \mathbb{F}_7 \cup \{\infty\}$, a_i all distinct. Note that A_8 acts by conjugation on this set of permutations. Denote by \mathcal{S} the set of seven nontrivial translations in (A.1), such that $\mathcal{S} \cup \{\mathbb{1}\}$ is the group of translations of V . It follows that $\mathcal{S} \cup \{\mathbb{1}\} \cong (\mathbb{Z}_2)^3$ is the maximal normal subgroup of the group $\text{Aff}(V) = (\mathbb{Z}_2)^3 \rtimes L_3(2)$ of linear affine transformations of V . In particular, \mathcal{S} is invariant under conjugation by this group. Since $\text{Aff}(V)$ acts by permutations on the eight points of V , labelled $\infty, 0, \dots, 6$ in (A.1), we realize this group as a subgroup of the alternating group A_8 of index 15. In summary, the set \mathcal{S} is a subset of cardinality seven in the set of 105 permutations in A_8 of cycle shape 2^4 , which is invariant under conjugation by $\text{Aff}(V) = (\mathbb{Z}_2)^3 \rtimes L_3(2) \subset A_8$. It follows that the remaining 98 permutations in A_8 of cycle shape 2^4 decompose into fourteen sets of seven elements each, the images of \mathcal{S} under conjugations by elements of $A_8 \setminus \text{Aff}(V)$. One thus obtains an array $\mathcal{A}_{\text{even}}$ of fifteen 7-sets such that

- (1) each 7-set \mathbf{X} , $X \in \{A, B, \dots, N, O\}$, together with the identity permutation, forms a multiplicative abelian group of order 8;
- (2) each 7-set \mathbf{X} , $X \in \{A, B, \dots, N, O\}$, displays a Fano plane structure, i.e. it is possible to label the seven involutions in each set by a distinct element $i \in \mathbb{F}_7$, such that each set contains seven triplets of involutions labelled $\{i, i + 2, i + 3\}$ which correspond to the lines in the Fano plane; indeed,

these triplets yield seven multiplicative abelian 4-groups $\{\mathbb{1}, i, i+2, i+3\}$. For instance, $\{\mathbb{1}, 3, 5, 6\}$ with the involutions labelled $\{3, 5, 6\}$ in row **C** of Table A.1 is the 4-group

$$\{\mathbb{1}, (\infty 6)(05)(13)(24), (\infty 0)(14)(23)(56), (\infty 5)(06)(12)(34)\};$$

note that this leaves a freedom of choice for the labelling which amounts to the action of $L_3(2)$ in each row;

(3) the 7-set \mathcal{S} appears as row **A** of Table A.1.

The 45-dimensional space affording the representation of interest is obtained by taking 15 copies of a complex three-dimensional vector space to be described below. Each copy of this vector space is labelled by one of the letters A, \dots, O corresponding to the fifteen 7-sets with Fano plane structure displayed in Table A.1. This Fano plane structure on the rows **A**, \dots , **O** will be crucial to explain the action of M_{24} , or rather of its subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$, which we focus on in this appendix, on the representation space. Therefore we think of the fifteen 7-sets in Table A.1 as yielding the base of our representation: formally, we introduce a 15-dimensional complex Euclidean vector space \mathcal{B} and choose an orthonormal basis $\{P_A, P_B, \dots, P_N, P_O\}$ for it. It is immediate that the vector space \mathcal{B} carries a linear representation of A_8 . Indeed, A_8 acts by conjugation on the rows of the array $\mathcal{A}_{\text{even}}$ thus permuting the labels $\{A, B, \dots, N, O\}$ and thereby permuting our orthonormal basis of \mathcal{B} . We refer to \mathcal{B} as the BASE of Margolin's representation space.

