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MALNORMAL SUBGROUPS AND FROBENIUS GROUPS: BASICS AND EXAMPLES

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With an appendix by Denis Osin

Abstract. Malnormal subgroups occur in various contexts. We review a large number of examples, and compare the general situation to that of finite Frobenius groups of permutations.

In a companion paper [18], we analyse when peripheral subgroups of knot groups and 3-manifold groups are malnormal.

1. Introduction

A subgroup H of a group G is *malnormal* if $gHg^{-1} \cap H = \{e\}$ for all $g \in G$ with $g \notin H$. As far as we know, the term goes back to a paper by Benjamin Baumslag containing conditions for an amalgam $H *_L K$ (called a "generalized free product" in [3]) to be 2-free (i.e. such that any subgroup generated by two elements is free). Other authors write that H is *antinormal* or *conjugately separated*, instead of "malnormal" [22, 29].

The following question arose in discussions with Rinat Kashaev (see also [25] and [26]). We are grateful to him for this motivation.

Given a knot K in $\mathbf{S}^3$, when is the peripheral subgroup malnormal in the group $\pi_1(\mathbf{S}^3 \smallsetminus K)$ of K ?

The answer, for which we refer to [18], is that the peripheral subgroup is malnormal unless K is either a torus knot, or a cable knot, or a connected sum.

The main purpose of the present subsidiary paper is to collect in Section 3 several examples of pairs

(infinite group, malnormal subgroup)

which are classical. Section 2 is a reminder of basic elementary facts on malnormal subgroups. In Section 4, we allude to some facts concerning the more general notion of almost malnormal subgroup, important in the theory of relatively hyperbolic groups. We conclude in Section 5 by comparing malnormal subgroups in infinite groups with finite Frobenius groups.

2. General facts on malnormal subgroups

The two following propositions collect straightforward properties of malnormal subgroups. For a group G and an element $h \in G$, we denote by $C_G(h)$ the centraliser $\{g \in G \mid gh = hg\}$ of h in G.

Proposition 1. — *Let G be a group and H a subgroup; let X denote the homogeneous space G/H and let $x_0 \in X$ denote the class of H. The following properties are equivalent:*

(a) *H is malnormal in G;*
(b) *the natural action of H on $X \smallsetminus \{x_0\}$ is free;*
(c) *any $g \in G$, $g \neq e$, has zero or one fixed point on X.*

Moreover, if G contains a normal subgroup N such that G is the semi-direct product $N \rtimes H$, these properties are also equivalent to each of:

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(d) *$nh \neq hn$ for all $n \in N$, $n \neq e$, and $h \in H$, $h \neq e$;*
(e) *$C_G(h) = C_H(h)$ for all $h \in H$, $h \neq e$.*

The proof is an exercise; if necessary, see the proof of Theorem 6.4 in [21].

Following an "added in proof" of Peter Neumann in [30], we define a *Frobenius group* to be a group G which has a malnormal subgroup H distinct from $\{e\}$ and G. A *split Frobenius group* is a Frobenius group G containing a malnormal subgroup H and a normal subgroup N such that $G = N \rtimes H$. It follows that the restriction to N of the action of G on G/H is *regular*, i.e. transitive with trivial stabilisers (the latter condition means $\{n \in N \mid ngH = gH\} = \{e\}$ for all $gH \in G/H$).

In finite group theory, according to a famous result of Frobenius, Properties (a) to (c) *imply* the existence of a splitting normal subgroup N, so that any finite Frobenius group is split. See Section 5 for more details.

Proposition 2. — *Let G be a group.*

(i) *The trivial subgroups $\{e\}$ and G are malnormal in G. They are the only subgroups of G which are both normal and malnormal.*
(ii) *Let H be a malnormal subgroup in G; then gHg^{-1} is malnormal for all $g \in G$.*
More generally, if α is an automorphism of G, then $\alpha(H)$ is malnormal.
(iii) *Let H be a malnormal subgroup of G and K a malnormal subgroup of H; then K is malnormal in G.*
(iv) *Let H be a malnormal subgroup and S be a subgroup of G; then $H \cap S$ is malnormal in S.*
(v) *Let $(H_\iota)_{\iota \in I}$ be a family of malnormal subgroups of G; then $\bigcap_{\iota \in I} H_\iota$ is malnormal in G.*
(vi) *Let H_1 and H_2 be two groups; then H_1 is malnormal in the free product $H_1 * H_2$.*
(vii) *Let H be a non-trivial subgroup of G; if the centre of G is non-trivial, then H is not malnormal in G.*
(viii) *Let H be a non-trivial subgroup of G containing at least 3 elements; if G contains a normal subgroup C which is infinite cyclic, then H is not malnormal in G.*
In particular, a group G without 2-torsion containing a normal infinite cyclic subgroup (such as the fundamental group of a Seifert manifold not covered by $\mathbf{S}^3$) does not contain any non-trivial malnormal subgroup.

