



CONFLUENTES MATHEMATICI LYON

Jérémie GUILHOT, Cédric LECOUEY, and Pierre TARRAGO

Basics on positively multiplicative graphs and algebras

Volume 18 (2026), p. 31–69.

<https://doi.org/10.5802/cml.104>

© Les auteurs, 2026.

Les articles des *Confluentes Mathematici* sont mis à disposition sous la license Creative Commons Attribution-NonCommercial-NoDerivs (CC-BY-NC-ND) 4.0

<http://creativecommons.org/licenses/by-nc-nd/4.0/>



Publication membre du centre
Mersenne pour l'édition scientifique ouverte

<http://www.centre-mersenne.org/>

e-ISSN : 1793-7434

BASICS ON POSITIVELY MULTIPLICATIVE GRAPHS AND ALGEBRAS

JÉRÉMIE GUILHOT[†], CÉDRIC LECOUEY, AND PIERRE TARRAGO

Abstract. An oriented graph is said positively multiplicative when its adjacency matrix A embeds in a matrix algebra admitting a basis $\mathbf{b}$ with nonnegative structure constants in which the matrix of the multiplication by A coincides with A . The goal of this paper is to present basic properties of this notion and explain, through various simple examples, how it relates to highly non trivial problems like the combinatorial description of fusion rules, the description of the minimal boundary of graded graphs or the study of random walks on alcove tilings.

1. INTRODUCTION

The aim of this paper is to study positively multiplicative algebras and positively multiplicative graphs, two notions which provide a unified background to many problems at the interaction between algebra, combinatorics and probability. Given a finite set Z of indeterminates and a field $\mathbb{K}$ equal to $\mathbb{R}$ or $\mathbb{C}$, a positively multiplicative algebra is a unital $\mathbb{K}[Z^{\pm 1}]$ -algebra $\mathcal{A}$ admitting a basis $\mathbf{B}$ such that $1 \in \mathbf{B}$ and the structure constants of $\mathcal{A}$ with respect to $\mathbf{B}$ lie in $\mathbb{R}_+[Z^{\pm 1}]$, the subset of $\mathbb{K}[Z^{\pm 1}]$ of Laurent polynomials with nonnegative real coefficients. The set Z can be empty and then it is just required that the structure constants are nonnegative. This class of algebras contains the fusion algebras [6], the group algebras or the character algebras associated to finite groups, the character algebras of simple Lie algebras, but also homology or cohomology rings defined from algebraic varieties related in particular to Schubert calculus or to the geometry of affine Grassmannians, see [15, 16].

Due to the positivity of the structure constants of $\mathcal{A}$ with respect to $\mathbf{B}$, the matrix of the multiplication by any positive linear combination s of elements in $\mathbf{B}$ expressed in the basis $\mathbf{B}$ can be considered as an adjacency matrix of an oriented weighted directed graph with set of vertices in bijection with the set $\mathbf{B}$. Conversely, starting from a finite oriented graph Γ with weights in $\mathbb{R}_+[Z^{\pm 1}]$, it is a natural question to look for an underlying positively multiplicative algebra structure. We will say that Γ is positively multiplicative at a vertex v_{i_0} when its adjacency matrix A_Γ can be embedded in a matrix algebra which is positively multiplicative with respect to a basis $\mathbf{B}$ such that (1) the i_0^{th} element of $\mathbf{B}$ is equal to the identity and (2) the matrix of the multiplication by A_Γ expressed in the basis $\mathbf{B}$ is the matrix A_Γ itself. One

2020 *Mathematics Subject Classification*: 05E10, 15B51, 60J10.

Keywords: Associative algebras, graphs, harmonic functions, fusion rules.

(*) The first author was supported by the Agence Nationale de la Recherche funding ANR JCJC Project ANR-18-CE40-0001 and by the Australian Research Council discovery project DP200100712. The three authors are supported by the Agence Nationale de la Recherche funding ANR CORTIPOM 21-CE40-001.

can notice here that we have two types of constraints for a graph to be positively multiplicative which are of very different nature:

- (1) The first one relates to linear algebra: does there exist a matrix algebra $\mathcal{A}$ with a basis $\mathbf{B}$ containing 1 and such that the matrix of the multiplication by A is A ? If so, we will say that Γ is multiplicative.
- (2) The second one is of geometric nature: does there exist an algebra as in (1) whose cone $\mathcal{C}(\mathbf{B})$ of nonnegative linear combinations of elements in $\mathbf{B}$ is stable by multiplication?

We shall see that the answer to Question (1) is most of the time positive when the graph Γ is of maximal dimension, that is when the algebra $\mathbb{K}[\mathbb{Z}^{\pm 1}][A_\Gamma]$ has dimension the number of vertices in Γ . In this case, the algebra $\mathcal{A}$ and its basis $\mathbf{B}$ are unique when they exist and can be computed by elementary linear algebra techniques. In contrast, it is highly non trivial to find general conditions sufficient to guarantee a positive answer to Question (2). Moreover, even when a graph Γ is positively multiplicative, it is often very difficult to get a combinatorial description of the structure constants of the basis $\mathbf{B}$, for example by counting paths in Γ . Depending on the situation, this problem may be equivalent to the description of tensor product multiplicities in representation theory, the determination of combinatorial fusion rules or that of computing the structure constants in the homology rings of affine Grassmannians. For tensor product multiplicities of Kac–Moody algebras, an elegant description exists in terms of the Littelmann path model [20] or the combinatorics of crystal graphs [13]. For the symmetric groups, these multiplicities (Kronecker coefficients) are much less understood. The determination of combinatorial descriptions for the fusions rule in conformal field theory [1] or the structure constants in the homology rings of affine Grassmannians [15] is still an unsolved problem.

The goal of the paper is to propose a unified approach in the study of positively multiplicative graphs independent of the algebraic or geometric context where they naturally appear. The results, presented in an expository style, gather basic facts on these notions that we will need in future works and for which we did not find explicit references in the literature. Such results will be used in particular in [10, 11] to study respectively random walks on alcoves tilings and convergences of random particle systems on a discrete circle. Although they use quite elementary tools, we believe that they deserved to be written down. We will explain in particular, when the graph is of maximal dimension, how to decide whether a graph is multiplicative and then provide simple procedures to compute the associated basis. When a graph Γ is positively multiplicative, we will also give a general construction (called expansion) yielding an infinite graded graph Γ_e defined from Γ for which it is easy to get a complete description of the extremal positive harmonic functions. These functions are essential tools in the study of random walks on graphs or alcoves tilings (see for example [14, 19]).

In each of the following sections, we have chosen to present numerous examples of the different notions we introduce. We hope they will be sufficiently helpful for the reader. The paper itself is organised as follows. Section 2 is devoted to the notion of positively multiplicative algebras and its connection with fusion algebras. In Section 3, we define and study multiplicative graphs Γ . When Γ is of maximal dimension, we give a simple procedure to check whether or not it is multiplicative

at a given vertex and then to compute its associated basis $\mathbf{B}$. The positively multiplicative graphs are presented in Section 4. In Section 5, we introduce the column and row Kirillov–Reshetikhin crystals of affine type A. These are two important examples of positively multiplicative graphs whose definition is elementary but with structure constants of high combinatorial complexity. In Section 6, we detail the expansion procedure and explain how to get a description of the minimal boundary (i.e. of the extremal positive harmonic functions) on each extended graph coming from a positively multiplicative graph. Their generalisations and their study will be our main objective in [11]. In Section 7, we study commutative positively multiplicative (possibly infinite dimensional) algebras over $\mathbb{C}$ and show that the subset of elements of the distinguished basis $\mathbf{B}$ that are generalised permutations has the structure of an abelian group. In the case of a positively multiplicative algebra over $\mathbb{C}$ coming from a positively multiplicative graph, this subset is precisely the set of vertex v of Γ for which Γ is positively multiplicative at v .

2. POSITIVELY MULTIPLICATIVE ALGEBRAS

We start by introducing some notation that we will use all along the paper:

- $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$,
- $\mathbf{Z} = \{z_1, \dots, z_N\}$ with $N \in \mathbb{N}$ is a set (possibly empty) of formal indeterminates,
- $\mathbb{K}[\mathbf{Z}^{\pm 1}]$ is the Laurent polynomial ring in $\mathbf{Z}$ over $\mathbb{K}$,
- $\mathbb{R}_+[\mathbf{Z}^{\pm 1}]$ is the set of Laurent polynomials with nonnegative real coefficients,
- $\mathbb{K}(\mathbf{Z}^{\pm 1})$ is the ring of fractions of $\mathbb{K}[\mathbf{Z}^{\pm 1}]$.

In this section, $\mathcal{A}$ denotes a unital associative algebra over $\mathbb{K}[\mathbf{Z}^{\pm 1}]$.

DEFINITION 2.1. — The algebra $\mathcal{A}$ is said to be *positively multiplicative* (PM for short) when it admits a $\mathbb{K}[\mathbf{Z}^{\pm 1}]$ -basis $\mathbf{B} = \{\mathbf{b}_i, i \in I\}$ indexed by a countable set I satisfying the following two conditions:

- (1) $1 \in \mathbf{B}$,
- (2) for any $i, j \in I$, we have $\mathbf{b}_i \mathbf{b}_j = \sum_{k \in I} c_{i,j}^k \mathbf{b}_k$ with $c_{i,j}^k \in \mathbb{R}_+[\mathbf{Z}^{\pm 1}]$.

When such a basis $\mathbf{B}$ exists, we say that it is a positively multiplicative basis for $\mathcal{A}$ (PM-basis for short) and that $\mathcal{A}$ is positively multiplicative with respect to $\mathbf{B}$. The algebra $\mathcal{A}$ is said to be *strongly positively multiplicative* when it is positively multiplicative and for any $(j, k) \in I^2$ there exists at least one index i in I such that $c_{i,j}^k \neq 0$.

Examples 2.2.

- (1) The algebra $\mathbb{K}^n$ is PM with respect to the basis $\mathbf{B} = \{\mathbf{b}_i \mid i = 1, \dots, n\}$ with $\mathbf{b}_1 = (1, \dots, 1)$ and $\mathbf{b}_i = e_{i-1}$ for $i = 2, \dots, n$ where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{K}^n$. It is easy to construct other PM-basis for $\mathbb{K}^n$, for instance one could change $\mathbf{b}_2$ to $(0, 1, \dots, 1)$ in the basis $\mathbf{B}$. In general a PM basis is not unique.
- (2) Any finite-dimensional $\mathbb{C}$ -algebra $\mathcal{A}$ generated by one element is PM. Indeed, if we set $\mathcal{A} = \mathbb{C}[a]$ and denote by μ_a the minimal polynomial of a , we can factorise μ_a

$$\mu_a(X) = \prod_{i=1}^k (X - \alpha_i)^{m_i}$$

where $\alpha_1, \dots, \alpha_k$ are the k distinct complex roots of μ_a . Then, using the Chinese remainder theorem, we get

$$\mathcal{A} \cong \prod_{i=1}^k \mathbb{C}[X]/(X - \alpha_i)^{m_i}.$$

The algebra on the right hand side is PM because each algebra $\mathbb{C}[X]/(X - \alpha_i)^{m_i}$ is with respect to the basis $\{1, X - \alpha_i, \dots, (X - \alpha_i)^{m_i-1}\}$.

- (3) Regarded as a two-dimensional $\mathbb{R}$ -algebra, the field $\mathbb{C}$ of complex numbers is not positively multiplicative. Indeed if such a basis $(1, z)$ existed, we could assume that $z^2 = a + bz$ with $(a, b) \in \mathbb{R}_{\geq 0}^2$ and get that z is a non real root of the polynomial $X^2 - bX - a$. Since its discriminant is positive, this yields a contradiction.
- (4) Let $\mathcal{A}$ be a finite-dimensional commutative subalgebra of $\text{Mat}_n(\mathbb{C})$ (the algebra of $n \times n$ complex matrices) stable by the adjoint operation. Then, each matrix A in $\mathcal{A}$ is diagonalisable because $AA^* = A^*A$. Moreover, since $\mathcal{A}$ is commutative, there is a common basis of diagonalisation. Therefore, the algebra $\mathcal{A}$ is isomorphic to $\mathbb{C}^n$ which is PM.
- (5) Consider a polynomial $P(X) = X^n - a_{n-1}X^{n-1} - \dots - a_0$ such that $(a_{n-1}, \dots, a_0) \in \mathbb{R}_+^n$ and A its companion matrix. Then $\mathbb{K}[A]$ is a n -dimensional PM subalgebra of $\text{Mat}_n(\mathbb{K})$ with respect to the basis $\{A^k \mid k = 0, \dots, n-1\}$.
- (6) For any finite group G , the group algebra $\mathbb{C}[G]$ is a PM algebra with respect to the basis $\{g \mid g \in G\}$.
- (7) For any finite group G , its complex character ring $\mathcal{R}[G]$ is a commutative PM algebra with respect to the basis of irreducible characters. Observe that $\mathcal{R}[G]$ is also isomorphic to $\mathbb{C}^n$ just by considering the basis of indicator functions associated to the conjugacy classes of G .
- (8) Homology and cohomology rings associated to algebraic varieties are other important examples of PM algebras.

Examples (6)–(7) above are particular cases of fusion algebras (see [6, Section 5.1] for a detailed introduction to fusion algebras). A fusion algebra is an algebra over $\mathbb{C}$ (here we take $Z = \emptyset$) with a positively multiplicative basis $B = \{b_i, i \in I\}$ and an involutive anti-automorphism $*$ (that induces an involution $i \mapsto i^*$ on I by setting $b_i^* = b_{i^*}$) that satisfy $c_{i,j}^k = c_{i^*,j^*}^{k^*}$ for all $i, j, k \in I$.