	c_0	c_1	c_2	c_3	c_4	c_5	c_6
A	$\infty 0.15.23.46^0$	$\infty 1.05.26.34^1$	$\infty 2.03.16.45^2$	$\infty 3.02.14.56^3$	$\infty 4.06.13.25^4$	$\infty 5.01.24.36^5$	$\infty 6.04.12.35^6$
B	$\infty 2.03.14.56^2$	$\infty 1.06.24.35^3$	$\infty 3.02.15.46^4$	$\infty 0.16.23.45^5$	$\infty 4.05.12.36^0$	$\infty 5.04.13.26^1$	$\infty 6.01.25.34^6$
C	$\infty 3.02.16.45^4$	$\infty 1.04.25.36^1$	$\infty 0.14.23.56^5$	$\infty 2.03.15.46^2$	$\infty 4.01.26.35^0$	$\infty 5.06.12.34^6$	$\infty 6.05.13.24^3$
D	$\infty 0.12.36.45^5$	$\infty 3.06.14.25^0$	$\infty 2.01.35.46^3$	$\infty 4.05.13.26^4$	$\infty 1.02.34.56^6$	$\infty 5.04.16.23^2$	$\infty 6.03.15.24^1$
E	$\infty 0.16.24.35^4$	$\infty 4.02.13.56^5$	$\infty 2.04.15.36^2$	$\infty 1.06.25.34^0$	$\infty 3.05.14.26^1$	$\infty 5.03.12.46^3$	$\infty 6.01.23.45^6$
F	$\infty 0.14.26.35^1$	$\infty 1.04.23.56^0$	$\infty 4.01.25.36^5$	$\infty 3.05.12.46^6$	$\infty 5.03.16.24^2$	$\infty 2.06.13.45^4$	$\infty 6.02.15.34^3$
G	$\infty 0.12.34.56^0$	$\infty 1.02.35.46^1$	$\infty 5.06.13.24^4$	$\infty 3.04.15.26^3$	$\infty 2.01.36.45^5$	$\infty 4.03.16.25^2$	$\infty 6.05.14.23^6$
H	$\infty 0.13.26.45^0$	$\infty 1.03.25.46^1$	$\infty 2.06.15.34^2$	$\infty 5.04.12.36^6$	$\infty 4.05.16.23^4$	$\infty 6.02.14.35^3$	$\infty 3.01.24.56^5$
I	$\infty 0.16.25.34^2$	$\infty 1.06.23.45^1$	$\infty 2.05.13.46^4$	$\infty 6.01.24.35^6$	$\infty 4.03.15.26^0$	$\infty 3.04.12.56^3$	$\infty 5.02.14.36^5$
J	$\infty 4.06.12.35^6$	$\infty 1.03.24.56^3$	$\infty 2.05.14.36^5$	$\infty 3.01.26.45^1$	$\infty 6.04.15.23^0$	$\infty 5.02.16.34^2$	$\infty 0.13.25.46^4$
K	$\infty 6.04.13.25^6$	$\infty 1.02.36.45^3$	$\infty 2.01.34.56^2$	$\infty 3.05.16.24^5$	$\infty 0.12.35.46^0$	$\infty 5.03.14.26^1$	$\infty 4.06.15.23^4$
L	$\infty 1.05.24.36^5$	$\infty 5.01.23.46^3$	$\infty 2.06.14.35^0$	$\infty 3.04.16.25^2$	$\infty 4.03.12.56^1$	$\infty 0.15.26.34^6$	$\infty 6.02.13.45^4$
M	$\infty 5.01.26.34^3$	$\infty 0.15.24.36^4$	$\infty 2.04.13.56^2$	$\infty 3.06.12.45^3$	$\infty 4.02.16.35^5$	$\infty 1.05.23.46^1$	$\infty 6.03.14.25^0$
N	$\infty 0.13.24.56^4$	$\infty 2.04.16.35^2$	$\infty 6.05.12.34^6$	$\infty 3.01.25.46^3$	$\infty 4.02.15.36^5$	$\infty 5.06.14.23^0$	$\infty 1.03.26.45^1$
O	$\infty 0.14.25.36^5$	$\infty 6.03.12.45^6$	$\infty 1.04.26.35^1$	$\infty 3.06.15.24^3$	$\infty 4.01.23.56^0$	$\infty 5.02.13.46^4$	$\infty 2.05.16.34^2$

Table A.1: Array $\mathcal{A}_{\text{even}}$ of 105 permutations of cycle shape 2^4 in A_8 with labelling relative to the square torus symmetry, where $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8$ denotes the permutation $(a_1, a_2)(a_3, a_4)(a_5, a_6)(a_7, a_8)$.