Proof. — Claims (i) to (v) follow from the definition. Claim (vi) follows from the usual normal form in free products, and appears formally as Corollary 4.1.5 of [28].

For (vii), we distinguish two cases for the centre Z of G. First case: $Z \not\subseteq H$; for $z \in Z$ with $z \notin H$, we have $zHz^{-1} \cap H = H \neq \{e\}$, so that H is not malnormal. Second case: $Z \subset H$; for $g \in G$ with $g \notin H$, we have $\{e\} \neq Z \subset H \cap gHg^{-1}$.

Claim (viii) is obvious if $H \cap C \neq \{e\}$, so that we can assume that $H \cap C = \{e\}$. Choose $c \in C$, $c \neq e$ and $h \in H$, $h \neq e$; observe that $h^{-1}ch = c^{\pm 1}$.

If $h^{-1}ch = c$, then $e \neq h = c^{-1}hc \in H \cap c^{-1}Hc$, and H is not malnormal. Since H is not of order two, there exist $h_1, h_2 \in H \smallsetminus \{e\}$ with $k \doteqdot h_2 h_1^{-1} \neq e$. If $h_j^{-1} c h_j = c$ for at least one of $j = 1, 2$, the previous argument applies. Otherwise $k^{-1}ck = c$, so that H is not malnormal for the same reason. □

About (viii), note that the infinite dihedral group D_∞ contains an infinite cyclic subgroup of index 2, and that any subgroup of order 2 in D_∞ is malnormal.

Corollary 3. — *Let G be a group.*

(ix) *Any subgroup H of G has a* malnormal hull[1]*, which is the smallest malnormal subgroup of G containing H.*

[1] Or *malnormal closure*, as in Definition 13.5 of [23]

(x) *Any group H is isomorphic to a malnormal subgroup of some group.*
(xi) *Let $\pi : G \longrightarrow Q$ be a projection onto a quotient group and let H be a malnormal subgroup in G; then $\pi(H)$ need not be malnormal in Q.*

Proof. — Claim (ix) is straightforward from (v), and (x) from (vi).

Consider a factor $\mathbf{Z}$ of the free product $G = \mathbf{Z} * \mathbf{Z}$ of two infinite cyclic groups, and the projection π of G on its abelianization. Then Claim (xi) follows from (vi) and (ix). □

Claim (ix) above suggest the following construction, potentially useful for the work of Rinat Kashaev. Given a group $\mathcal{G}$ and a subgroup $\mathcal{H}$, let $\mathcal{N}$ be the largest normal subgroup of $\mathcal{G}$ contained in $\mathcal{H}$; set $H = \mathcal{H}/\mathcal{N}$, $G = \mathcal{G}/\mathcal{N}$. Then H has a malnormal hull, say G_0, in G. There are interesting cases in which H is malnormal in $G_0 = G$.

For example, let p, q be a pair of coprime integers, $p, q \geqslant 2$, and let $a, b \in \mathbf{Z}$ be such that $ap + bq = 1$. Then $\mathcal{G} = \langle s, t \mid s^p = t^q \rangle$ is a torus knot group; a possible choice of meridian and parallel is $\mu = s^b t^a$ and $\lambda = s^p \mu^{-pq}$, which generate the peripheral subgroup $\mathcal{P} \simeq \mathbf{Z}^2$ of $\mathcal{G}$ (as in Proposition 3.28 of [7]). Let $\mathcal{N} = \langle s^p \rangle$ denote the centre of $\mathcal{G}$; if u and v denote respectively the images of s^b and t^a in $G := \mathcal{G}/\mathcal{N}$, then $G = \langle u, v \mid u^p = v^q = e \rangle \simeq C_p * C_q$ is the free product of two cyclic groups of orders p and q, and $P := \mathcal{P}/\mathcal{N} = \langle uv \rangle \simeq \mathbf{Z}$ is the infinite cyclic group generated by uv. As we will see below (Remark 8.B), P is malnormal in G.

Let G be a group and H, H', K be subgroups such that $K \subset H' \subset H \subset G$. If K is malnormal in H, then K is malnormal in H', by (iv). It follows that it makes sense to speak about maximal subgroups of G in which K is malnormal. We will see in Example 9 that K may be malnormal in *several* such maximal subgroups of G; in other words, one cannot define *one* largest subgroup of G in which K would be malnormal.

3. Examples of malnormal subgroups of infinite groups

Proposition 2 provides a sample of examples. Here are a few others.

Example 4 (translation subgroup of the affine group of the line). — Let $\mathbf{k}$ be a field and let $G = \begin{pmatrix} \mathbf{k}^* & \mathbf{k} \\ 0 & 1 \end{pmatrix}$ be its affine group, where the subgroup $\mathbf{k}^*$ of G is identified with the isotropy subgroup of the origin for the usual action of G on the affine line $\mathbf{k}$. Then $\mathbf{k}^*$ is malnormal in G.