PROPOSITION 2.3. — *Every fusion algebra is strongly positively multiplicative.*

Proof. — Let $\mathcal{A}$ be a fusion algebra with basis $B = \{b_i, i \in I\}$. Since $*$ is an anti-automorphism, we have $c_{i,j}^k = c_{j^*,i^*}^{k^*} = c_{j,k^*}^{i^*}$ using the property of fusion algebras. Assume there exists $(j, k) \in I^2$ such that $c_{i,j}^k = 0$ for all $i \in I$. By the previous argument, we have $c_{j,k^*}^{i^*} = 0$ for all $i \in I$ and thus $c_{j,k^*}^i = 0$ for all $i \in I$ since the map $i \mapsto i^*$ is a bijection on I . We get the equality $b_j b_{k^*} = 0$. This implies that $b_{j^*} b_j b_{k^*} b_k = 0$. Now observe that $c_{j^*,j}^{i_0} = c_{j,i_0}^j = 1$ and similarly $c_{k^*,k}^{i_0} = 1$, where $b_{i_0} = 1 \in B$. Since the structure constants of the fusion algebra $\mathcal{A}$ are nonnegative, the coefficient of 1 in the product $b_{j^*} b_j b_{k^*} b_k$ is positive. Hence $b_{j^*} b_j b_{k^*} b_k \neq 0$ and we get the desired contradiction. $\square$

PROPOSITION 2.4. — *Let $\mathcal{A}$ be a finite-dimensional, commutative and positively multiplicative algebra. Then $\mathcal{A}$ is integral over $\mathbb{K}[Z^{\pm 1}]$.*

Proof. — Let $B = \{b_i, i \in I\}$ be a PM-basis for $\mathcal{A}$. Since $\mathcal{A}$ is generated over $\mathbb{K}[Z^{\pm 1}]$ by the elements of B , it suffices to show that the elements b_i are integral over $\mathbb{K}[Z^{\pm 1}]$. For any such b_i , the matrix M of the multiplication by b_i in $\mathcal{A}$ in the basis B has coefficients in $\mathbb{K}[Z^{\pm 1}]$ (in fact in $\mathbb{R}_+[Z^{\pm 1}]$). Thus the characteristic polynomial of M has coefficients in $\mathbb{K}[Z^{\pm 1}]$. This implies that its minimal polynomial, which is also the minimal polynomial of b_i also has coefficients in $\mathbb{K}[Z^{\pm 1}]$. Hence b_i is integral over $\mathbb{K}[Z^{\pm 1}]$. $\square$

Remark 2.5. — Given a PM-algebra $\mathcal{A}$ with $Z \neq \emptyset$, any morphism $\theta : \mathbb{K}[Z^{\pm 1}] \rightarrow \mathbb{K}$ defines a specialisation of $\mathcal{A}$ that we shall denote $\mathcal{A}_\theta$. When $\theta(z) \in \mathbb{R}_{>0}$ for any $z \in Z$, the algebra $\mathcal{A}_\theta$ remains positively multiplicative. Further if $\mathcal{A}$ is strongly positively multiplicative, so is $\mathcal{A}_\theta$.

3. MULTIPLICATIVE GRAPH

Let $\Gamma = (V, E, \omega)$ be a finite oriented weighted graph where V is the finite set of vertices, E is the set of edges E and ω is the edge weight function taking values in some algebra. We will assume that Γ does not have multiple edges, so that the set of edges can be viewed as a subset of $V \times V$: a pair $(v, v') \in V \times V$ represents an edge starting at v and ending at v' . Finally, it will be convenient to extend the edge weight function ω to the full set $V \times V$ simply by setting $\omega(v, v') = 0$ if there is no edge from v to v' . Note that with this setting, the sum of the weights of all edges starting at a given vertex v is equal to $\sum_{v' \in V} \omega(v, v')$.

Unless explicitly specified otherwise, all the graphs considered in this paper are finite oriented weighted graph with edge weight function taking values in $\mathbb{K}[Z^{\pm 1}]$. We denote by $\text{Graph}_n(\mathbb{K}[Z^{\pm 1}])$ the set of all such graphs that contains n vertices. In most examples and applications the indeterminates in $Z^{\pm 1}$ will be eventually specialised to positive reals.

In this section, Γ denotes a graph in $\text{Graph}_n(\mathbb{K}[Z^{\pm 1}])$ whose set of vertices is $\{v_1, \dots, v_n\}$ and edge weight function ω . We denote by $A_\Gamma = (a_{i,j})_{1 \leq i,j \leq n}$ the adjacency matrix of Γ (which lies in $\text{Mat}_n(\mathbb{K}[Z^{\pm 1}])$), that is the coefficient $a_{i,j}$ is equal to $\omega(v_j, v_i)$. The adjacency algebra $\mathcal{A}_\Gamma$ of Γ is the algebra $\mathbb{K}(Z^{\pm 1})[A_\Gamma]$ of polynomials in A_Γ with coefficients in $\mathbb{K}(Z^{\pm 1})$. Note that the adjacency matrix and the adjacency algebra of Γ are defined up to an ordering of the vertices of Γ and thus up to a conjugation of A_Γ by a permutation matrix.

In this section, we introduce the notion of multiplicative graph which lies at the heart of the paper. We then study this notion in the case where the graph is of maximal dimension, that is when the adjacency algebra is of dimension the number of vertices of Γ , where we give an explicit criterion to decide whether or not the graph is multiplicative.

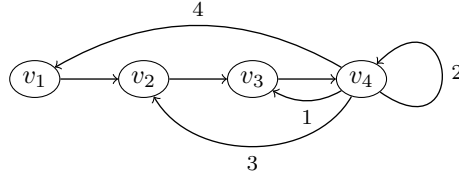
3.1. Multiplicative graph, roots and matrix realisation. Given any finite dimensional associative algebra $\mathcal{A}$, a basis B of $\mathcal{A}$ and $x \in \mathcal{A}$, we denote by m_x the endomorphism of $\mathcal{A}$ defined by $a \mapsto xa$ and by $\text{Mat}_B(m_x)$ the matrix of m_x in the basis B . We write $B[i]$ for the i^{th} element of the basis B .

DEFINITION 3.1.

- (1) We say that $(\mathcal{A}, \mathbf{B})$ is a matrix realisation of Γ if $\mathcal{A}$ is a subalgebra of $\text{Mat}_n(\mathbb{K}(\mathbb{Z}^{\pm 1}))$ of dimension n containing $\mathcal{A}_\Gamma$ and $\mathbf{B}$ is a basis of $\mathcal{A}$ such that $A_\Gamma = \text{Mat}_{\mathbf{B}}(m_{A_\Gamma})$.
- (2) We say that Γ is multiplicative at v_{i_0} if there exists a matrix realisation $(\mathcal{A}, \mathbf{B})$ of Γ with $\mathbf{B}[i_0] = I_n$.
- (3) We say that Γ is multiplicative if there exists a vertex v_{i_0} such that Γ is multiplicative at v_{i_0} .
- (4) The set $\mathcal{R}_\Gamma$ of roots of Γ is the set of vertices v_{i_0} such that Γ is multiplicative at v_{i_0} .

We will see in Example 3.15 that it is possible to have a graph Γ that admits a matrix realisation but which is not multiplicative. Note that if $(\mathcal{A}, \mathbf{B})$ is a matrix realisation of Γ then $(\mathcal{A}, \mathbf{B}x)$ is also a matrix realisation of Γ for all invertible elements $x \in \mathcal{A}^\times$. In particular, a graph that admits a matrix realisation $(\mathcal{A}, \mathbf{B})$ such that there exists $\mathbf{b} \in \mathbf{B} \cap \mathcal{A}^\times$ is multiplicative. Indeed the pair $(\mathcal{A}, \mathbf{B}\mathbf{b}^{-1})$ is a matrix realisation of Γ and $\mathbf{B}\mathbf{b}^{-1}$ contains the identity.

Example 3.2. — Let $P = X^n - a_{n-1}X^{n-1} - \dots - a_0 \in \mathbb{K}[X]$ and let A be the companion matrix of P . Let Γ be the weighted graph with n vertices $\{v_1, \dots, v_n\}$ and adjacency matrix A_Γ equal to A (here $\mathbb{Z} = \emptyset$). For example, if $P = X^4 - 2X^3 - X^2 - 3X - 4$, the graph Γ and the matrix A_Γ are given by



$$A_\Gamma = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

The minimal polynomial of A_Γ is P so that $\mathbb{K}[A]$ is of dimension n and the pair $(\mathbb{K}[A], \mathbf{B})$ where $\mathbf{B} = \{I_n, A_\Gamma, \dots, A_\Gamma^{n-1}\}$ is a matrix realisation of Γ . The graph Γ is multiplicative at v_1 .

PROPOSITION 3.3. — *The graph Γ is multiplicative at v_{i_0} if and only if there exists a n -dimensional associative algebra $\mathcal{A}$ over $\mathbb{K}(\mathbb{Z}^{\pm 1})$, an element $x \in \mathcal{A}$ and a basis $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of $\mathcal{A}$ such that $\mathbf{b}_{i_0} = 1$ and $\text{Mat}_{\mathbf{B}}(m_x) = A_\Gamma$.*

Proof. — Assume that there exists such an algebra $\mathcal{A}$. The map

$$\begin{aligned} M_{\mathbf{B}} : \mathcal{A} &\longrightarrow \text{Mat}_n(\mathbb{K}(\mathbb{Z}^{\pm 1})) \\ y &\longmapsto \text{Mat}_{\mathbf{B}}(m_y) \end{aligned}$$

is an injective morphism of algebras. Then $M_{\mathbf{B}}(\mathcal{A})$ is a subalgebra of $\text{Mat}_n(\mathbb{K}(\mathbb{Z}^{\pm 1}))$. We claim that $(M_{\mathbf{B}}(\mathcal{A}), \mathbf{B})$ where $\mathbf{B} = \{M_{\mathbf{B}}(\mathbf{b}_i) \mid i = 1, \dots, n\}$ is a matrix realisation

of Γ . First, $M_B(x) = A_\Gamma$ by construction, so that $A_\Gamma \subset M_B(\mathcal{A})$. Then, we have $xb_j = m_x(b_j) = \sum_{i=1}^n a_{i,j} b_i$ so that $m_{xb_j} = \sum a_{i,j} m_{b_i}$. Hence,

$$\begin{aligned} M_B(x) M_B(b_j) &= A_\Gamma M_B(b_j) = \text{Mat}_B(m_x) \text{Mat}_B(m_{b_j}) \\ &= \text{Mat}_B(m_x m_{b_j}) \\ &= \text{Mat}_B(m_{xb_j}) \\ &= \sum_{i=1}^n a_{i,j} \text{Mat}_B(m_{b_i}) \\ &= \sum_{i=1}^n a_{i,j} M_B(b_i). \end{aligned}$$

Since $b_{i_0} = 1$, we have $\text{Mat}_B(m_{b_{i_0}}) = I_n$ and we get that Γ is multiplicative at v_{i_0} . The converse is obvious by definition of a multiplicative graph. $\square$

Example 3.4. — Let Γ be the Cayley graph of the symmetric group $\mathfrak{S}_n$ associated to the transpositions $\tau_i = (1, i)$ with $i = 2, \dots, n$. That is, the set of vertices of Γ is $\{\sigma \mid \sigma \in \mathfrak{S}_n\}$ and there is an edge (with weight 1) between σ and σ' if and only if $\sigma' = \tau_i \sigma$ for some i . We show that Γ is multiplicative at $\mathbf{e}$ where $\mathbf{e}$ denotes the identity of $\mathfrak{S}_n$. Let $\mathcal{A} = \mathbb{K}[\mathfrak{S}_n]$ be the group algebra of the symmetric group $\mathfrak{S}_n$ with basis $B = \{e_\sigma \mid \sigma \in \mathfrak{S}_n\}$ and let $x = \sum_{2 \leq i \leq n} e_{(1,i)}$. Then we have $A_\Gamma = \text{Mat}_B(m_x)$ hence showing that Γ is multiplicative at $\mathbf{e}$. In fact, Γ is multiplicative at any vertex ρ by considering the basis $B e_{\rho^{-1}}$.

Given $i, j, k \in \{1, \dots, n\}$, let $m_{k \rightarrow i}^j \in \mathbb{K}[\mathbb{Z}^{\pm 1}]$ be the sum of the weights of all paths of length j that start at v_k and finish at v_i . For $i_0 \in \{1, \dots, n\}$, we denote by M_{i_0} the matrix with coefficients $(m_{i_0 \rightarrow i}^j)_{1 \leq i, j \leq n}$.

Assume that Γ is multiplicative graph at v_{i_0} and let $(\mathcal{A}, B)$ be a matrix realisation of Γ where $B = \{b_1, \dots, b_n\}$ and $b_{i_0} = I_n$. By induction, for any integer $j \geq 0$ and $k \in \{1, \dots, n\}$ we have

$$A_\Gamma^j b_k = \sum_{i=1}^n m_{k \rightarrow i}^j b_i. \quad (3.1)$$

We denote by $c_{i,j}^k$ where $i, j, k \in \{1, \dots, n\}$ the structure constants of $\mathcal{A}$ with respect to B , that is

$$b_i b_j = \sum_{k=1}^n c_{i,j}^k b_k.$$

LEMMA 3.5. — *Assume that Γ is strongly connected. Then, for all $1 \leq j, k \leq n$, there exists at least one $i \in \{1, \dots, n\}$ such that $c_{i,j}^k \neq 0$.*

Proof. — Assume that we have $1 \leq j_0, k_0 \leq n$ such that $c_{i,j_0}^{k_0} = 0$ for any $i \in \{1, \dots, n\}$. This means that the right ideal $\mathcal{A} b_{j_0}$ generated by b_{j_0} in $\mathcal{A}$ is contained in $\bigoplus_{k \neq k_0} \mathbb{K}(\mathbb{Z}^{\pm 1}) b_k$. Thus for any nonnegative integer ℓ , we have $A^\ell b_{j_0} \in \bigoplus_{k \neq k_0} \mathbb{K}(\mathbb{Z}^{\pm 1}) b_k$. By (3.1), this implies that there cannot exist a path in Γ from v_{j_0} to v_{k_0} which contradicts our assumption that Γ is strongly connected. $\square$

3.2. Multiplicative graphs of maximal dimension. Recall that the adjacency algebra $\mathcal{A}_\Gamma$ of Γ is defined to be $\mathbb{K}(\mathbb{Z}^{\pm 1})[A_\Gamma]$ where $A_\Gamma \in \text{Mat}_n(\mathbb{K}[\mathbb{Z}^{\pm 1}])$ is the adjacency matrix of Γ .