The Fano plane structure also appears within an irreducible 3-dimensional representation of $L_3(2)$ constructed from a rank 3 sublattice Λ_3^{b7} of the icosian Leech lattice that appears in Wilson's description¹² of the maximal subgroup $L_3(2) \rtimes \mathbb{Z}_2$ of the Hall-Janko group J_2 [32]. To understand the Fano plane structure in this context, consider the following 21 pairs of root vectors¹³:

$$\pm(2, 0, 0)^\sigma, \quad \pm(0, b7, \pm b7)^\sigma, \quad \pm(\pm \overline{b7}, 1, -1)^\sigma \quad \pm(\overline{b7}, \pm 1, \pm 1)^\sigma \quad (\text{A.2})$$

in \mathbb{C}^3 , where $b7 := \frac{1}{2}(-1 + \sqrt{-7})$ and $\overline{b7} := \frac{1}{2}(-1 - \sqrt{-7})$, and $(a, b, c)^\sigma$ means that all cyclic permutations of $\{a, b, c\}$ should also be considered. These root vectors

¹²This lattice is also very closely related to the lattice generated from the root graph $J_3(4)$ introduced in [5], whose associated complex reflection group preserves the Klein quartic $F(x, y, z) = xy^3 + yz^3 + zx^3 = 0$ [20].

¹³We follow Margolin's conventions and scale all root vectors to length 2.

generate the lattice $\Lambda_3^{b7} \subset \mathbb{C}^3$ over $\mathbb{Z}[b7]$. The automorphism group of this lattice is $\mathbb{Z}_2 \times L_3(2)$, where $L_3(2)$ can be generated by the 21 reflections in the root vectors (A.2). This fact can be used to define an action of $L_3(2)$ on the underlying vector space \mathbb{C}^3 : The three root vectors $(2, 0, 0)^\sigma$ form a coordinate frame, that is an orthonormal basis of \mathbb{C}^3 , where $(2, 0, 0)$ now represents the pair of root vectors $\pm(2, 0, 0)$, etc. The remaining 18 pairs of root vectors can be partitioned into six other coordinate frames as follows:

0	1	2	3	4	5	6
2 0 0	-1 $\overline{b7}$ 1	$-\overline{b7}$ 1 -1	$\overline{b7}$ 1 1	$b7$ $b7$ 0	$-b7$ 0 $b7$	-1 -1 $\overline{b7}$
0 2 0	-1 $-\overline{b7}$ 1	0 $b7$ $b7$	$\overline{b7}$ -1 -1	1 -1 $\overline{b7}$	1 $\overline{b7}$ 1	$b7$ $-b7$ 0
0 0 2	$b7$ 0 $b7$	$\overline{b7}$ 1 -1	0 $b7$ $-b7$	1 -1 $-\overline{b7}$	-1 $\overline{b7}$ -1	1 1 $\overline{b7}$

Table A.2: Point frames of the lattice Λ_3^{b7} .

The labels 0, . . . , 6 in Table A.2 indicate the structure of the points in a Fano plane on these seven orthonormal bases of \mathbb{C}^3 : any pair a, b of coordinate frames is fixed, up to signs, by a nontrivial automorphism in $L_3(2)$, and all automorphisms fixing this pair fix a third frame c . The frames a, b, c then yield the points on a line in the Fano plane Figure A.1(b). These seven frames are therefore called POINT FRAMES.

Another partition of the pairs of root vectors into seven different coordinate frames is possible. The first frame consists of the first root vector pair of point frame 0, the second root vector pair of point frame 2, and the third root vector pair of point frame 3. It is labelled 023 and called a LINE FRAME. This way the seven line frames of Table A.3 correspond to the lines of the Fano plane whose points are the point frames. We can now uniquely specify every root vector in Λ_3^{b7} up to sign