Observe that the field $\mathbf{k}$ need not be commutative (in other words, $\mathbf{k}$ can be a division algebra). More generally, if V is a $\mathbf{k}$-module, then the subgroup $\mathbf{k}^* = \begin{pmatrix} \mathbf{k}^* & 0 \\ 0 & 1 \end{pmatrix}$ of the group $\begin{pmatrix} \mathbf{k}^* & V \\ 0 & 1 \end{pmatrix} = \mathbf{k}^* \ltimes V$ is malnormal. To check Example 4, it seems appropriate to use (c) in Proposition 1, with $\mathbf{k}^* \ltimes V$ acting on $\mathbf{k}$ by $((a, b), x) \longmapsto ax + b$.

Subgroups of G provide notable examples. If $\mathbf{k} = \mathbf{R}$, for $n \in \mathbf{Z}$, $|n| \geqslant 2$, the soluble Baumslag-Solitar group

$$\Gamma = \mathrm{BS}(1, n) = \begin{pmatrix} n^{\mathbf{Z}} & \mathbf{Z}[\frac{1}{|n|}] \\ 0 & 1 \end{pmatrix} \simeq \langle a, t \mid tat^{-1} = a^n \rangle$$

is a subgroup of G. The intersection $\mathrm{BS}(1, n) \cap \mathbf{R}^*$ is the infinite cyclic group generated by $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$, and is malnormal in $\mathrm{BS}(1, n)$; see Claim (iv) of Proposition 2.

Example 5 (parabolic subgroup of a Fuchsian group). — Consider a discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ which is not elementary and which contains a parabolic element γ_0; denote by ξ the fixed point of γ_0 in the circle $\mathbf{S}^1$. Then the parabolic subgroup

$$P = \{\gamma \in \Gamma \mid \gamma(\xi) = \xi\}$$

corresponding to ξ is malnormal in Γ.

For Example 5, we recall the following standard facts. The group $\mathrm{PSL}_2(\mathbf{R})$ is identified with the connected component of the isometry group of the Poincaré half plane $\mathcal{H}^2 = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$. It acts naturally on the boundary $\partial\mathcal{H}^2 = \mathbf{R} \cup \{\infty\}$ identified with the circle $\mathbf{S}^1$. Any $\gamma \in \Gamma$, $\gamma \neq e$, is either hyperbolic, with exactly two fixed points on $\mathbf{S}^1$, or parabolic, with exactly one fixed point on $\mathbf{S}^1$, or elliptic, without any fixed point on $\mathbf{S}^1$. For such a group Γ containing at least one parabolic element fixing a point $\xi \in \mathbf{S}^2$, "non-elementary" means $P \neq \Gamma$. Since Γ is discrete in $\mathrm{PSL}_2(\mathbf{R})$, a point in the circle cannot be fixed by both a parabolic element and a hyperbolic element in Γ.

It follows that the action of P on the complement of $\{\xi\}$ in the orbit $\Gamma\xi$ satisfies Condition (b) of Proposition 1, so that P is malnormal in Γ.

In particular, in $\mathrm{PSL}_2(\mathbf{Z})$, the infinite cyclic subgroup P generated by the class $\gamma_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ is malnormal (case of $\xi = \infty$). In anticipation of Section 5, let us point out here that there *cannot* exist a normal subgroup N of $\mathrm{PSL}_2(\mathbf{Z})$ such that $\mathrm{PSL}_2(\mathbf{Z}) = N \rtimes P$, because this would imply the existence of a surjection $\mathrm{PSL}_2(\mathbf{Z}) \longrightarrow P \simeq \mathbf{Z}$, but this is impossible since the abelianised group of $\mathrm{PSL}_2(\mathbf{Z})$ is finite (cyclic of order 6).

Example 6 (parabolic subgroup of a torsion-free Kleinian group). — Consider a discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbf{C})$ which is not elementary and torsion-free. Suppose Γ contains a parabolic element γ_0, with fixed point ξ in the sphere $\mathbf{S}^2$. Then the parabolic subgroup

$$P = \{\gamma \in \Gamma \mid \gamma(\xi) = \xi\}$$

corresponding to ξ is malnormal in Γ.

In case the group Γ of Example 6 is the group of a hyperbolic knot, P is the *peripheral subgroup* of Γ. Part of Example 6 carries over to a much larger setting, see Proposition 13 below.

The argument indicated in Example 5 carries over to the case of Example 6. The group $\mathrm{PSL}_2(\mathbf{C})$ is identified with the connected component of the isometry group of the hyperbolic 3-space $\mathcal{H}^3 = \{(z,t) \in \mathbf{C} \times \mathbf{R} \mid t > 0\}$; it acts naturally on the boundary $\partial\mathcal{H}^3 = \mathbf{C} \cup \{\infty\}$ identified with the sphere $\mathbf{S}^2$.