DEFINITION 3.6. — We say that Γ is of *maximal dimension* if $\dim \mathcal{A}_\Gamma = n$.

Let Γ be a graph of maximal dimension and assume that there exists a matrix realisation $(\mathcal{A}, \mathbf{B})$ of Γ . Then since $\mathcal{A}$ contains A_Γ and is of dimension n , we must have $\mathcal{A} = \mathcal{A}_\Gamma$. In particular, $\mathcal{A}$ is commutative. Note also that, in this case, the centralizer of A_Γ is $\mathcal{A}_\Gamma$.

Examples 3.7.

- (1) The graph in Example 3.4 is not of maximal dimension. Indeed, if it was, the algebra $\mathbb{K}[\mathfrak{S}_n]$ would be isomorphic to a commutative algebra.
- (2) The graphs Γ constructed from a polynomial P as in Example 3.2 are of maximal dimension since the minimal polynomial of A_Γ is P and the degree of P is the number of vertices in Γ .
- (3) Let $\mathcal{A}$ be the complex character algebra of a finite group G with positively multiplicative basis the set $\mathbf{B} = \{\chi_1, \dots, \chi_n\}$ of irreducible characters. Let

$$\varphi = a_1\chi_1 + \dots + a_n\chi_n \quad \text{with} \quad (a_1, \dots, a_n) \in \mathbb{R}^n.$$

Let Γ be the graph with n vertices such that $A_\Gamma = \text{Mat}_{\mathbf{B}}(m_\varphi)$. In other words, the vertices of Γ are in bijection with the set of irreducible characters and the weights encode the tensor product multiplicities. The basis $\mathbf{B}' = \{1_{C_1}, \dots, 1_{C_n}\}$ of characteristic functions associated to the conjugacy classes of G is another positively multiplicative basis of $\mathcal{A}$. Using $\mathbf{B}'$, we see that $\mathcal{A}$ is isomorphic to the algebra $\mathbb{C}^n$. It follows that Γ is of maximal dimension if and only if φ takes distinct values on each conjugacy class. Indeed the element $x = (x_1, \dots, x_n)$ of $\mathbb{C}^n$ generates $\mathbb{C}^n$ if and only if its minimal polynomial has degree n . Since this polynomial coincides with that of the multiplication by x in $\mathbb{C}^n$ whose eigenvalues are the coordinates $x_i, i = 1, \dots, n$, this is equivalent to say that the x_i 's are pairwise distinct.

We first show that if Γ is of maximal dimension then it admits a matrix realisation.

PROPOSITION 3.8. — Assume that Γ is of maximal dimension. The set

$$\mathfrak{B} = \{\mathbf{B} \mid (\mathcal{A}_\Gamma, \mathbf{B}) \text{ is a matrix realisation of } \Gamma\}$$

is non-empty and the invertible elements $\mathcal{A}_\Gamma^\times$ of $\mathcal{A}_\Gamma$ acts transitively by left multiplication on this set.

Proof. — We prove that the set $\mathfrak{B}$ is non empty. Since $\dim \mathcal{A}_\Gamma = n$, the set $\mathbf{B}' = \{A^i, i = 0, \dots, n-1\}$ is a basis of $\mathcal{A}_\Gamma$ and $\text{Mat}_{\mathbf{B}'}(m_{A_\Gamma}) = \mathcal{C}_{\mu_{A_\Gamma}}$ where μ_{A_Γ} is the minimal polynomial of A_Γ and $\mathcal{C}_{\mu_{A_\Gamma}}$ is the companion matrix of μ_{A_Γ} . The matrices $\mathcal{C}_{A_\Gamma}$ and A_Γ have entries in $\mathbb{K}(\mathbb{Z}^{\pm 1})$ and are conjugate in $\text{Mat}_n(\mathbb{K}(\mathbb{Z}^{\pm 1}))$ since $\dim \mathbb{K}(\mathbb{Z}^{\pm 1})[A_\Gamma] = n$ is the degree of μ_{A_Γ} . Thus $A_\Gamma = P\mathcal{C}_{\mu_A}P^{-1}$ where P is an invertible matrix P with entries in $\mathbb{K}(\mathbb{Z}^{\pm 1})$. If we define $\mathbf{B}$ to be the basis of $\mathcal{A}_\Gamma$ such that the change of basis matrix from $\mathbf{B}'$ to $\mathbf{B}$ is equal to P , we then have $\text{Mat}_{\mathbf{B}}(m_{A_\Gamma}) = A_\Gamma$ as required.

We prove that $\mathcal{A}_\Gamma^\times$ acts by multiplication on $\mathfrak{B}$. Let $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathbf{B}' = \{\mathbf{b}'_1, \dots, \mathbf{b}'_n\}$ be two bases in $\mathfrak{B}$. Let $Q = (q_{i,j})_{1 \leq i,j \leq n}$ be the change of basis matrix from $\mathbf{B}'$ to $\mathbf{B}$, that is $\mathbf{b}'_j = \sum_i q_{i,j} \mathbf{b}_i$. We have $\text{Mat}_{\mathbf{B}'}(m_{A_\Gamma}) = Q^{-1} \text{Mat}_{\mathbf{B}}(m_{A_\Gamma}) Q$ so that $A_\Gamma = Q^{-1} A_\Gamma Q$ and A_Γ commutes with Q . But Γ is of maximal dimension so the centralizer of A_Γ is equal to $\mathcal{A}_\Gamma$, therefore $Q \in \mathcal{A}_\Gamma$. Let $U \in \mathbb{K}(\mathbb{Z}^{\pm 1})[X]$ be such that $Q = U(A_\Gamma)$. Since $\text{Mat}_{\mathbf{B}}(m_{A_\Gamma}) = A_\Gamma$, we get that $\text{Mat}_{\mathbf{B}}(m_Q) = Q$. But by definition, we have

$$\mathbf{b}'_j = \sum_{i=1}^n q_{i,j} \mathbf{b}_i = Q \mathbf{b}_j.$$

This shows that $\mathbf{B}$ can be obtained from $\mathbf{B}'$ by multiplying by the invertible element Q .

Finally, it is clear that if $\mathbf{B} \in \mathfrak{B}$ then $x\mathbf{B} \in \mathfrak{B}$ for all invertible elements $x \in \mathcal{A}_\Gamma^\times$. Therefore, we have a transitive action of the abelian group $\mathcal{A}_\Gamma^\times$ on $\mathfrak{B}$. $\square$

We keep the notation of the proposition and we fix $i \in \{1, \dots, n\}$. The matrices in $\{\mathbf{B}[i] \mid \mathbf{B} \in \mathfrak{B}\}$ are either all invertible or all not invertible. When they are all invertible, there exists a unique basis $\mathbf{B}$ such that $(\mathcal{A}_\Gamma, \mathbf{B})$ is a matrix realisation of Γ and $\mathbf{B}[i] = I_n$. The graph Γ is then multiplicative at v_i . When they are not invertible, the graph Γ is not multiplicative at v_i . In other word the previous proposition implies the following result.

COROLLARY 3.9. — *Assume that Γ is of maximal dimension. If Γ is multiplicative at v_{i_0} then there exists a unique matrix representation $(\mathcal{A}, \mathbf{B})$ such that $\mathbf{B}[i_0] = I_n$.*

Remark 3.10. — In the proof of the previous proposition, we have used the fact that $\text{Mat}_{\mathbf{B}}(m_{A_\Gamma}) = A_\Gamma$ implies that $\text{Mat}_{\mathbf{B}}(m_{U(A_\Gamma)}) = U(A_\Gamma)$ for each polynomial $U \in \mathbb{K}(\mathbb{Z}^{\pm 1})[X]$. In general, any n -dimensional algebra embeds in the algebra of its linear endomorphisms just by considering the multiplication by each element (which is a linear map). This embedding is not surjective (the algebras have dimensions n and n^2). We thus warn the reader that in general, the action of a linear endomorphism of $\mathcal{A}$ does not coincide with the multiplication by its matrix in the basis $\mathbf{B}$ (which does not necessarily commute with $\mathcal{A}$). This is in particular the case when we consider generalised permutation matrices associated to the basis $\mathbf{B}$.

The situation simplifies even further when the minimal polynomial μ_{A_Γ} of the adjacency matrix of Γ is irreducible and the graph Γ is strongly connected.

PROPOSITION 3.11. — *Assume that Γ is a strongly connected graph and that μ_{A_Γ} is irreducible over $\mathbb{K}(\mathbb{Z}^{\pm 1})$. Then Γ is of maximal dimension and is multiplicative at any vertex.*

Proof. — Consider the Frobenius reduction of the matrix A_Γ in the field $\mathbb{K}(\mathbb{Z}^{\pm 1})$. Denote its invariant factors by $\mu_1 = \mu_{A_\Gamma}, \mu_2, \dots, \mu_r$ with $\mu_{i+1} \mid \mu_i$ for any $i = 1, \dots, r-1$. Since μ_1 is irreducible we must have $\mu_1 = \mu_2 = \dots = \mu_r$. The matrix A_Γ is conjugate in $\mathbb{K}(\mathbb{Z}^{\pm 1})$ to a matrix B with r identical blocks equal to $\mathcal{C}_{\mu_{A_\Gamma}}$ the companion matrix of μ_{A_Γ} . Let us write $B = PAP^{-1}$. The matrix P has entries in $\mathbb{K}(\mathbb{Z}^{\pm 1})$ so it can be written under the form $P = \frac{1}{d}P'$ where P' has coefficients in $\mathbb{K}[\mathbb{Z}^{\pm 1}]$ and $d \in \mathbb{K}[\mathbb{Z}^{\pm 1}]$. As a consequence, there exists a specialisation f_θ such that $f_\theta(z) > 0$ for any $z \in \mathbb{Z}$ and $f_\theta(d) \neq 0$. The matrix A_θ (the specialisation

of A_Γ via θ) remains irreducible because Γ is strongly connected, thus its Perron–Frobenius eigenvalue λ corresponds to an eigenspace of dimension 1. This implies that $r = 1$. Indeed, A_θ is conjugate in $\mathbb{C}$ to the matrix B_θ (the specialisation of B via θ) with r identical blocks, each of them having the eigenvalue λ because it is a root of $\theta(\mu_{A_\Gamma})$. Since $r = 1$, the minimal polynomial μ_{A_Γ} has degree n and we can apply Proposition 3.8 to get a basis $\mathbf{B}$ such that $(\mathcal{A}_\Gamma, \mathbf{B})$ is a matrix representation of Γ . Further since $\mathbb{K}(\mathbb{Z}^{\pm 1})[A_\Gamma] \simeq \mathbb{K}(\mathbb{Z}^{\pm 1})[X]/\langle \mu_{A_\Gamma} \rangle$ we know that $\mathbb{K}(\mathbb{Z}^{\pm 1})[A_\Gamma]$ is a field. All the elements of $\mathbf{B}$ are non-zero (otherwise $\mathbf{B}$ is not a basis), hence they are invertible in $\mathbb{K}(\mathbb{Z}^{\pm 1})[A_\Gamma]$ and this implies that Γ is multiplicative at any vertex. $\square$

Recall that the coefficient $m_{i,j} = m_{i_0 \rightarrow i}^j$ is the sum of the weights of all paths in Γ of length j from the vertex v_{i_0} to the vertex v_i and that $M_{i_0} = (m_{i_0 \rightarrow i}^j)_{1 \leq i, j \leq n}$.

The matrix M_{i_0} gives a simple combinatorial criterion to decide whether or not the matrix $\mathbf{B}[i_0]$ is invertible in a matrix realisation $(\mathcal{A}, \mathbf{B})$ of Γ (recall that this does not depend on the matrix realisation). Further, when $\mathbf{b}_{i_0}$ is invertible, the matrix M_{i_0} allows one to compute the corresponding unique normalised basis $\mathbf{B}$ such that $\mathbf{B}[i_0] = I_n$ (see Corollary 3.9).

THEOREM 3.12. — *The graph Γ is of maximal dimension and is multiplicative at v_{i_0} if and only if the matrix M_{i_0} is invertible. Moreover, in this case, the columns of $M_{i_0}^{-1}$ give the vectors of the normalised basis $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_{i_0} = 1, \dots, \mathbf{b}_n\}$ expressed in the basis $\{1, A_\Gamma, \dots, A_\Gamma^{n-1}\}$. In particular, the entries of the matrices in the basis $\mathbf{B}$ belong to $\frac{1}{\det M_{i_0}} \mathbb{K}[\mathbb{Z}^{\pm 1}]$.*

Proof. — Assume that Γ is of maximal dimension and is multiplicative at v_{i_0} . Then there exists a basis $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of $\mathcal{A}_\Gamma$ such that $(\mathcal{A}_\Gamma, \mathbf{B})$ is a matrix realisation of Γ and $\mathbf{b}_{i_0} = I_n$. Using (3.1) we get

$$A_\Gamma^j \mathbf{b}_{i_0} = A_\Gamma^j = \sum_{i=1}^n m_{i_0 \rightarrow i}^j \mathbf{b}_i.$$

Since $\dim \mathcal{A}_\Gamma = n$, this equation shows that the matrix M_{i_0} is the change of basis matrix from the basis $\{I_n, A_\Gamma, \dots, A_\Gamma^{n-1}\}$ to the basis $\mathbf{B}$. Therefore, it is invertible and the columns of $M_{i_0}^{-1}$ express the vectors $\mathbf{B}_{i_0}$ in the basis $\{I_n, A_\Gamma, \dots, A_\Gamma^{n-1}\}$.