023	134	245	356	460	501	612
2 0 0	-1 $\overline{b7}$ 1	$-\overline{b7}$ 1 -1	$\overline{b7}$ 1 1	$b7$ $b7$ 0	$-b7$ 0 $b7$	-1 -1 $\overline{b7}$
0 $b7$ $b7$	$\overline{b7}$ -1 -1	1 -1 $\overline{b7}$	1 $\overline{b7}$ 1	$b7$ $-b7$ 0	0 2 0	-1 $-\overline{b7}$ 1
0 $b7$ $-b7$	1 -1 $-\overline{b7}$	-1 $\overline{b7}$ -1	1 1 $\overline{b7}$	0 0 2	$b7$ 0 $b7$	$\overline{b7}$ 1 -1

Table A.3: Line frames of the lattice Λ_3^{b7} .

by the line frame, that is by the line in the Fano plane that it belongs to, along with a point on that line, that is by a point frame. In other words, pairs of root vectors in Λ_3^{b7} are in 1 : 1 correspondence with lines in $\mathbb{P}(V)$ with one marked point. The group $L_3(2)$ of lattice automorphisms of Λ_3^{b7} of determinant one acts faithfully on the point frames and on the line frames, and it preserves the projective structure of the Fano plane. We obtain an induced irreducible representation of $L_3(2)$ on the vector space $\mathcal{V} \cong \mathbb{C}^3$ generated by Λ_3^{b7} over \mathbb{C} : consider the orthonormal basis $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$ of \mathbb{C}^3 and specify each of these root vectors in terms of a point and a line in the Fano plane, that is, $(0, 023)$, $(0, 501)$, $(0, 460)$, respectively. The images of the three basis vectors under $g \in L_3(2)$ are specified, up to a sign, by the images of these points and lines under the permutation by which g acts on the seven points of the Fano plane. The correct signs of the images follow from the requirement that g maps the pairs of root vectors in point frame a to the pairs of root vectors in point frame $g(a)$ for all $a \in \mathbb{F}_7$.

The 45-dimensional space of the irreducible representation of M_{24} that we are interested in is obtained by assigning to each row of the array $\mathcal{A}_{\text{even}}$ in Table A.1 one of 15 mutually orthogonal copies of the complex vector space \mathcal{V} generated by Λ_3^{b7} over \mathbb{C} [23]. In other words, the representation space is given by

$$V_{45} := \mathcal{V}_A \oplus \mathcal{V}_B \oplus \dots \oplus \mathcal{V}_N \oplus \mathcal{V}_O, \quad (\text{A.3})$$

where each \mathcal{V}_X is a copy of \mathcal{V} which carries the irreducible representation of $L_3(2)$ described above, and $X \in \{A, B, \dots, N, O\}$ with A, B, \dots, N, O labelling the rows as in Table A.1. Using the vector space \mathcal{B} with orthonormal basis

$$\{P_A, P_B, \dots, P_N, P_O\}$$

that was introduced above, we have $V_{45} = \mathcal{V} \otimes \mathcal{B}$. We refer to \mathcal{B} as the BASE of the representation space V_{45} , while the \mathcal{V}_X are referred to as the FIBERS.

It remains to identify how the group M_{24} acts on this space. Margolin constructs an irreducible representation $M: M_{24} \rightarrow \text{End}_{\mathbb{C}}(V_{45})$ in [23]. Here we only discuss the action of the maximal subgroup $(\mathbb{Z}_2)^4 \rtimes A_8$, as this is of primary relevance to our work. Margolin gives an explicit construction of the extra group element that generates M_{24} together with the copy of $(\mathbb{Z}_2)^4 \rtimes A_8$ we describe below.