The number of conjugacy classes of subgroups of the type P is equal to the number of orbits of Γ on the subset of $\mathbf{S}^2$ consisting of points fixed by some parabolic element of Γ.

In Example 6, the hypothesis that Γ is torsion-free cannot be omitted. For example, consider the Picard group $\mathrm{PSL}_2(\mathbf{Z}[i])$, its subgroup P of classes of matrices of the form $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$, and the subgroup Q of P of classes of matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Set

$$g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{PSL}_2(\mathbf{Z}[i]) \qquad \text{(observe that } g^2 = e,\ g \notin P),$$

and

$$h = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in P \qquad \text{(observe that } h^2 = e,\ h \notin Q).$$

As $ghg^{-1} = h^{-1} \in P$, the subgroup P is not malnormal in $\mathrm{PSL}_2(\mathbf{Z}[i])$). As $h \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} h^{-1} \in Q$ for all b, the subgroup Q is not malnormal in P (and *a fortiori* not malnormal in $\mathrm{PSL}_2(\mathbf{Z}[i])$).

Note that, in the group Γ of Example 5, torsion is allowed, because each element $\gamma \neq e$ of finite order in $\mathrm{PSL}_2(\mathbf{R})$ acts without fixed point on $\mathbf{S}^1$. But the element h above, of order 2, has fixed points on $\mathbf{S}^2$.

Example 7 is a variation on Examples 5 and 6, related to boundary fixed points of hyperbolic elements rather than of parabolic elements.

Example 7 (virtually cyclic subgroup of a torsion-free Gromov hyperbolic group). Consider a Gromov hyperbolic group Γ which is not elementary, an element $\gamma_0 \in \Gamma$ of infinite order, and one of the two points in the boundary $\partial\Gamma$ fixed by γ_0, say ξ. Set

$$P = \{\gamma \in \Gamma \mid \gamma(\xi) = \xi\}.$$

Assume moreover that Γ is torsion-free. Then P is malnormal in Γ.

For the background of Example 7, see [15], in particular Theorem 30 of Chapter 8. Recall that any element in P fixes also the other fixed point $\xi_- \in \partial\Gamma$ of γ_0, that the infinite cyclic subgroup of Γ generated by γ_0 is of finite index in P, and that P itself is infinite cyclic.

In torsion-free non-elementary hyperbolic groups, subgroups of the form P are precisely the maximal abelian subgroups. (See also Example 10.)

To check the claim of Example 7, consider $\gamma \in \Gamma$ such that $\gamma \notin P$, i.e. such that $\gamma(\xi) \neq \xi$. If one had $\gamma(\xi) = \xi_-$, every element of $\gamma P \gamma^{-1}$ would fix ξ_-, and therefore also ξ; it would follow that $\gamma(\xi_-) = \xi$, so that γ^2 would fix ξ and ξ_-; since Γ is torsion-free, this would imply $\gamma = 1$, in contradiction with the hypothesis. Hence $\gamma(\{\xi, \xi_-\})$ and $\{\xi, \xi_-\}$ are disjoint, and it follows that $\gamma P \gamma^{-1} \cap P = \{e\}$.

Remark 8. — Here are illustrations and variations on Example 7.

(A) In the free group F_2 on two generators a and b, any primitive element, for example $a^k b a^\ell b^{-1}$ with $k, l \in \mathbf{Z} \smallsetminus \{0\}$, generates an infinite cyclic subgroup which is malnormal.

(An element γ in a group Γ is *primitive* if there does not exist any pair (δ, n), with $\delta \in \Gamma$ and $n \in \mathbf{Z}$, $|n| \geqslant 2$, such that $\gamma = \delta^n$.)

(B) Here is an old-fashioned example: consider a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{R})$, and a hyperbolic element $h \in \Gamma$ fixing two distinct points $\alpha, \omega \in \mathbf{S}^1$. Then

$$P = \{\gamma \in \Gamma \mid \gamma(\alpha) = \alpha\}$$

is malnormal in Γ.

We particularize further, as referred to at the end of Section 2. Consider two integers $p \geqslant 2$ and $q \geqslant 3$; denote by C_p and C_q the finite cyclic groups of order p and q. In the hyperbolic plane, consider a rotation u of angle $2\pi/p$ and a rotation v of angle $2\pi/q$. If the hyperbolic distance between the fixed points of u and v is large enough, the group generated by u and v is a free product

$$\Gamma = \langle u, v \mid u^p = v^q = e \rangle \simeq C_p * C_q,$$

by the theorem of Poincaré on polygons generating Fuchsian groups [36]; moreover, the product uv is hyperbolic and primitive. It follows that the infinite cyclic group generated by uv is malnormal in Γ.