Assume now that M_{i_0} is invertible. Let $(x_0, \dots, x_{n-1}) \in \mathbb{K}(\mathbb{Z}^{\pm 1})^n$ such that $\sum_{j=0}^{n-1} x_j A_\Gamma^j = 0$. The j^{th} column C_j of M_{i_0} coincides with the i_0^{th} column of A_Γ^j thus we have $\sum_{j=0}^{n-1} x_j C_j = 0$ and since M_{i_0} is invertible, we must have $x_i = 0$ for all i . It follows that $\dim \mathcal{A}_\Gamma = n$ and Γ is of maximal dimension. By Proposition 3.8, there exists a basis $\mathbf{B}$ such that $(\mathcal{A}_\Gamma, \mathbf{B})$ is a matrix realisation of Γ . The family $\{A_\Gamma^j \mathbf{b}_{i_0} \mid j = 0, \dots, n-1\}$ is also a basis of $\mathcal{A}_\Gamma$ (the change of basis matrix from $\{A_\Gamma^j \mathbf{b}_{i_0} \mid j = 0, \dots, n-1\}$ to $\mathbf{B}$ is the invertible matrix M_{i_0}). But this can only be true if $\mathbf{b}_{i_0}$ is invertible. Therefore Γ is multiplicative at v_{i_0} . $\square$

Remark 3.13. — Theorem 3.12 provides a simple procedure to:

- (1) decide whether or not a graph Γ of maximal dimension is multiplicative at a given vertex and,
- (2) compute the unique corresponding matrix realisation.

For instance, we can compute explicitly all the structure constants of the homology ring of affine Grassmannians for the affine Weyl groups of type G_2 [9].

The corollary below follows easily from Theorem 3.12 and Proposition 3.11.

COROLLARY 3.14. — Assume that Γ is strongly connected and that μ_{A_Γ} is irreducible. Then the matrix M_{i_0} is invertible for all i_0 .

Example 3.15. — Consider the graph Γ with adjacency matrix A_Γ given as follows:



The graph Γ is strongly connected and since we have

$$\mu_A(X) = (X + 1)(X^2 - X - z_1 - z_2),$$

Γ is of maximal dimension. We compute

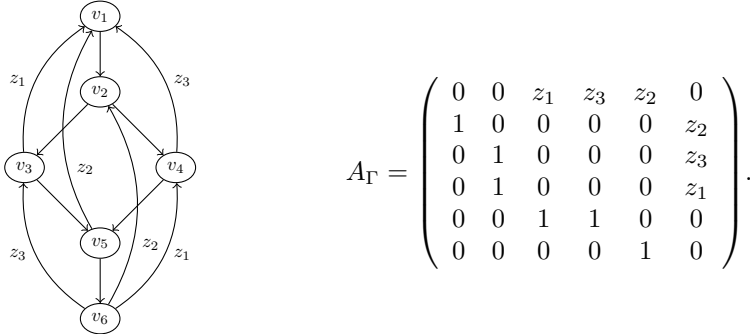
$$M_1 = \begin{pmatrix} 1 & 0 & z_1 + z_2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & z_1 & z_2 \\ 1 & 0 & z_1 + 1 \\ 0 & 1 & z_1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & z_2 & z_1 \\ 0 & 1 & z_2 \\ 1 & 0 & z_2 + 1 \end{pmatrix}$$

and

$$\det M_1 = 0, \quad \det(M_2) = z_2 - z_1^2, \quad \det(M_3) = z_2^2 - z_1.$$

It follows that Γ is multiplicative at v_2 (respectively at v_3) if and only if $z_2 \neq z_1^2$ (respectively $z_1 \neq z_2^2$). It is not multiplicative at v_1 . Observe that when $z_1 = z_2 = 1$, Γ is not multiplicative but, according to Proposition 3.8, it admits a matrix realisation.

Example 3.16. — Consider the graph Γ with adjacency matrix A_Γ given as follows:



We get $\mu_A(T) = T^6 - 2(z_1 + z_3)T^3 - 4z_2T^2 + (z_1 - z_3)^2$ which is irreducible. The matrices M_1 and M_1^{-1} are respectively

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & z_1 + z_3 & 2z_2 & 0 \\ 0 & 1 & 0 & 0 & z_1 + z_3 & 4z_2 \\ 0 & 0 & 1 & 0 & 0 & z_1 + 3z_3 \\ 0 & 0 & 1 & 0 & 0 & 3z_1 + z_3 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

and

$$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{z_1+z_3}{2} & -z_2 \\ 0 & 1 & \frac{2z_2}{z_1-z_3} & \frac{2z_2}{z_3-z_1} & 0 & -\frac{z_1+z_3}{2} \\ 0 & 0 & \frac{3z_1+z_3}{2z_1-2z_3} & \frac{z_1+3z_3}{2z_3-2z_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{-1}{2z_1-2z_3} & \frac{-1}{2z_3-2z_1} & 0 & 0 \end{pmatrix}.$$

The graph Γ is multiplicative at v_1 . One can check that all the entries in $\{\mathbf{b}_1 = I_n, \dots, \mathbf{b}_6\}$ belong to $\mathbb{Z}[z_1, z_2, z_3]$. For example, we have

$$\mathbf{b}_3 = \frac{1}{z_1 - z_3} \left(2z_2 A_\Gamma + \frac{1}{2} (3z_1 + z_3) A_\Gamma^2 - \frac{1}{2} A_\Gamma^5 \right) = \begin{pmatrix} 0 & z_1 & z_2 & 0 & 0 & z_1 z_3 \\ 0 & 0 & z_1 & 0 & z_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 & z_2 \\ 0 & 1 & 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Each element in $\mathcal{A}_\Gamma$ can be regarded as a linear map of $\mathbb{K}(Z^{\pm 1})^n = \bigoplus_{i=0}^{n-1} \mathbb{K}(Z^{\pm 1}) e_i$ where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{K}(Z^{\pm 1})^n$. In this context, saying that M_{i_0} is invertible is equivalent to saying that the vector e_{i_0} is cyclic¹ for A_Γ . Indeed, for any $1 \leq k \leq n$, the vector $A_\Gamma^k e_{i_0}$ is given by the i_0^{th} column of A_Γ^k which coincides with the k^{th} column of M_{i_0} . Thus M_{i_0} is invertible if and only if e_{i_0} is cyclic.

THEOREM 3.17. — *Assume that Γ is strongly connected, multiplicative at v_{i_0} and of maximal dimension. Let $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be the unique basis of $\mathcal{A}_\Gamma$ such that $\mathbf{b}_{i_0} = I_n$ and let $c_{i,j}^k$ be the structure constants $\mathcal{A}_\Gamma$ with respect to $\mathbf{B}$. We have*

- (1) $\mathbf{b}_i(e_{i_0}) = e_i$ for any $i = 1, \dots, n$, that is the i_0^{th} column of $\mathbf{b}_i$ is equal to e_i .
- (2) The j^{th} column of the matrix $\mathbf{b}_i$ is equal to $(c_{i,j}^k)_{k=1, \dots, n}$.
- (3) For any $1 \leq i, j \leq n$, the j^{th} column of $\mathbf{b}_i$ is equal to the i^{th} column of $\mathbf{b}_j$.
- (4) The entries of the matrices in the basis $\mathbf{B}$ and the coefficients $c_{i,j}^k$ all belong to $\frac{1}{\det M_{i_0}} \mathbb{K}[Z^{\pm 1}]$.

Proof. — We prove (1). Define the matrix C by the equations $Ce_i = \mathbf{b}_i(e_{i_0})$ for all $i \in \{1, \dots, n\}$. To prove that C belongs to $\mathcal{A}_\Gamma$, it suffices to show that C commutes with A_Γ (Γ is of maximal dimension therefore the centralizer of A_Γ is $\mathcal{A}_\Gamma$). Let $j \in \{1, \dots, n\}$. On the one hand, we have

$$CA_\Gamma e_j = C \left(\sum_{i=1}^n a_{i,j} e_i \right) = \sum_{i=0}^{n-1} a_{i,j} C e_i = \sum_{i=0}^{n-1} a_{i,j} \mathbf{b}_i e_{i_0}.$$

On the other hand

$$A_\Gamma C e_j = (A_\Gamma \mathbf{b}_j) e_{i_0} = \left(\sum_{i=0}^{n-1} a_{i,j} \mathbf{b}_i \right) e_{i_0} = \sum_{i=0}^{n-1} a_{i,j} \mathbf{b}_i e_{i_0}$$

where we use the fact that $\text{Mat}_{\mathbf{B}}(m_{A_\Gamma}) = A_\Gamma$ in the second equality. The matrix C is invertible because the set $\{\mathbf{b}_i e_{i_0} \mid 1 \leq i \leq n\}$ is a basis of $\mathbb{K}(Z^{\pm 1})^n$. Indeed, let $(x_1, \dots, x_n) \in \mathbb{K}(Z^{\pm 1})^n$ be such that $\sum_{1 \leq i \leq n} x_i \mathbf{b}_i e_{i_0} = 0$. Then $\sum_{1 \leq i \leq n} x_i \mathbf{b}_i = 0$

¹Recall that a vector v in K^n is cyclic for A when $\{A^m v \mid 0 \leq m < n\}$ is a basis of K^n . Equivalently, the linear map $K[A] \rightarrow K^n$ sending any B on Bv is injective.

because e_{i_0} is a cyclic vector for A_Γ but this implies that $x_i = 0$ for all i . This shows that (A_Γ, B') where $B' = C^{-1}B = (b'_1, \dots, b'_n)$ is a matrix realisation of Γ . Moreover we have

$$b'_i e_{i_0} = C^{-1} b_i e_{i_0} = C^{-1} C e_i = e_i \quad \text{for all } i \in \{1, \dots, n\}.$$

It follows that $b'_i b'_{i_0}(e_{i_0}) = b'_i(e_{i_0}) = e_i$ for all $i \in \{1, \dots, n\}$ and that $b'_i b'_{i_0} = b'_i$ since e_{i_0} is cyclic for A_Γ . Since B' is a basis of A_Γ , this implies that $b'_{i_0} = I_n$. By Corollary 3.9, we get $B = B'$ and $C = I_n$ as desired.

We prove (2). We have

$$b_i b_j e_{i_0} = \sum_{k=1}^n c_{i,j}^k b_k e_{i_0} \iff b_i e_j = \sum_{k=0}^{n-1} c_{i,j}^k e_k$$

hence the result.

Assertion (3) follows from the fact that $b_i b_j = b_j b_i$ in the commutative algebra A_Γ .

We prove (4). Since Γ is multiplicative at v_{i_0} and is of maximal dimension, Theorem 3.12 implies that the matrix M_{i_0} is invertible and that the columns of M_{i_0} belong to $\frac{1}{\det M_{i_0}} \mathbb{K}[Z^{\pm 1}]$. Assertion (2) implies that the coefficients $c_{i,j}^k$ also belong to $\frac{1}{\det M_{i_0}} \mathbb{K}[Z^{\pm 1}]$. $\square$

4. POSITIVELY MULTIPLICATIVE GRAPH

In this section, Γ is a graph in $\text{Graph}_n(\mathbb{R}_+[Z^{\pm 1}])$ with set of vertices $\{v_1, \dots, v_n\}$.

4.1. Definitions and examples.

DEFINITION 4.1.

- (1) We say that Γ is positively multiplicative at v_{i_0} if there exists a matrix realisation (A, B) of Γ such that A is a PM algebra with respect to B and $B_{[i_0]} = I_n$. We then say Γ is PM at v_{i_0} with respect to the matrix realisation (A, B) .
- (2) We say that Γ is PM if there exists v_{i_0} such that Γ is PM at v_{i_0} .
- (3) The set $\mathcal{R}_\Gamma^+$ of positive roots of Γ is the set of vertex v_{i_0} such that Γ is PM at v_{i_0} .

Remark 4.2. — Assume that Γ is PM with respect to the matrix realisation (A, B) . Then the algebra $\mathcal{A}$ can be defined over the ring $\mathbb{K}[Z^{\pm 1}]$ and we have

$$\mathcal{A} = \bigoplus \mathbb{K}[Z^{\pm 1}] b_i \quad \text{where } B = \{b_1, \dots, b_n\}.$$

Therefore, we can specialise the indeterminates in Z to any positive real numbers.

Given a graph Γ , it is not easy in general to decide whether or not it is (positively) multiplicative. In the case where Γ is of maximal dimension, one can use Theorem 3.12 to decide whether or not the graph is multiplicative and, when it is, to compute the unique corresponding matrix realisation (A, B) . Then, according to Theorem 3.17, the graph Γ will be positively multiplicative if and only if all the coefficients appearing in the matrices in basis B are in $\mathbb{R}_+[Z^{\pm 1}]$.

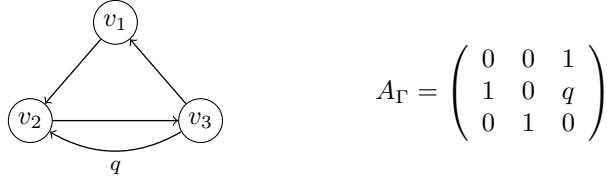
It is however fairly easy to construct PM graph from PM algebras. Indeed, let $\mathcal{A}$ be a finite dimensional PM algebra with PM basis $\mathbf{B} = \{1 = \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ and let

$$s = \sum_{i=1}^n \beta_i \mathbf{b}_i \quad \text{where } \beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}_+^n. \quad (4.1)$$

Let Γ be the graph with set of vertices $\{v_1, \dots, v_n\}$ defined by the equation $A_\Gamma = \text{Mat}_{\mathbf{B}}(m_s)$ (recall here that m_s is the left multiplication by s). In general the graph Γ is not strongly connected or even connected. To insure the strong connectivity, we need additional hypotheses on the algebra $\mathcal{A}$. For example, if $\mathcal{A}$ is strongly positive and $\beta \in (\mathbb{R}_+^*)^n$, the graph Γ is strongly connected and coincides in fact with a complete weighted graph.