The group A_8 acts on the 45-dimensional space V_{45} as follows: the fifteen rows of $\mathcal{A}_{\text{even}}$ are permuted under conjugation by elements of A_8 , so let $\tau \in A_8$ and $M(\tau)(X) := Y$ if $\tau \mathbf{X} \tau^{-1} = \mathbf{Y}$ for rows of $\mathcal{A}_{\text{even}}$ labelled X, Y (X may be equal to Y). Since τ maps 4-groups to 4-groups, the Fano plane structure of the rows \mathbf{X} and \mathbf{Y} is preserved under τ . To describe the induced action $\mathcal{V}_X \rightarrow \mathcal{V}_Y$, as above we use the fact that every root vector in Λ_3^{b7} (up to a sign) is specified by a line $p_1 p_2 p_3$ of the Fano plane $\mathbb{P}_X(\mathbb{F}_2^3)$ associated with row X and a point p_k , $k \in \{1, 2, 3\}$, on that line. The label of the point p_k is one of the labels attached to the involutions of $\mathcal{A}_{\text{even}}$ in row X . The permutation τ therefore induces a map $m_\tau^{(X,Y)}: \mathbb{P}_X(\mathbb{F}_2^3) \rightarrow \mathbb{P}_Y(\mathbb{F}_2^3)$. The root vector image in \mathcal{V}_Y is then reconstructed from $m_\tau^{(X,Y)}(p_k)$ and the line $m_\tau^{(X,Y)}(p_1) m_\tau^{(X,Y)}(p_2) m_\tau^{(X,Y)}(p_3)$.

For instance, suppose one conjugates $\mathcal{A}_{\text{even}}$ by $\tau = (\infty, 0)(1, 5)$. The corresponding permutation on the 15 rows is given by $(B, C)(D, O)(E, N)(F, H)(G, I)(J, K)$, i.e. τ maps $\mathcal{V}_A, \mathcal{V}_L$ and \mathcal{V}_M to themselves, \mathcal{V}_B to \mathcal{V}_C , \mathcal{V}_D to \mathcal{V}_O etc. To determine the precise action on these spaces, one reads off the permutation on the seven points of the Fano plane encoded in the labelling of involutions within $\mathcal{A}_{\text{even}}$. In the case of the labelling displayed in Table A.1, the permutation (B, C) corresponds to the map $m_\tau^{(B,C)}: \mathbb{P}_B(\mathbb{F}_2^3) \rightarrow \mathbb{P}_C(\mathbb{F}_2^3)$ with $m_\tau^{(B,C)}(0) = 1$, $m_\tau^{(B,C)}(1) = 0$, $m_\tau^{(B,C)}(2) = 4$, $m_\tau^{(B,C)}(3) = 3$, $m_\tau^{(B,C)}(4) = 2$, $m_\tau^{(B,C)}(5) = 5$, $m_\tau^{(B,C)}(6) = 6$. We encode this map succinctly and mnemonically as the ‘‘permutation’’ $m_\tau^{(B,C)} = (1, 0)(2, 4)$, which governs how \mathcal{V}_B is mapped to \mathcal{V}_C . Specifically, the root vector in \mathcal{V}_B corresponding to point 0 within the line 023 of $\mathbb{P}_B(\mathbb{F}_2^3)$ is mapped on \mathcal{V}_C to the root vector 1 within line 134 of $\mathbb{P}_C(\mathbb{F}_2^3)$. In other words, Margolin’s representation $M: A_8 \rightarrow \text{End}_{\mathbb{C}}(V_{45})$ enforces $M((\infty, 0)(1, 5))((2, 0, 0)_B) = \pm(-1, \bar{b}7, 1)_C$. Since A is fixed under τ , the corresponding permutation of $\mathbb{P}_A(\mathbb{F}_2^3)$ is $m_\tau^{(A,A)} = (2, 3)(4, 6)$, which induces a map from \mathcal{V}_A to itself, and so on.

We note that the maps $m_\tau^{(X,Y)}$ and $m_\tau^{(U,V)}$ for two pairs of rows need not be identical. This prompts us to introduce the following

DEFINITION A.1. — *Let $\mathcal{A}_{\text{even}}$ be an array as in Table A.1, with fixed labelling, and let $\tau \in A_8$ act by conjugation on the 15 rows of $\mathcal{A}_{\text{even}}$. If the induced maps m_τ between Fano planes associated with the permuted rows are not all identical, then the array is said to exhibit the action of τ on the 45-dimensional space V_{45} with a TWIST.*