(C) Let us mention another variation: let $\mathbb{G}$ be a connected semisimple real algebraic group without compact factors, let d denote its real rank, and let Γ be a torsion-free uniform lattice in $G := \mathbb{G}(\mathbf{R})$. Then $\mathbb{G}$ contains a maximal torus $\mathbb{T}$ such that $A := \Gamma \cap \mathbb{T}(\mathbf{R}) \simeq \mathbf{Z}^d$ is malnormal in Γ; see [38], building up on a result of Prasad and Rapinchuk, and motivated by the construction of an example in operator algebra theory. For an earlier use of malnormal subgroups in operator algebra theory, see [37], in particular Corollary 4.4, covered by our Example 7.

The following example supports the last claim of Section 2.

Example 9. — There exists a group G containing two distinct maximal subgroups B_+, B_- and a subgroup $T \subset B_+ \cap B_-$ which is malnormal in each of B_+, B_-, but not in G.

Set $G = PGL_2(\mathbf{C})$, and let $\pi : GL_2(\mathbf{C}) \longrightarrow G$ denote the canonical projection. Define the subgroups

$$T = \pi \begin{pmatrix} \mathbf{C}^* & 0 \\ 0 & \mathbf{C}^* \end{pmatrix}, \quad B_+ = \pi \begin{pmatrix} \mathbf{C}^* & \mathbf{C} \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad B_- = \pi \begin{pmatrix} \mathbf{C}^* & 0 \\ \mathbf{C} & 1 \end{pmatrix}$$

of G. Then T is malnormal in B_+ and in B_- (see Example 4 and Proposition 2.ii), but not in G, since T is strictly contained in its normalizer $N_G(T)$, the quotient $N_G(T)/T$ being the Weyl group of order 2. Moreover, B_+ and B_- are maximal subgroups in G; this can be checked in an elementary way, and is also a consequence of general properties of parabolic subgroups (see [5], Chapter IV, § 2, n^o 5, Théorème 3).

Example 10 (CSA). — A group is said to be CSA if all its maximal abelian subgroups are malnormal. The following groups are known to be CSA :

(i) torsion-free hyperbolic groups;
(ii) groups acting freely and without inversions on Λ-trees (in particular on trees);
(iii) universally free groups.

For (i), see Example 7. For (ii), see Corollary 1.9 in [2]. And (iii) follows; for CSA groups, see [29], [16], and other papers by the same authors.

The nature of the two next examples is more combinatorial than geometric.

Example 11 (M. Hall). — Let F be a free group and H a finitely generated subgroup of F; then there exist a subgroup of finite index F_0 in F which contains H and a subgroup K of F such that $F_0 = H * K$.

In particular, H is malnormal in F_0.

This is a result due to M. Hall and often revisited: see [17], [9], [39], or Lemma 15.22 on Page 181 of [19].

Rank 2 malnormal subgroups of free groups are characterised in [13].

Example 12 (B.B. Newman [31]). — Let $G = \langle X; r \rangle$ be a one-relator group with torsion, let Y be a subset of X which omits at least one generator occuring in r, and let H be the subgroup of G generated by Y. Then H is malnormal in G.

Let us quote a few more known facts about malnormal subgroups:

- There exist hyperbolic groups for which there is no algorithm to decide which finitely generated subgroups are malnormal [6].
- If H is a finitely generated subgroup of a finitely generated free group F, the malnormal closure K of H in F has been investigated in [23]; in particular, K is finitely generated (part of Theorem 13.6 in [23]).
- Other papers on malnormal groups include [4] and [24].

4. Almost malnormal subgroups

A subgroup H of a group G is *almost malnormal* if $gHg^{-1} \cap H$ is finite for all $g \in G$ with $g \notin H$, equivalently if the following condition holds: for any pair of distinct points $x, y \in G/H$, the subgroup $\{g \in G \mid gx = x, gy = y\}$ is finite.

Note that, inside torsion-free groups, almost malnormal subgroups are preisely the malnormal subgroups.

Almost malnormal subgroups are important in relatively hyperbolic groups. Without entering much details, we remind the following snippets. Let G be a group and $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G. There is a notion of *finite relative presentation of* G *with respect to* $\{H_\lambda\}_{\lambda\in\Lambda}$, say $\langle X, \{H_\lambda\}_{\lambda\in\Lambda} \mid R = 1 \ \ \forall R \in \mathcal{R}\rangle$, where X is a finite subset of G and $\mathcal{R}$ a finite subset of $(*_{\lambda\in\Lambda}H_\lambda) * F_X$ (where F_X stands for the free group on X), and of a notion of *relative Dehn funtion* $f : \mathbf{N} \longrightarrow \mathbf{N}$ *with respect to* $\{H_\lambda\}_{\lambda\in\Lambda}$ for such a presentation (such an f exists for *some* presentations only). We refer to [33], of which we quote now part of Proposition 2.36:

PROPOSITION 13. — *Let G and $\{H_\lambda\}_{\lambda\in\Lambda}$ be as above. Assume that*

(a) *G has a finite relative presentation with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$;*

(b) *there exists a relative Dehn function for this presentation.*

Then H_λ is almost malnormal in G for all $\lambda \in \Lambda$.