We now give a series of example of positively multiplicative graphs.

Example 4.3. — In this example, we show that the set of roots $\mathcal{R}_\Gamma$ and the set of positive roots $\mathcal{R}_\Gamma^+$ can be different. Consider the graph Γ with adjacency matrix A_Γ given as follows where $q > 0$:



We compute

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & q \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & q & 1 \\ 1 & 0 & q \end{pmatrix}.$$

All these matrices are invertible, hence showing that $\mathcal{R}_\Gamma = \{v_1, v_2, v_3\}$ by Theorem 3.12. Further, for $i = 1, 2, 3$, there exists a unique matrix realisation $(\mathcal{A}_\Gamma, \mathbf{B}_i)$ such that $\mathbf{B}_i[i] = I_3$. Again by Theorem 3.12, we get

$$\begin{aligned} \mathbf{B}_1 &= \{I_3, A_\Gamma, A_\Gamma^2\}, \\ \mathbf{B}_2 &= \{-qI_3 + A_\Gamma^2, I_3, A_\Gamma\}, \\ \mathbf{B}_3 &= \{q^2 + A_\Gamma - qA_\Gamma^2, -q + A_\Gamma^2, I_3\}. \end{aligned}$$

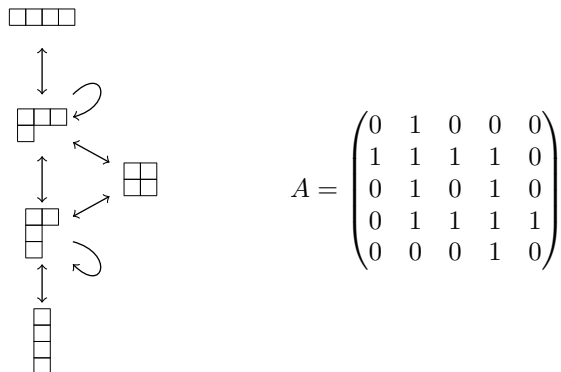
Both matrices $-qI_3 + A_\Gamma^2$ and $-q + A_\Gamma^2$ have negative coefficients, hence by Theorem 3.17, the graph Γ is not PM at v_2 and v_3 . It is PM at v_1 , hence showing that $\mathcal{R}_\Gamma^+ = \{v_1\}$.

Example 4.4. — In Example 3.16, one can check that the matrices of the elements in the given basis have coefficients in $\mathbb{R}_+[z_1, z_2, z_3]$. Thus the graph is positively multiplicative at v_1 .

Example 4.5. — We give an example of a positively multiplicative graph which is not of maximal dimension. Recall that the characters of the symmetric group $\mathfrak{S}_n$ are parametrised by the partitions λ of n . Let χ_λ be the character associated to λ . It was proved by Gamba and Radicati [8] (see also [12, Section 7.13] for a simpler proof) that

$$\chi_\lambda \times \chi_{(n-1,1)} = (l_\lambda - 1)\chi_\lambda + \sum_{\mu \neq \lambda} \chi_\mu$$

where the sum runs over the partitions μ obtained by moving a box in the Young diagram of λ and l_λ is the number of distinct parts in λ . The algebra $\mathcal{A}$ of characters for $\mathfrak{S}_n$ is PM for the basis $\mathcal{B} = \{\chi_\lambda \mid n \vdash \lambda\}$. This implies that the graph Γ whose adjacency matrix is that of the multiplication by $\chi_{(n-1,1)}$ in the basis $\mathcal{B}$, called the Hamermesh graph, is PM. The graph Γ is not of maximal dimension in general. For example, for $n = 4$, we get the following graph and adjacency matrix



whose minimal polynomial has degree 4.

In general, if we define a graph Γ as in the example above starting from the algebra $\mathcal{A}$ of complex characters of a finite group G and its basis $\mathcal{B}$ of irreducible characters, it is known that the graph Γ will be strongly connected if and only if the character s in (4.1) is the character of a faithful representation.

4.2. Relabelling of vertices and weight changes on PM graphs. Assume that Γ is positively multiplicative graph at v_{i_0} with respect to the matrix realisation $(\mathcal{A}, \mathcal{B})$. It is natural to consider a larger class of graphs constructed from Γ by authorizing the two following variations (we will see that they preserve the property of being positively multiplicative):

- (1) for a permutation $\sigma \in \mathfrak{S}_n$, switch the labels of the vertices in Γ : the label i becomes $\sigma(i)$. This gives a new adjacency matrix $A^\sigma = T_\sigma^{-1} A T_\sigma$ where T_σ is the permutation matrix associated to σ , that is the (i, j) -coefficient of T_σ is $\delta_{i, \sigma(j)}$.
- (2) for $\lambda_1, \dots, \lambda_n$ positive monomials in $\mathbb{R}_+[Z^{\pm 1}]$ and for any vertex v_j in Γ , change each edge $v_j \xrightarrow{a_{i,j}} v_i$ into $v_j \xrightarrow{(\lambda_j/\lambda_i)a_{i,j}} v_i$. This gives a new adjacency matrix $A^D = D^{-1} A D$ where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.

We will denote by $\mathcal{G}_n$ the group of generalised permutation matrices of size n . These are matrices in which each row and each column contains exactly one nonzero entry equal to a monomial in $\mathbb{R}_+[Z^{\pm 1}]$. Any generalised permutation matrix of $\mathcal{G}_n$ can be written under the form $P = D T_\sigma$. There is a natural action of $\mathcal{G}_n$ on the set $\text{Graph}_n(\mathbb{K}[Z^{\pm 1}])$: we define the graph Γ^P by the equation $A_{\Gamma^P} = P^{-1} A_\Gamma P$. We are going to see that this action of $\mathcal{G}_n$ (by conjugation on the adjacency matrices) restricts to the set of positively multiplicative graphs.

First, we have for all $j = 1, \dots, n$

$$Ab_j = \sum_{i=1}^n a_{i,j} b_i \iff Ab_{\sigma(j)} = \sum_{i=1}^n a_{\sigma(i), \sigma(j)} b_{\sigma(i)} \iff A^\sigma b'_{\sigma(j)} = \sum_{i=1}^n a_{i,j}^\sigma b'_{\sigma(i)}$$

where $b'_i = T_\sigma^{-1} b_i T_\sigma$ for $i = 1, \dots, n$ and $A^\sigma = (a_{i,j}^\sigma)$. Thus, by setting $b_i^\sigma = b'_{\sigma(i)}$ we obtain that the graph Γ^σ with adjacency matrix A^σ is positively multiplicative at $v_{\sigma^{-1}(i_0)}$ with respect to $(\mathcal{A}, B_\sigma)$ where $B_\sigma = \{b_i^\sigma \mid i = 1, \dots, n\}$.

Similarly, we get $A^D = (a_{i,j} \times \frac{\lambda_j}{\lambda_i})$ and the equivalences

$$\begin{aligned} Ab_j &= \sum_{i=1}^n a_{i,j} b_i \iff A(\lambda_j b_j) \\ &= \sum_{i=1}^n \frac{\lambda_j}{\lambda_i} a_{i,j} (\lambda_i b_i) \iff A^D(\lambda_j b_j'') = \sum_{i=1}^n \frac{\lambda_i}{\lambda_j} a_{i,j} (\lambda_i b_i'') \end{aligned}$$

where $b_i'' = D^{-1} b_i D$ for any $i = 0, \dots, n$. Hence the graph Γ^D with adjacency matrix A^D is positively multiplicative at v_{i_0} with respect to $(\mathcal{A}, B_D)$ where $B_D = \{b_i^D \mid i = 1, \dots, n\}$ and $b_i^D = \lambda_i D^{-1} b_i D$.

PROPOSITION 4.6. — *With the previous notation, for any $P = DT_\sigma \in \mathcal{G}_n$, the graph Γ is positively multiplicative at v_{i_0} with respect to $(\mathcal{A}, B)$ if and only if Γ^P is positively multiplicative at $v_{\sigma(i_0)}$ with respect to $(\mathcal{A}, B^P)$ where $B^P = \{b_1^P, \dots, b_n^P\}$ and $b_i^P = \lambda_{\sigma(i)} P^{-1} b_{\sigma(i)} P$ for all $i = 1, \dots, n$.*

Remark 4.7. — When P commutes with each element of the algebra $\mathcal{A}_\Gamma$ (thus in particular with the adjacency matrix A_Γ), we get $b_i^P = \lambda_{\sigma(i)} b_{\sigma(i)}$ and thus B^P is just the image of B by the generalised permutation matrix P .

5. COLUMN AND ROW KIRILLOV–RESHETIKHIN CRYSTALS OF AFFINE TYPE A

The goal of this section is to present two particular families of positively multiplicative graphs that are closely connected with the representation theory of affine quantum groups (see [4]). In contrast to their combinatorial definition which is very simple, their associated multiplicative algebras have sophisticated structure constants for which no simple combinatorial description is known. They will provide us interesting illustrations of the notions and results presented in the previous sections of the paper.

5.1. Partitions, column and row tableaux. A partition of length k is a sequence of integers $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$. It is conveniently identified with its Young diagram as illustrated below

$$\lambda = (5, 3, 3, 2) \longleftrightarrow \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}.$$

By a column tableau of length k in the alphabet $\{1, 2, \dots, n\}$, we mean a filling of the Young diagram of shape $(1, 1, \dots, 1)$ (with k ones) by strictly increasing integers in $\{1, 2, \dots, n\}$ from top to bottom. We denote by $\text{Col}_k(n)$ the set of such tableaux. By a row tableau of length k in the alphabet $\{1, 2, \dots, n\}$, we mean a filling of the

Young diagram of shape (k) by increasing integers in $\{1, 2, \dots, n\}$ from left to right. We denote by $\text{Row}_k(n)$ the set of such tableaux.

Example 5.1. — The following two tableaux are column and row tableau, respectively

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 4 & 5 \\ \hline \end{array}.$$

5.2. The Kirillov–Reshetikhin (KR) column crystals. Given integers $n \geq 2$ and $1 \leq k \leq n$, the KR crystal $B_{k,n}^c$ is the oriented graph with vertices the set of vertices $\text{Col}_k(n)$ and edges as follows:

- there is an edge between C and C' if there exists $i \in \{1, \dots, n\}$ such that $i \in C$, $i+1 \notin C'$ and C' is the column tableau obtained by replacing i by $i+1$ in C ;
- there is an edge between C and C' if $n \in C$, $1 \notin C'$ and C' is the column tableau obtained by replacing n by 1 in C and by reordering the entries.

There is a simple bijection between $\text{Col}_k(n)$ and the set of partitions contained in the rectangle $(n-k)^k$. It associates to each column tableau $C = \{c_k > c_{k-1} > \dots > c_1\}$ the partition $\lambda = (c_k - k, c_{k-1} - k + 1, \dots, c_1 - 1)$.

Examples 5.2.

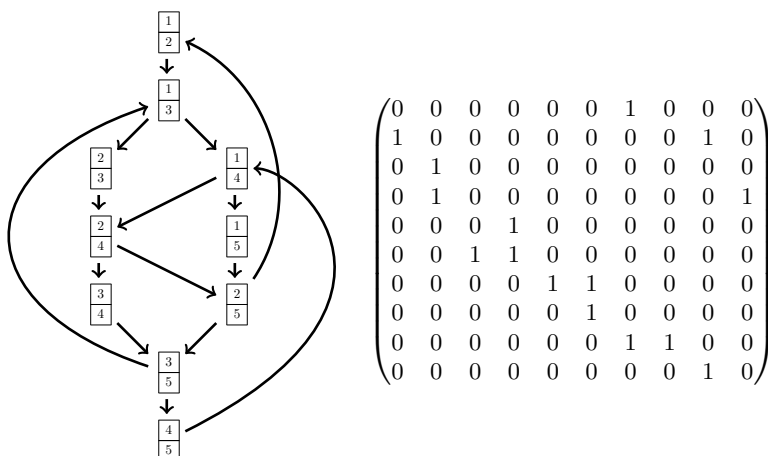
- (1) Assume $n = 5$ and $k = 2$. The set of tableaux in $\text{Col}_2(5)$ is

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 5 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 5 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}.$$

They correspond to the partitions (included in the box $(3, 3)$):

$$\emptyset \quad \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}.$$

The graph $B_{2,5}^c$ and the corresponding adjacency matrix $A_{B_{2,5}^c}$ with the ordering of $\text{Col}_2(5)$ above are:



The minimal polynomial of $B_{2,5}^c$ is $X^{10} - 11X^5 - 1$ so it is of maximal dimension.

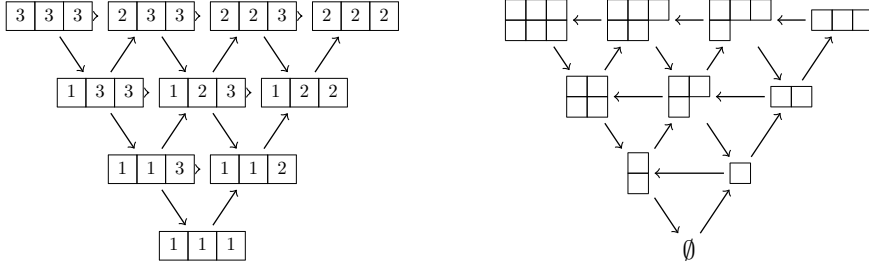
- (2) When $n = 6$ and $k = 2$, one can check that the minimal polynomial of $B_{2,6}^c$ is $X^{13} - 26X^7 - 27X$ so that the graph $B_{2,6}^a$ is not of maximal dimension.