For a subgroup $G \subset A_8$, assume that there exists a labelling of the array $\mathcal{A}_{\text{even}}$ which is compatible with the linear structure of the Fano planes $\mathbb{P}_X(\mathbb{F}_2^3)$ for all

c'_0	c'_1	c'_2	c'_3	c'_4	c'_5	c'_6
$A' \infty 0.15.23.46$	$A \infty 5.01.26.34$	$M \infty 3.02.16.45$	$C \infty 2.03.14.56$	$B \infty 6.04.13.25$	$K \infty 1.05.24.36$	$L \infty 4.06.12.35$
$B' \infty 3.02.14.56$	$A \infty 6.01.24.35$	$I \infty 2.03.15.46$	$C \infty 0.16.23.45$	$B \infty 5.04.12.36$	$H \infty 4.05.13.26$	$D \infty 1.06.25.34$
$C' \infty 2.03.16.45$	$A \infty 4.01.25.36$	$F \infty 0.14.23.56$	$C \infty 3.02.15.46$	$B \infty 1.04.26.35$	$O \infty 6.05.12.34$	$N \infty 5.06.13.24$
$D' \infty 0.12.36.45$	$D \infty 6.03.14.25$	$M \infty 1.02.35.46$	$G \infty 5.04.13.26$	$B \infty 2.01.34.56$	$K \infty 4.05.16.23$	$H \infty 3.06.15.24$
$E' \infty 0.16.24.35$	$E \infty 2.04.13.56$	$M \infty 4.02.15.36$	$N \infty 6.01.25.34$	$B \infty 5.03.14.26$	$K \infty 3.05.12.46$	$F \infty 1.06.23.45$
$F' \infty 0.14.26.35$	$F \infty 4.01.23.56$	$O \infty 1.04.25.36$	$C \infty 5.03.12.46$	$E \infty 3.05.16.24$	$K \infty 6.02.13.45$	$L \infty 2.06.15.34$
$G' \infty 0.12.34.56$	$G \infty 2.01.35.46$	$D \infty 6.05.13.24$	$C \infty 4.03.15.26$	$I \infty 1.02.36.45$	$5, K \infty 3.04.16.25$	$L \infty 5.06.14.23$
$H' \infty 0.13.26.45$	$H \infty 3.01.25.46$	$N \infty 6.02.15.34$	$F \infty 4.05.12.36$	$B \infty 5.04.16.23$	$D \infty 2.06.14.35$	$L \infty 1.03.24.56$
$I' \infty 0.16.25.34$	$I \infty 6.01.23.45$	$E \infty 5.02.13.46$	$O \infty 1.06.24.35$	$B \infty 3.04.15.26$	$G \infty 4.03.12.56$	$L \infty 2.05.14.36$
$J' \infty 6.04.12.35$	$A \infty 3.01.24.56$	$H \infty 5.02.14.36$	$I \infty 1.03.26.45$	$N \infty 4.06.15.23$	$K \infty 2.05.16.34$	$O \infty 0.13.25.46$
$K' \infty 4.06.13.25$	$A \infty 2.01.36.45$	$G \infty 1.02.34.56$	$D \infty 5.03.16.24$	$F \infty 0.12.35.46$	$K \infty 3.05.14.26$	$E \infty 6.04.15.23$
$L' \infty 5.01.24.36$	$A \infty 1.05.23.46$	$M \infty 6.02.14.35$	$H \infty 4.03.16.25$	$G \infty 3.04.12.56$	$I \infty 0.15.26.34$	$L \infty 2.06.13.45$
$M' \infty 1.05.26.34$	$A \infty 0.15.24.36$	$M \infty 4.02.13.56$	$E \infty 6.03.12.45$	$O \infty 2.04.16.35$	$N \infty 5.01.23.46$	$L \infty 3.06.14.25$
$N' \infty 0.13.24.56$	$N \infty 4.02.16.35$	$M \infty 5.06.12.34$	$C \infty 1.03.25.46$	$H \infty 2.04.15.36$	$E \infty 6.05.14.23$	$G \infty 3.01.26.45$
$O' \infty 0.14.25.36$	$O \infty 3.06.12.45$	$M \infty 4.01.26.35$	$C \infty 6.03.15.24$	$D \infty 1.04.23.56$	$F \infty 2.05.13.46$	$I \infty 5.02.16.34$

Table A.4: Array \mathcal{A}_{odd} obtained from \mathcal{A}_{even} through conjugation by $(0, \infty)$.