The group G is *hyperbolic relatively to* $\{H_\lambda\}_{\lambda\in\Lambda}$ if (a) and (b) above hold, and if moreover the corresponding relative Dehn function is linear. A motivating example is that of the fundamental group $G = \pi_1(M)$ of a complete finite-volume Riemannian manifold with pinched negative sectional curvature, with $\{H_\lambda\}_{\lambda\in\Lambda}$ the collection of the cusp subgroups [12]. Also, a Gromov hyperbolic group is hyperbolic relatively to the collection reduced to the trivial subgroup $\{e\}$.

There are various conditions which ensure an abundance of almost malnormal subgroups. For example, from [35, Theorem 5.6 and Lemma 5.2]:

Example 14 (an ℓ^2-criterion). — Consider a countable group G with non-vanishing first ℓ^2-Betti number $\beta_1^{(2)}(G) \neq 0$, and an infinite subgroup K of G with $\beta_1^{(2)}(K) < \beta_1^{(2)}(G)$ (for example $K \simeq \mathbf{Z}$). Then there exists a proper subgroup H of G which contains K and which is almost malnormal.

Almost malnormal subgroups appear in criteria for residual finiteness, in several papers by Daniel T. Wise, such as [42, 43, 44, 45]. Here is a result from the first of these:

the free product of two virtually free groups amalgamating a finitely generated almost malnormal subgroup is residually finite.

The malnormality condition is necessary! indeed: *there exists a free group F and a subgroup E of finite index such that the amalgamated product $F *_E F$ is an infinite simple group* [8, Theorem 5.5].

5. Comparison with malnormal subgroups of finite groups

Let G be a Frobenius group (as defined in Section 2) and H a malnormal subgroup of G; we assume that $H \neq \{e\}$ and $H \neq G$. Let N denote the *Frobenius kernel*, which is by definition the union of $\{e\}$ and of the complement in G of $\bigcup_{g\in G} gHg^{-1}$; observe that $N \smallsetminus \{e\}$ is the set of elements in G without any fixed point on G/H. The subgroup H is called the *Frobenius complement.*

For the case of a *finite group* G, let us quote the following three important results, for which we refer to Theorems V.7.6, V.8.7, and V.8.17 in [20]; see also [1] or [11].

(F1) *N is a normal subgroup of G, its size $|N|$ coincides with the index $[G : H]$, and G is a semi-direct product $N \rtimes H$* (Frobenius [14]); *moreover* $|N| \equiv 1 \pmod{|H|}$.

(F2) *N is a nilpotent group; moreover, if H is of even order, then N is abelian* (Thompson [40, 41]).

(F3) *Let H' be another malnormal subgroup in G, neither $\{e\}$ nor G, and let N' be the corresponding Frobenius complement; then H' is conjugate to H and $N' = N$.*

Moreover, $N' = N$ coincides with the "Fitting subgroup" of G, i.e. the largest nilpotent normal subgroup of G.

There are known non-trivial examples showing that H need not be solvable, and that N need not be abelian.

These facts do not carry over to infinite groups, as already noted in several places including [10] and Page 90 in [11].

In what follows, G is a group with malnormal subgroup H, neither $\{e\}$ nor G, and Frobenius complement N.

5.1. **N need not be a subgroup of G.** For example, let K be a non-trivial knot in the 3-sphere, G_K its group, and P_K its peripheral subgroup. Assume that K is prime, and neither a torus knot nor a cable knot, so that P_K is malnormal in G_K [18]. Since the abelianisation of G_K is $\mathbf{Z}$ (by Poincaré duality), the Frobenius kernel is not a subgroup (otherwise $P_K \simeq \mathbf{Z}^2$ would be a quotient of $G_K^{\mathrm{ab}} \simeq \mathbf{Z}$, which is preposterous).

Another example is provided by H, malnormal in $G = H * K$, with H and K non-trivial and H not of order 2 (see Proposition 2.vi). Again, the Frobenius kernel is not a subgroup; indeed, for $h_1, h_2 \in H \smallsetminus \{e\}$ with $h_1 h_2 \neq e$ and $k \in K \smallsetminus \{e\}$, both $h_1 k$ and $k^{-1} h_2$ are in the Frobenius complement, but $h_1 h_2$ is not.