5.3. The KR row crystals. Given integers $n \geq 2$ and $1 \leq \ell \leq n$, the KR crystal $B_{\ell,n}^r$ is the oriented graph with vertices the set of vertices $\text{Row}_\ell(n)$ and edges as follows:

- there is an edge between L and L' if there exists $i \in \{1, \dots, n\}$ such that $i \in L$ and L' is the row tableau obtained by replacing i by $i + 1$ in L ;
- there is an edge between L and L' if $n \in L$ and L' is the column tableau obtained by replacing n by 1 in L and by reordering the entries.

There is a simple bijection between $\text{Row}_k(n)$ and the set of partitions contained in the rectangle $(\ell, \dots, \ell)$ ($n - 1$ terms). It associates to each row tableau $L = \{c_k \geq c_{k-1} \geq \dots \geq c_1\}$ the partition λ containing a column of length $c_m - 1$ for all $1 \leq m \leq k$. In other words, the image of λ is $(c_k - 1, \dots, c_1 - 1)$.

Example 5.3. — The graph $B_{3,3}^s$ is as follows. On the left we have indexed the vertices with the set $\text{Row}_k(n)$ and on the right by the corresponding partitions using the bijection described above.



5.4. Quotients of the algebra of symmetric polynomials. Let $\Lambda_k = \text{Sym}[x_1, \dots, x_k] = \mathbb{Q}[e_1, \dots, e_k]$ be the algebra of symmetric polynomials in the k variables $x_1, \dots, x_k$. Here,

$$e_m = \sum_{1 \leq i_1 < \dots < i_m \leq k} x_{i_1} \dots x_{i_m} \quad \text{for any } 1 \leq m \leq k.$$

Write $\mathcal{P}_k$ for the set of partitions with at most k parts. It is well-known that Λ_k admits the two distinguished bases

$$\{e_{\lambda'} \mid \lambda \in \mathcal{P}_k\} \quad \text{and} \quad \{s_{\lambda} \mid \lambda \in \mathcal{P}_k\}$$

where for any partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we have $e_{\lambda'} = e_{\lambda'_1}, \dots, e_{\lambda'_k}$ and s_{λ} is the Schur polynomial associated to λ . Here λ' is the conjugate partition of λ whose parts are the heights of the columns in its Young diagram. For any positive integer ℓ , consider $\mathcal{I}_{k,\ell}$ the ideal of Λ_k defined by

$$\mathcal{I}_{k,\ell} = \langle s_{\lambda} \mid \lambda \text{ has } \ell + 1 \text{ columns of height less than } k \rangle.$$

Write $\mathcal{P}_k^{\ell}$ for the set of partitions in $\mathcal{P}_k$ with at most ℓ columns of height less than k . Recall that for any nonnegative integer m , the m^{th} homogeneous symmetric

polynomial is defined by

$$h_m = \sum_{1 \leq i_1 \leq \dots \leq i_m \leq k} x_{i_1} \dots x_{i_m}.$$

LEMMA 5.4 (See [16, Section 4]). — We have $I_{k,\ell} = \langle h_{\ell+1}, \dots, h_{\ell+k-1} \rangle$.

We define the ideals

$$I_{k,\ell}^q = \langle h_{\ell+1}, \dots, h_{\ell+k-1}, h_{\ell+k} - (-1)^{k-1}q \rangle \subset \mathbb{Q}[q] \otimes \Lambda_k$$

and

$$J_{k,\ell} = \langle h_{\ell+1}, \dots, h_{\ell+k-1}, e_k - 1 \rangle$$

and set

$$\mathcal{Q}_{k,\ell} := \Lambda_k / \mathcal{I}_{k,\ell}, \quad \mathcal{A}_{k,\ell}^q := \mathbb{Q}[q] \otimes \Lambda_k / I_{k,\ell}^q \quad \text{and} \quad \mathcal{S}_{k,\ell} := \Lambda_k / J_{k,\ell}.$$

Clearly the algebras $\mathcal{Q}_{k,\ell}$ and $\mathcal{A}_{k,\ell}^q$ are isomorphic by sending $h_{\ell+k} \bmod \mathcal{I}_{k,\ell}$ on $(-1)^{k-1}q$. They were considered in [2, 16] from where we can extract some of their important properties.

THEOREM 5.5 (see [16]). — Set $n = \ell + k$.

- (1) The set $\mathcal{B}_{k,\ell}^a = \{b_\lambda := s_\lambda \bmod \mathcal{A}_{k,\ell}^q \mid \lambda \subset \ell^k\}$ is a basis of $\mathcal{A}_{k,\ell}^q$ so that $\dim \mathcal{A}_{k,\ell}^q = \binom{n}{k}$.
- (2) After specialising $q = 1$, we have in $\mathcal{A}_{k,\ell}^1 := \mathcal{A}_{k,\ell}^q / \langle q = 1 \rangle$ for any $\lambda \subset \ell^k$

$$b_{(1)} \cdot b_\lambda = \sum_{\nu \subset \ell^k} b_\nu$$

where ν is obtained by adding one box to λ or by deleting the first row and the first column when $\lambda_1 = \ell$ and λ has k rows (i.e. by removing the unique possible hook of length $\ell + k - 1$ if any). In particular, in the algebra $\mathcal{A}_{k,\ell}^1$, the matrix of the multiplication by $b_{(1)}$ in the basis b_λ is the adjacency matrix of the KR-crystal $B_{k,n}$.

- (3) The structure constants of $\mathcal{A}_{k,\ell}^1$ with respect to $\mathcal{B}_{k,\ell}^a$ are nonnegative integers.

COROLLARY 5.6. — The graph $B_{k,n}^a$ with $n = \ell + k$ is positively multiplicative.

Proof. — The previous theorem together with Proposition 3.3 (taking $\mathcal{A} = \mathcal{A}_{k,\ell}^1$, $\mathcal{B} = \mathcal{B}_{k,\ell}^a$ and $x = b_{(1)}$) shows that the graph $B_{k,n}$ is multiplicative. Since the structure constants of $\mathcal{A}_{k,\ell}^1$ with respect to $\mathcal{B}_{k,\ell}^a$ are nonnegative integers, the result follows. $\square$

Now, let us turn to the algebra $\mathcal{S}_{k,\ell} := \Lambda_k / J_{k,\ell} = \mathcal{Q}_{k,\ell} / \langle e_k = 1 \rangle$.

THEOREM 5.7 (See [1, 2]). — In the algebra $\mathcal{S}_{k,\ell}$ the following statements hold.

- (1) The set $\mathcal{B}_{k,\ell}^s = \{b_\lambda \bmod J_{k,\ell} \mid \lambda \subset \ell^{k-1}\}$ is a basis of $\mathcal{S}_{k,\ell}$ so that

$$\dim \mathcal{S}_{k,\ell} = \binom{\ell + k - 1}{\ell}.$$

- (2) In $\mathcal{S}_{k,\ell}$, we have for any $\lambda \subset \ell^{k-1}$

$$b_{(1)} \cdot b_\lambda = \sum_{\nu \subset \ell^{k-1}} b_\nu$$

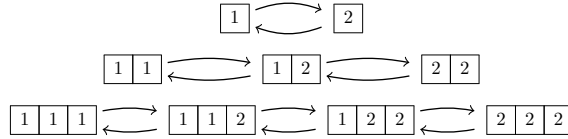
where ν is obtained by adding one box to λ , next by deleting a column of height k if such a column appears. In particular, in the algebra $\mathcal{S}_{k,\ell}$, the matrix of the multiplication by b_1 in the basis $\mathbf{B}_{k,\ell}^s$ is the adjacency matrix of the KR-crystal labelled by the rows of length ℓ on $\{1, \dots, k\}$.

- (3) The structure constants associated of $\mathcal{S}_{k,\ell}$ with respect to $\mathbf{B}_{k,\ell}^s$ are nonnegative integers.

COROLLARY 5.8. — The graph $B_{l,k}^s$ is positively multiplicative.

Proof. — The previous theorem together with Proposition 3.3 (taking $\mathcal{A} = \mathcal{S}_{k,\ell}$, $\mathbf{B} = \mathbf{B}_{k,\ell}^s$ and $x = b_{(1)}$) shows that the graph $B_{l,k}^s$ is multiplicative. Since the structure constants of $\mathcal{S}_{k,\ell}$ with respect to $\mathbf{B}_{k,\ell}^s$ are nonnegative integers, the result follows. $\square$

5.5. An example: the graph $B_{\ell,2}^s$. We consider KR crystal $B_{\ell,2}^s$ with one row and ℓ columns filled with integers in $\{1, 2\}$. The first examples of these graphs for $\ell = 1, 2$ and 3 are:



The adjacency matrix of $B_{\ell,2}^s$ is the $(\ell + 1) \times (\ell + 1)$ -matrix

$$A_{B_{\ell,2}^s} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

The algebra $\mathcal{S}_{2,\ell}$ is $\text{Sym}[x_1, x_2]/(e_2 = 1, h_{\ell+1} = 0)$. Since $e_2 = x_1 x_2$, we just have $x_2 = x_1^{-1}$ and by setting $x = x_1$, we see that the algebra $\mathcal{S}_{2,\ell}$ is equal to the algebra $\mathcal{L}[x^{\pm 1}]$ of Laurent polynomials P in the indeterminate x such that $P(x^{-1}) = P(x)$. Now in $\mathcal{L}[x^{\pm 1}]$, we have

$$h_a = \sum_{\substack{1 \leq i \leq j \leq \ell \\ i+j=a}} x^i x^{-j} = x^{-a} \sum_{i=0}^a x^{2i} = \frac{x^{a+1} - x^{-a-1}}{x - x^{-1}}.$$

Therefore $\mathcal{S}_{2,\ell}$ is isomorphic to the algebra $\mathcal{L}[x^{\pm 1}]/\langle x^{\ell+2} - x^{-(\ell+2)} \rangle$. A simple computation shows that

$$h_i \times h_j = \sum_{k=0}^j h_{i+j-2k} \quad \text{for all } i \geq j \in \mathbb{N}.$$

Let $\bar{x}$ and $\mathbf{b}_i$ be the image of x and h_i respectively in the quotient $\mathcal{L}[x^{\pm 1}]/\langle x^{\ell+2} - x^{-(\ell+2)} \rangle$ and define $\mathbf{B} = \{\mathbf{b}_0 = 1, \dots, \mathbf{b}_\ell\}$. We have $\bar{x}^{\ell+2} = \bar{x}^{-\ell-2}$. Therefore, for any $0 \leq a \leq \ell - 1$, we get

$$\mathbf{b}_{\ell+1+a} = \frac{\bar{x}^{\ell+a+2} - \bar{x}^{-\ell-a-2}}{\bar{x} - \bar{x}^{-1}} = \frac{\bar{x}^{-\ell-2+a} - \bar{x}^{\ell+2-a}}{\bar{x} - \bar{x}^{-1}} = -\mathbf{b}_{\ell+1-a}.$$

This gives assuming that $i \geq j$:

$$\mathbf{b}_i \mathbf{b}_j = \mathbf{b}_{i-j} + \mathbf{b}_{i-j+2} + \dots + \mathbf{b}_{i+j} \quad \text{when } i + j \leq \ell$$

and

$$\begin{aligned} \mathbf{b}_i \mathbf{b}_j &= \sum_{k=0}^j \mathbf{b}_{i+j-2k} = \mathbf{b}_{i-j} + \dots + \mathbf{b}_\ell + 0 - \mathbf{b}_\ell - \dots - \mathbf{b}_{i+j-\ell-1} \\ &= \mathbf{b}_{i-j} + \dots + \mathbf{b}_{2(\ell+1)-i-j-1} \quad \text{when } i + j > \ell. \end{aligned}$$

This can be summarised by the rules

$$\mathbf{b}_i \mathbf{b}_j = \begin{cases} \mathbf{b}_{|i-j|} + \dots + \mathbf{b}_{i+j} & \text{if } i + j \leq \ell, \\ \mathbf{b}_{|i-j|} + \dots + \mathbf{b}_{2(\ell+1)-i-j-1} & \text{if } i + j > \ell. \end{cases}$$

One may recognise the fusion rules for the fusion algebra $\widehat{su}_2$ (see [5]) at level ℓ . This construction is related to *Conformal Field Theory*. The Perron–Frobenius eigenvalue of the adjacency matrix of $B_{\ell,2}^c$ is $\lambda = 2 \cos \frac{\pi}{\ell+2}$ with normalised associated eigenvector

$$\frac{1}{\sin \frac{\pi}{\ell+2}} \left(\sin \frac{a\pi}{\ell+2} \right)_{1 \leq a \leq \ell+1}.$$

6. INFINITE PM GRAPHS AND HARMONIC FUNCTIONS

The goal of this section is to present a general combinatorial construction (called expansion) yielding infinite graphs from finite ones. To simplify the exposition, we will also assume that all the algebras considered in this section are commutative.

Let Γ be a possibly infinite oriented graph with set of vertices V . We say that Γ is graded if there exists a partition $\{V_i \mid i \in \mathbb{N}\}$ of V such that for any $i \geq 1$, the edges in Γ which start at V_i finish at V_{i+1} . In addition, when V_0 is a singleton, the graph Γ is said rooted and graded.

Recall that $\mathbf{Z} = \{z_1, \dots, z_N\}$ with $N \in \mathbb{N}$ is a set (possibly empty) of formal indeterminates. For $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{Z}^N$, we define $z^\beta = z_1^{\beta_1} z_2^{\beta_2} \dots z_N^{\beta_N}$. Then the set $\{z^\beta \mid \beta \in \mathbb{Z}^N\}$ form a $\mathbb{K}$ -basis of $\mathbb{K}[\mathbf{Z}^{\pm 1}]$. Given $a \in \mathbb{K}[\mathbf{Z}^{\pm 1}]$, we denote by $a[\beta]$ the coefficient of z^β in the expansion of a in this basis. In other words, we have

$$a = \sum_{\beta \in \mathbb{Z}^N} a[\beta] z^\beta.$$

We write $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^N$.