$X \in \{A, B, \dots, N, O\}$ and which exhibits a twist for no $\tau \in G$. In other words, the action of G factorizes according to $V_{45} = \mathcal{V} \otimes \mathcal{B}$. Then we say that G acts WITHOUT A TWIST.

Another partition of the 105 permutations of cycle shape 2^4 in A_8 is useful for exhibiting the action of the normal subgroup $(\mathbb{Z}_2)^4$ in $(\mathbb{Z}_2)^4 \rtimes A_8 \subset M_{24}$, that is, to exhibit $M: (\mathbb{Z}_2)^4 \rightarrow \text{End}_{\mathbb{C}}(V_{45})$. This action is obtained from the array \mathcal{A}_{even} by conjugation with an element of $S_8 \setminus A_8$. We choose this element to be $(0, \infty)$ and call the conjugate array \mathcal{A}_{odd} , displayed in Table A.4. We label each of the permutations in the array by a letter A, B, \dots, N , or O , according to the row in which this permutation occurs in \mathcal{A}_{even} . Note that under conjugation by $(0, \infty)$, the seven boldfaced involution of column c_1 of \mathcal{A}_{even} are interchanged with the seven involutions of row A , leaving $(\infty, 0)(1, 5)(2, 3)(4, 6)$ invariant. Similarly, $(0, \infty)$ interchanges the seven boldfaced involutions of columns c_i , $i = 2, \dots, 6$ with the involutions of rows M, C, B, K, L, J respectively, leaving

$$\begin{aligned}
 &(\infty, 0)(1, 5)(2, 4)(3, 6), \quad (\infty, 0)(1, 4)(2, 3)(5, 6), \quad (\infty, 0)(1, 6)(2, 3)(4, 5), \\
 &(\infty, 0)(1, 2)(3, 5)(4, 6), \quad (\infty, 0)(1, 5)(2, 6)(3, 4), \quad \text{and} \quad (\infty, 0)(1, 3)(2, 5)(4, 6)
 \end{aligned}$$

invariant. In fact, each row X' with $X' \in \{A', B', \dots, N', O'\}$ of \mathcal{A}_{odd} contains involutions from seven different rows of \mathcal{A}_{even} , and therefore specifies seven of the 15 copies of \mathcal{V} . To each row X' of \mathcal{A}_{odd} , Margolin associates an automorphism of V_{45} (call it X' as well) that fixes these seven copies of \mathcal{V} pointwise, and acts as $-\mathbb{1}$ on the other eight. In other words, X' acts linearly on the base \mathcal{B} of $V_{45} = \mathcal{V} \otimes \mathcal{B}$; it fixes the seven vectors P_Y in our orthonormal basis of \mathcal{B} which have labels Y occurring in the line X' of the dual array \mathcal{A}_{odd} , while multiplying the other eight basis vectors by -1 . X' acts trivially on \mathcal{V} . The resulting automorphisms A', B', \dots, N', O' are captured by Table A.5, which can easily be identified as the character table of an abelian group of order 2^4 . Since this group is normalised by the action of A_8 , altogether one obtains an action of $(\mathbb{Z}_2)^4 \rtimes A_8$ on the 45-dimensional space V_{45} of (A.3) and thus $M: (\mathbb{Z}_2)^4 \rtimes A_8 \rightarrow \text{End}_{\mathbb{C}}(V_{45})$. The representation of the full group M_{24} on V_{45} is described in [23].

1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	
		A'	B'	C'	D'	E'	F'	G'	H'	I'	J'	K'	L'	M'	N'	O'
1	A	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	
1	B	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	
1	C	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	1	1	
1	D	-1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	
1	E	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	1	-1	
1	F	-1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	1	
1	G	-1	-1	1	1	-1	-1	1	-1	1	-1	1	1	-1	1	
1	H	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	
1	I	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	
1	J	1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	1	1	
1	K	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	
1	L	1	-1	-1	-1	-1	1	1	1	-1	-1	1	1	1	-1	
1	M	1	-1	-1	1	1	-1	-1	-1	-1	-1	1	1	1	1	
1	N	-1	-1	1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	
1	O	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	1	

Table A.5: The action of $(\mathbb{Z}_2)^4$ on the 45-dimensional space V_{45} .

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