The example of the cyclic subgroup H generated by $x^{-1}y^{-1}xy$ in the free group G on two generators x and y, which is a malnormal subgroup of which the Frobenius complement is not a subgroup, appears on Page 51 of [27]; see also (A) in Remark 8.

5.2. **There are examples with $N = \{e\}$.** The content of this subsection was shown to us by D. Osin.

Consider one of the finitely generated torsion-free infinite group with exactly two conjugacy classes constructed in [34, Corollary 1.3], say G. Then G is a limit of an infinite sequence of torsion-free groups and epimorphisms $G_1 \to G_2 \to \cdots$, where each G_i is hyperbolic relative to a subgroup H_i and the epimorphism $G_i \to G_{i+1}$ maps H_i isomorphically onto H_{i+1}. Let H be the limit of H_i in G; it is a non-trivial subgroup of G, indeed H is not finitely generated.

The group H is malnormal in G. To check this, consider $g \in G$ and $h, k \in H \smallsetminus \{e\}$ with $g^{-1}hg = k$. The same equality holds in some G_i for some preimages of g, h, k. Since G_i is torsion-free and hyperbolic relative to H_i, the group H_i is malnormal in G_i. Hence the preimage of g in G_i belongs to H_i. Consequently $g \in H$.

Obviously the kernel N is trivial in this case as every element of G is conjugate to an element of H.

5.3. **When N is a subgroup of G, it need not be nilpotent.** This is shown by the example of the wreath product $G = S \wr \mathbf{Z}$, with S a simple group. The subgroup $H = \mathbf{Z}$ is malnormal, and the corresponding Frobenius kernel $N = \bigoplus_{i \in \mathbf{Z}} S_i$, with each S_i a copy of S, is not nilpotent. More generally, given a group H acting on a set X in such a way that $h^{\mathbf{Z}}x$ is infinite for all $h \in H$, $h \neq e$, and $x \in X$, as well as a group $S \neq \{e\}$, the permutational wreath product $G = S \wr_X H$ contains H as a malnormal subgroup, with Frobenius kernel $\bigoplus_{x \in X} S_x$.

5.4. **Malnormal subgroups need not be conjugate.** This is clear with a non-trivial free product $G = H * K$ as in Proposition 2, where H and K are both malnormal subgroups, and are clearly non-conjugate. If $G = H * K * L$, with non-trivial factors, the malnormal subgroup H is strictly contained in the malnormal subgroup $H * K$.

5.5. **On centralisers.** Let $G = N \rtimes H$ be a semi-direct product. If $X = G/H$ (as in Proposition 1) is identified with N, note that the natural action of G can be written like this: $g = mh \in G$ acts on $n \in N$ to produce $mhnh^{-1} \in N$. Consider the two following conditions, the first being as in Proposition 1:

(a) H is malnormal in G;

(f) $C_G(n) = C_N(n)$ for any $n \in N$, $n \neq e$.

Then (f) implies (a). Indeed, for any $n \in N$, $n \neq e$, Condition (f) implies $C_H(n) = H \cap C_G(n) \subset H \cap N = \{e\}$. In other words, for $h \in H$, the equality $hnh^{-1} = n$ implies $h = e$; this is Condition (d) of Proposition 1. Hence Condition (a) of the same proposition also holds.

When G is finite, then, conversely, (a) implies (f); see Theorem 6.4 in [21]. In a previous version of this paper, we asked wether this carries over to the general case. In fact, it does not; this was shown to us by Denis Osin, by a general construction, and independently by Daniel Allcock.

Here is Allcock's simple example. Consider an infinite dihedral group

$$G = \langle r, t \mid r^2 = e,\ rtr = t^{-1} \rangle$$

and its infinite cyclic subgroup T generated by t; we think of r as a half-turn and of t as a translation of the real line. Let π be the homomorphisms of G onto the group $\{1, -1\}$ of order two mapping r and t onto -1, and set $N = \ker \pi$. On the one hand, N is also an infinite dihedral group, generated by $s = rt$ and $u = t^2$, which satisfy the relations $s^2 = e$ and $sus = u^{-1}$; we denote by U the infinite cyclic subgroup of N generated by u. On the other hand, G is the semi-direct product $N \rtimes H$, where $H = \{e, r\}$ is a subgroup of order 2 which is malnormal in G. Observe that U is of index 2 in both N and T, and of index 4 in G. For any $u \in U$, $u \neq e$, we have

$$U = C_N(u) \neq C_G(u) = T.$$

Thus Condition (a) is satisfied, but Condition (f) is not.

More generally, Denis Osin has shown us that the two centralisers which appear in Condition (f) can be very different from each other. We reproduce his construction in the following appendix.