6.1. Expansion of a graph. In this section, $\Gamma \in \mathbf{Graph}_n(\mathbb{R}_+[Z^{\pm 1}])$ denotes a finite strongly connected graph with set of vertices $V = \{v_1, \dots, v_n\}$, edge weight function ω and adjacency matrix $A_\Gamma = (a_{i,j})$ in $\mathbf{Mat}_n(\mathbb{R}_+[Z^{\pm 1}])$. Recall that ω takes values in $\mathbb{R}_+[Z^{\pm 1}]$.

A path π of length ℓ on Γ is a sequence of $\ell + 1$ vertices of Γ with two consecutive vertices being connected by an oriented edge. The weight $\text{wt}(\pi)$ of the path π is the product of the weights of the edges encountered. Therefore $\text{wt}(\pi)$ is a monomial of the form cz^β with $\beta \in \mathbb{Z}^N$ and $c \in \mathbb{R}_+$.

DEFINITION 6.1. — The expansion Γ_e of Γ at v_{i_0} is the rooted graded graph with vertices

$$V_e \subset \left\{ v_{i,z^\beta}^k \mid i \in \{1, \dots, n\}, \quad \beta \in \mathbb{Z}^N, \quad \ell \in \mathbb{N} \right\} \quad \text{where } v_{i,\beta}^k = (v_i, z^\beta, \ell)$$

and edge weight function $\omega_e : V_e \times V_e \rightarrow \mathbb{R}_+$ both constructed by induction as follows:

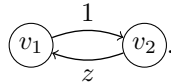
- $v_{i_0,0}^0$ is the unique vertex of V_e^0 ,
- if $v_{i,\beta}^k \in V_e^\ell$ with $\ell \geq 0$ and $\omega(v_i, v_j)[\delta] \in \mathbb{R}_+^*$ for some $\delta \in \mathbb{Z}^N$ then $v_{j,\beta+\delta}^{\ell+1} \in V_e^{\ell+1}$ and $\omega_e(v_{i,\beta}^k, v_{j,\beta+\delta}^{\ell+1}) = \omega(v_i, v_j)[\delta]$.

Remark 6.2.

- (1) By construction of Γ_e , for all $i \in \{1, \dots, n\}$, $\beta \in \mathbb{Z}^N$ and $\ell \in \mathbb{N}$, we have $v_{i,\beta}^\ell \in V_e$ if and only if there is a path π in Γ of length ℓ from v_{i_0} to v_i such that $\text{wt}(\pi)[\beta] \neq 0$. Moreover, given $v_{i,\beta}^\ell \in V_e^\ell$, we have $v_{j,\delta}^{\ell+1} \in V_e^{\ell+1}$ if and only if we have $\omega(v_i, v_j)[\delta - \beta] \neq 0$ (in particular there is an edge from v_i to v_j).
- (2) There is a strong connection between the edge weight functions ω_e and ω . More precisely, for any vertex $v_{i,\beta}^\ell \in V_e^\ell$ we have

$$\begin{aligned} \sum_{v_{j,\delta}^{\ell+1} \in V_e^{\ell+1}} \omega_e(v_{i,\beta}^\ell, v_{j,\delta}^{\ell+1}) z^{\delta-\beta} &= \sum_{v_{j,\delta}^{\ell+1} \in V_e^{\ell+1}} \omega(v_i, v_j)[\delta - \beta] z^{\delta-\beta} \\ &= \sum_{j \in \{1, \dots, n\}} \sum_{\delta \in \mathbb{Z}^N} \omega(v_i, v_j)[\delta - \beta] z^{\delta-\beta} \quad \text{using (1)} \\ &= \sum_{j \in \{1, \dots, n\}} \omega(v_i, v_j). \end{aligned}$$

Example 6.3. — Let Γ be the graph given by



The expansion Γ_e of Γ at v_1 is

$$v_{1,0}^0 \xrightarrow{1} v_{2,0}^1 \xrightarrow{1} v_{1,1}^2 \xrightarrow{1} v_{2,1}^3 \xrightarrow{1} v_{1,2}^4 \xrightarrow{1} \dots$$

Assume that Γ be a positively multiplicative with respect to the matrix realisation $(\mathcal{A}, \mathbf{B})$ where $\mathbf{B} = \{b_1 = 1, b_2, \dots, b_n\}$. By definition $\mathcal{A}$ is an algebra over

$\mathbb{K}(Z^{\pm 1})$ whose structure constants $c_{i,j}^k$ with respect to $\mathbf{B}$ belong to $\mathbb{R}_+[Z^{\pm 1}]$. In other words, we have

$$c_{i,j}^k = \sum_{\beta \in \mathbb{Z}^N} c_{i,j}^k[\beta] z^\beta \quad \text{where } \beta = (\beta_1, \dots, \beta_N) \in \mathbb{Z}^N \quad \text{and} \quad c_{i,j}^k[\beta] \in \mathbb{R}_+.$$

The algebra $\mathcal{A}$ can also be viewed as an infinite dimensional $\mathbb{K}$ -algebra $\mathcal{A}_{\mathbb{K}}$ with basis

$$\{z^\beta \mathbf{b}_i \mid i = 1, \dots, n \quad \text{and} \quad \beta \in \mathbb{Z}^N\}.$$

In order to extend the notion of positively multiplicative graphs to the expansion Γ_e of Γ , we will in fact need the larger algebra $\mathcal{A}'_e = \mathcal{A}_{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{K}[q]$ where q is a new indeterminate distinct from $z_1, \dots, z_N$. The powers of q will record the lengths of the paths starting from the vertex v_{i_0} of Γ chosen to construct the expansion. The set

$$\mathbf{B}'_e = \{q^\ell z^\beta \mathbf{b}_i \mid i = 1, \dots, n, \beta \in \mathbb{Z}^N, \ell \in \mathbb{N}\}$$

is a $\mathbb{K}$ -basis of $\mathcal{A}'_e$. Finally we define $\mathcal{A}_e$ to be the subspace of $\mathcal{A}'_e$ with basis $\mathbf{B}_e = \{q^\ell z^\beta \mathbf{b}_i \mid \mathbf{v}_{i,\beta}^\ell \in V_e\}$.

Given $\mathbf{v}_{i,\beta}^\ell$ where $i \in \{1, \dots, n\}$, $\beta \in \mathbb{Z}^N$ and $\ell \in \mathbb{N}$, we set $\mathbf{b}_{i,\beta}^\ell = q^\ell z^\beta \mathbf{b}_i \in \mathbf{B}'_e$. Note that $1 = \mathbf{b}_{1,0}^0$.

PROPOSITION 6.4. — *Let Γ_e be the expansion of Γ at v_1 .*

(1) *For any vertex $\mathbf{v}_{j,\beta}^\ell \in V_e$, we have in $\mathcal{A}_e$*

$$qA_\Gamma \times \mathbf{b}_{j,\beta}^\ell = \sum_{\mathbf{v}_{i,\delta}^{\ell+1} \in V_{\ell+1}} \omega_e(\mathbf{v}_{j,\beta}^\ell, \mathbf{v}_{i,\delta}^{\ell+1}) \mathbf{b}_{i,\delta}^{\ell+1}$$

In particular, $qA_\Gamma \in \mathbf{B}'_e$.

- (2) *$\mathbf{B}_e$ is the subset of $\mathbf{B}'_e$ containing the elements $q^\ell z^\beta \mathbf{b}_i$ with $\ell \geq 0$ which appear with a non-zero coefficient in the expansions of the powers $(qA_\Gamma)^\ell$, $\ell \geq 0$ on the basis $\mathbf{B}'_e$.*
- (3) *The element $\mathbf{b}_{1,0}^0 = 1$ belongs to $\mathbf{B}_e$ and the product of two elements in the basis $\mathbf{B}_e$ expands on $\mathbf{B}_e$ with nonnegative real coefficients. In particular $\mathcal{A}_e$ is a subalgebra of $\mathcal{A}'_e$ with PM basis $\mathbf{B}_e$.*

Proof. — We prove (1). Let $\mathbf{v}_{j,\beta}^\ell \in V_e$. We have

$$\begin{aligned} qA_\Gamma \times \mathbf{b}_{j,\beta}^\ell &= q^{\ell+1} z^\beta A_\Gamma \mathbf{b}_j \\ &= q^{\ell+1} z^\beta \sum_{i=1}^n \omega(v_j, v_i) \mathbf{b}_i \\ &= q^{\ell+1} z^\beta \sum_{i=1}^n \sum_{\delta \in \mathbb{Z}^N} \omega(v_j, v_i) [\delta] z^\delta \mathbf{b}_i \\ &= \sum_{i=1}^n \sum_{\delta \in \mathbb{Z}^N} \omega(v_j, v_i) [\delta] q^{\ell+1} z^{\beta+\delta} \mathbf{b}_i \\ &= \sum_{i=1}^n \sum_{\delta' \in \mathbb{Z}^N} \omega(v_j, v_i) [\delta' - \beta] \mathbf{b}_{i,\delta'}^{\ell+1}. \end{aligned}$$

But $\omega(v_j, v_i)[\delta' - \beta] \neq 0$ exactly when there is an edge from $v_{j,\beta}^\ell$ to $v_{i,\delta'}^{\ell+1}$ of weight $\omega(v_j, v_i)[\delta' - \beta]$. Hence the result.

We prove (2). Since $\mathbf{b}_{1,1}^0 = 1 \in V_e$, we get by induction using (1):

$$(qA_\Gamma)^k = \sum_{\substack{\pi, \ell(\pi)=k \\ \pi \text{ starts at } v_{1,1}^0 \\ \text{and ends at } v_{i,\beta}^k}} \text{wt}(\pi) \mathbf{b}_{i,\beta}^k$$

We have $\mathbf{b}_{i,\beta}^k \in B_e$ (or equivalently $v_{i,\beta}^k \in V_e$) if and only if there exists a path from $v_{1,1}^0$ to $v_{i,\beta}^k$ in Γ_e . The result follows.

We prove (3). Given $\mathbf{b}_{i,\beta}^\ell$ and $\mathbf{b}_{j,\gamma}^s$ in the basis B'_e , we have

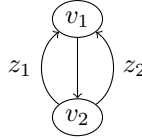
$$\mathbf{b}_{i,\beta}^k \times \mathbf{b}_{j,\gamma}^s = q^\ell z^\beta \mathbf{b}_i \times q^s z^\gamma \mathbf{b}_j = \sum_{k=1}^n c_{i,j}^k q^{\ell+s} z^{\beta+\gamma} \mathbf{b}_k = \sum_{k=1}^n \sum_{\delta \in \mathbb{Z}^N} c_{i,j}^k[\delta] \mathbf{b}_{k,\delta+\beta+\gamma}^{\ell+s}$$

which shows that $\mathbf{b}_{i,\beta}^\ell \times \mathbf{b}_{j,\gamma}^s$ expands positively on B'_e . Now if $\mathbf{b}_{i,\beta}^\ell$ and $\mathbf{b}_{j,\gamma}^s$ lie in B_e , then $\mathbf{b}_{i,\beta}^\ell$ appears in $(qA_\Gamma)^\ell$ and $\mathbf{b}_{j,\gamma}^s$ appears in $(qA_\Gamma)^s$. Since $A_\Gamma \in \text{Mat}_n(\mathbb{R}_+[Z^{\pm 1}])$, it follows that all the elements that appear in the product $\mathbf{b}_{i,\beta}^\ell \times \mathbf{b}_{j,\gamma}^s$ actually appears in $(qA_\Gamma)^{\ell+s}$ and hence belong to B_e . $\square$

Remark 6.5. — In fact, the interesting finite graphs Γ have often additional properties which simplify the definition and the study of their expansion. This is the case when:

- (1) The weights of Γ are only nonnegative reals.
- (2) The length of a path is completely determined by its weight and by its end point in Γ . In this case, the indeterminate q in the previous construction is redundant and can be omitted. This is the case for the graph considered in the example below.

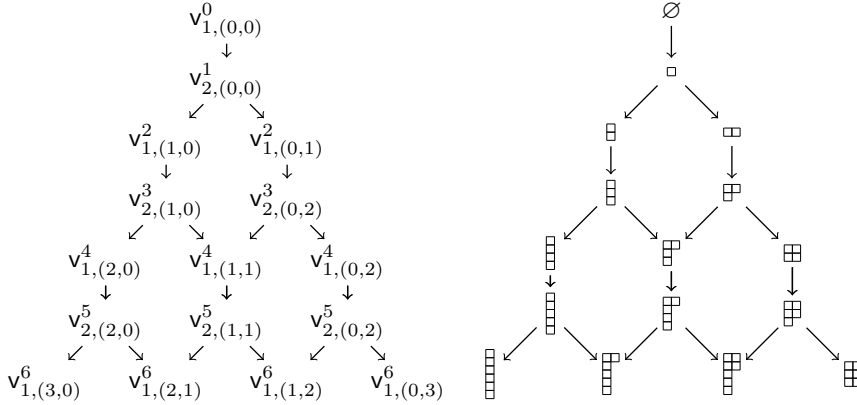
Example 6.6. — The graph Γ



with adjacency matrix $A_\Gamma = \begin{pmatrix} 0 & z_1 + z_2 \\ 1 & 0 \end{pmatrix}$ is PM with respect to the matrix realisation $(\mathcal{A}, \mathbf{B})$ where $\mathcal{A} = \mathbb{R}I_2 \oplus \mathbb{R}A$ and $\mathbf{B} = \{I_2, A\}$. We draw the graph Γ_e of the expansion of Γ at v_1 below on the left handside. The indeterminate q can be omitted, since when $v_{i,\beta}^\ell \in \Gamma_e$ with $\beta = (\beta_1, \beta_2)$ we must have $\ell = 2(\beta_1 + \beta_2) + 1$ if $v = v_2$ and $\ell = 2(\beta_1 + \beta_2)$ if $v = v_1$. We have

$$\mathcal{A}_e = \mathbb{R}[z_1, z_2]I_2 \oplus \mathbb{R}[z_1, z_2]A.$$

Note that the graph Γ_e can also be labeled by 2-bounded partitions (that is partition with parts less than or equal to 2) as illustrated on the right handside.