Appendix: A construction by Denis Osin

Observe that for any split extension $G = N \rtimes H$ and for any element $z \in N$, the group $C_N(z)$ is normal in $C_G(z)$, and $C_G(z)/C_N(z)$ is isomorphic to a subgroup of H. It turns out that *every* subgroup of H can be realized as $C_G(z)/C_N(z)$ for some $z \in N$ and $G = N \rtimes H$ with H malnormal. Below we prove this for finitely generated torsion-free groups. The proof of the general case is a bit longer. It is based on the same idea but uses some additional technical results about van Kampen diagrams and small cancellation quotients of relatively hyperbolic groups.

Theorem 15. — *For any finitely generated torsion-free group H and any finitely generated subgroup $Q \leqslant H$, there exists a group N, a split extension $G = N \rtimes H$, and a nontrivial element $z \in N$ such that H is malnormal in G and $C_G(z)/C_N(z) \simeq Q$.*

To prove the theorem we will need some tools from small cancellation theory over relatively hyperbolic groups. Let G be a group hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. An element of G is *loxodromic* if it is not conjugate to an element of $\bigcup_{\lambda \in \Lambda} H_\lambda$ and has infinite order. A group is *elementary* if it contains a cyclic subgroup of finite index. It is proved in [32] that, for every loxodromic element $g \in G$, there is a unique maximal elementary subgroup $E_G(g) \leqslant G$ containing g. Two loxodromic elements f, g are *commensurable* (in G) if f^k is conjugate to g^l in G for some non-zero k, l. A subgroup S of G is called *suitable* if it contains two non-commensurable loxodromic elements g, h such that $E_G(g) \cap E_G(h) = \{1\}$.

The next result follows from [34, Theorem 2.4] and its proof.

Theorem 16. — *Let G_0 be a group hyperbolic relative to a collection $\{H_\lambda\}_{\lambda \in \Lambda}$ and S a suitable subgroup of G_0. Then for every finite subset $T \subset G_0$, there exists a set of elements $\{s_t \mid t \in T\} \subset S$ such that the following conditions hold.*

(a) *Let* $G = \langle G_0 \mid t = s_t, t \in T \rangle$. *Then the restriction of the natural homomorphism* $\varepsilon\colon G_0 \to G$ *to every* H_λ *is injective.*
(b) G *is hyperbolic relative to* $\{\varepsilon(H_\lambda)\}_{\lambda\in\Lambda}$.
(c) *If* G_0 *is torsion-free, then so is* G.

Proof of Theorem 15. — Set
$$G_0 = (\langle z \rangle \times Q) * \langle a, b \rangle * H.$$
Clearly G_0 is hyperbolic relative to the collection $\{\langle z\rangle \times Q, H\}$. Let X and Y be finite generating sets of Q and H, respectively. We fix an isomorphic embedding $\iota\colon Q \to H$. Without loss of generality we can assume that $\iota(X) \subseteq Y$.

It is easy to see that $S = \langle a, b\rangle$ is a suitable subgroup of G_0. Indeed a and b are not commensurable in G_0 and $E_{G_0}(a) \cap E_{G_0}(b) = \{1\}$. We apply Theorem 16 to the finite set
$$T = \{z\} \cup \{a^y, b^y \mid y \in Y \cup Y^{-1}\} \cup \{x^{-1}\iota(x) \mid x \in X\}.$$
Let G be the corresponding quotient group. For simplicity we keep the same notation for elements of G_0 and their images in G. Part (a) of Theorem 16 also allows us to identify the subgroups $\langle z\rangle \times Q$ and H of G_0 with their (isomorphic) images in G.

Let N be the image of S in G. Note that, in the quotient group G, we have $t \in N$ for every $t \in T$. In particular we have $z \in N$ and $x^{-1}\iota(x) \in N$ for all $x \in X$. Hence the group G is generated by $\{a, b\} \cup Y$. Since $a^y, b^y \in N$ for all $y \in Y \cup Y^{-1}$, the subgroup N is normal in G and $G = NH$. Using Tietze transformations it is easy to see that the map $a \mapsto 1$ and $b \mapsto 1$ extends to a retraction $\rho\colon G \to H$ such that $\rho|_Q \equiv \iota$. In particular, $H \cap N = \{e\}$ and hence $G = N \rtimes H$.

Since G_0 is torsion-free, so is G by Theorem 16 (c). By Theorem 16 (b) and Proposition 13, the subgroups H and $\langle z\rangle \times Q$ are malnormal in G. In particular, $C_G(z) = \langle z\rangle \times Q$. Since $z \in N$ and $\rho|_Q \equiv \iota$, we obtain $\rho(z^n, q) = \iota(q)$ for every $(z^n, q) \in \langle z\rangle \times Q$. Hence
$$C_N(z) = C_G(z) \cap N = C_G(z) \cap \operatorname{Ker}\rho = \langle z\rangle \times \{e\}.$$
Therefore $C_G(z)/C_N(z) \simeq Q$. □

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