 FIGURE 6.1. The graph Γ_e and its labeling by 2-bounded partitions

6.2. Infinite PM graphs and harmonic functions. In this section, Γ denotes an infinite rooted graded (oriented) graph with set of vertices $\mathbf{V}$ where $\mathbf{V}$ is countable, edge weight function $\underline{\omega}$ that takes values in $\mathbb{R}_+$ and such that v_1 is the unique vertex at level 0.

DEFINITION 6.7. — The infinite graph Γ is *positively multiplicative* when there exists an algebra $\mathbf{A}$ over $\mathbb{K}$, PM with respect to a basis $\mathbf{B} = \{\mathbf{b}_v \mid v \in \mathbf{V}\}$ where $\mathbf{b}_{v_1} = 1$ and a distinguished element γ in $\mathbf{A}$ such that

$$\gamma \mathbf{b}_v = \sum_{v' \in \mathbf{V}} \underline{\omega}(v, v') \mathbf{b}_{v'} \quad \text{for any } v \in \mathbf{V}.$$

In particular, we have²

$$\gamma = \sum_{v \in \mathbf{V}} \underline{\omega}(v_1, v) \mathbf{b}_v.$$

DEFINITION 6.8. — A nonnegative (respectively positive) harmonic function on Γ is a map $f : \mathbf{V} \Rightarrow \mathbb{R}_+$ such that $f(v_1) = 1$ and for any vertex $v \in \mathbf{V}$

$$f(v) \geq 0 \text{ (resp. } f(v) > 0) \quad \text{and} \quad f(v) = \sum_{v' \in \mathbf{V}} \omega_e(v, v') f(v').$$

Observe that we have then

$$1 = f(v_1) = \sum_{v' \in \mathbf{V}} \underline{\omega}(v_1, v') f(v').$$

We denote by $\mathcal{H}(\Gamma)$ the set of nonnegative harmonic functions on Γ . This is a convex cone and we write $\mathcal{H}_\partial(\Gamma)$ for its subset of extremal points.

Remark 6.9. — The concept of positive harmonic functions on Γ are strongly connected to Markov chains. Indeed, if f is a positive harmonic function, we can define a Markov $\mathcal{H}$ chain on Γ with transition matrix Π defined by:

$$\Pi(v, v') = \underline{\omega}(v', v) \frac{f(v')}{f(v)} \quad \text{for all } v, v' \in \mathbf{V}.$$

²Note that γ is given by the fact that we must have $\gamma = \gamma \mathbf{b}_{v_1} = \sum_{v' \in \mathbf{V}} \underline{\omega}(v_1, v') \mathbf{b}_{v'}$.

It then becomes possible to study $\mathcal{H}$ (for example to get its drift, a law of large numbers etc.) when f is sufficiently simple, in particular when f is extremal (see [14, 17, 19] for examples).

Recall the following theorem essentially due to Kerov and Vershik (see [14] and also [18]).

THEOREM 6.10. — Assume that $\mathbb{K}(\mathbb{Z}^{\pm 1}) = \mathbb{R}$ and that Γ is positively multiplicative with associated algebra $\mathbf{A}$ and basis $\mathbf{B} = \{b_v \mid v \in \mathbf{V}\}$ with $1 = b_{v_1} \in \mathbf{B}$. Then the map $f : \mathbf{V} \rightarrow \mathbb{R}_+$ belongs to $\mathcal{H}_\partial(\Gamma)$ if and only if the linear form $\varphi : \mathbf{A} \rightarrow \mathbb{R}$ defined by $\varphi(b_v) = f(v)$ for any $v \in \mathbf{V}$ is a morphism of $\mathbb{R}$ -algebras satisfying $\varphi(\gamma) = 1$.

Example 6.11. — Let $\Gamma = (\mathbf{V}, \mathbf{E}, \underline{\omega})$ where $\mathbf{V} = \{(a, b) \in \mathbb{N}^2 \mid a = 0 \text{ or } b \leq 1\}$,

$$\mathbf{E} = \{((a, b), (a', b')) \in \mathbf{V} \times \mathbf{V} \mid (a', b') = (a + 1, b) \text{ or } (a', b') = (a, b + 1)\}$$

and $\underline{\omega}$ is constant equal to 1 on $\mathbf{E}$. This graph is positively multiplicative with respect to the algebra

$$\mathbf{A} = \mathbb{R}[x_1, x_2] / \langle x_1 x_2^2 \rangle \quad \text{and the basis} \quad \mathbf{B} = \{\overline{x_1^a x_2^b} \mid (a, b) \in \mathbf{V}\}$$

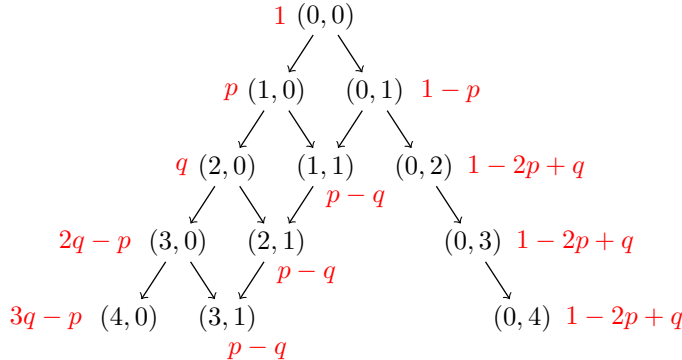
where $\overline{x_1}$ and $\overline{x_2}$ denotes the images of x_1 and x_2 in $\mathbf{A}$. Let f be an extremal harmonic function on Γ . Then according to the theorem above, we have $f(\gamma) = f(1, 0) + f(0, 1) = 1$ and the map φ from $\mathbf{A}$ to $\mathbb{R}_+$ defined by $\overline{x_1^a x_2^b} \rightarrow f(a, b)$ is a morphism of algebras. In particular $\overline{x_1 x_2^2}$ is sent to 0. But

$$\varphi(\overline{x_1 x_2^2}) = \varphi(\overline{x_1})\varphi(\overline{x_2})^2 = f(1, 0)f(0, 1) = 0$$

hence $f(1, 0) = 0$ ou $f(0, 1) = 0$. Finally we have

$$\begin{aligned} \mathcal{H}_\partial(\Gamma) &= \{f_1, f_2\} \quad \text{where } f_1(1, 0) = 1, \quad f_1(0, 1) = 0 \\ &\quad \text{and } f_2(1, 0) = 0 \quad \text{and } f_2(0, 1) = 1. \end{aligned}$$

Let us verify the theorem above “by hands”. Let f be an harmonic function on Γ . We have $f(1, 0) + f(0, 1) = 1$ and f is positive so that $f(1, 0) = p \in [0, 1]$ and $f(0, 1) = 1 - p \in [0, 1]$. Next we set $f(2, 0) = q \in [2p - 1, q]$. Below, we draw the graph Γ and we put the expected values of f in red close to each vertex.



It is a straightforward exercise to show that f satisfies $f(k, 0) = (k - 1)q - (k - 2)p \geq 0$ for all $k \geq 2$. Using the fact that f is positive, we get $p \in [0, 1]$ and $q \in \left[\frac{k-2}{k-1}p, p\right]$ for all $k \geq 2$. It follows that $q = p$ and $f(1, 1) = 0$. Finally, we get $f = pf_1 + (1 - p)f_2$ as expected.

Example 6.12. — Let $\mathcal{Y}_n$ be the Young lattice of partitions with at most n parts. Recall that its vertices are the Young diagrams λ associated to the partitions with at most n parts and we have an edge $\lambda \rightarrow \mu$ if the Young diagram of μ is obtained by adding a box to the Young diagram of λ . Let $\mathbb{A} = \text{Sym}[x_1, \dots, x_n]$ be the $\mathbb{R}$ -algebra of symmetric functions in the indeterminates $x_1, \dots, x_n$ and $\mathbb{B} = \{s_\lambda \mid \lambda \in \mathcal{Y}_n\}$ where s_λ is the Schur function associated to λ . Then $\mathcal{Y}_n$ is positively multiplicative with respect to $\mathbb{A}$ and the basis $\mathbb{B}$. Its extremal positive harmonic functions are the morphisms $\varphi : \mathbb{A} \rightarrow \mathbb{R}$ such that $\varphi(s_\lambda) \in \mathbb{R}_+^*$. One can show in this case that these morphisms are parametrised by the vectors $p = (p_1, \dots, p_n) \in \mathbb{R}_+^m$ with $p_1 + \dots + p_n = 1$, the morphism φ_p corresponding to p being the specialisation $x_i = p_i, i = 1, \dots, n$. The structure constants $c_{\lambda, \mu}^\nu$ are the Littlewood–Richardson coefficients and it is a classical result that $c_{\lambda, \mu}^\nu > 0$ only if ν can be reached from λ by a path in $\mathcal{Y}_n$ (i.e. $\lambda \subset \nu$).

6.3. Positive harmonic functions on expanded graphs. In this section, $\Gamma \in \text{Graph}_n(\mathbb{R}_+[Z^{\pm 1}])$ with set of vertices $\{v_1, \dots, v_n\}$ is a strongly connected and positively multiplicative graph with respect to the matrix realisation $(\mathcal{A}, \mathbb{B})$ where $\mathbb{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ and $\mathbf{b}_1 = 1$. Let Γ_e be the expansion of Γ at v_1 with set of vertices V_e . We denote by ω_e the edge weight function on Γ_e .

PROPOSITION 6.13. — *The infinite graph Γ_e is positively multiplicative with respect to the algebra $\mathcal{A}_e$ and the basis $\mathbb{B}_e$.*

Proof. — First of all, the graph Γ_e is rooted and graded with $v_{1,0}^0$ the unique vertex at level 0. We have $\mathbf{b}_{1,0}^0 = q^0 z^0 \mathbf{b}_1 = 1$ and by Proposition 6.4(1), we see that the element γ in Definition 6.7 is:

$$qA_\Gamma = qA_\Gamma \times \mathbf{b}_{1,0}^0 = \sum_{v_{i,\delta}^1 \in V_e^1} \omega_e(v_{1,0}^0, v_{i,\delta}^1) \mathbf{b}_{i,\delta}^1.$$

The result then follows from Proposition 6.4. □

Given $\mathbf{t} = (t_1, \dots, t_N) \in \mathbb{R}_+^*$ and $(\beta_1, \dots, \beta_N) \in \mathbb{Z}^N$ we write $\mathbf{t}^\beta = t_1^{\beta_1} \dots t_N^{\beta_N}$. With this notation, the specialisation that sends z_k to t_k for all $k \in \{1, \dots, N\}$ sends z^β to $\mathbf{t}^\beta$. We write $A_{\mathbf{t}}$ for the matrix obtained from A_Γ by applying this specialisation to all the coefficients of A_Γ .

THEOREM 6.14. — *Assume that $\mathbb{K}[Z^{\pm 1}, q] \subset \mathcal{A}_e$ and $\mathcal{A} = \mathbb{K}(Z^{\pm 1})[A_\Gamma]$ (i.e. Γ has maximal dimension). Then, the set $\mathcal{H}_\partial^+(\Gamma_e)$ of extremal positive harmonic functions is parametrised by a subset of $\mathbb{R}_+^{*N}$. More precisely, to any $\mathbf{t} = (t_1, \dots, t_N)$ in $\mathbb{R}_+^{*N}$ corresponds an extremal harmonic function φ on Γ_e such that*

$$\varphi(v_{i,\beta}^\ell) = t^\beta \lambda^{-\ell} \pi_i \quad \text{for } i = 1, \dots, n$$

where $\pi = (\pi_1, \dots, \pi_n)$ is the left Perron–Frobenius vector of the matrix $A_{\mathbf{t}}$ with eigenvalue

$$\lambda = \sum_{i=1}^n \varphi(a_{i,1}) \pi_i.$$

Moreover, all the elements in $\mathcal{H}_\partial^+(\Gamma_e)$ are obtained in this way.

Proof. — By Theorem 6.10, the elements of $\mathcal{H}_\theta^+(\Gamma_e)$ are determined by the morphisms of $\mathbb{R}$ -algebras $\varphi : \mathcal{A}_e \rightarrow \mathbb{R}$ which are positive on the basis B_e and such that $\varphi(\gamma) = 1$ where $\gamma = qA_\Gamma$.

Consider such a morphism φ . First of all, since $\mathbb{K}[Z^{\pm 1}, q] \subset \mathcal{A}_e$, we have $b_i \in \mathcal{A}_e$ for all $i = 1, \dots, N$. Since (A, B) is a matrix realisation of Γ and $b_1 = 1$, we have $A_\Gamma = \sum_{i=1}^n a_{i,1} b_i$ so that $A_\Gamma \in \mathcal{A}_e$. We set $\varphi(q) = \lambda^{-1} \in \mathbb{R}_{>0}$, $\varphi(z_k) = t_k \in \mathbb{R}_{>0}$ and $\varphi(b_i) = \pi_i \in \mathbb{R}$ for each $k = 1, \dots, N$. We have $\varphi(A_\Gamma) = \varphi(q^{-1}) = \lambda$ since $\varphi(\gamma) = 1$. Note that $\lambda = \sum_{i=1}^n \varphi(a_{i,1}) \pi_i$.

We show that $\pi_i \in \mathbb{R}_+^*$ for all $1 \leq i \leq n$. Let $i \in \{1, \dots, n\}$. Since Γ is strongly connected, there exists a path from v_1 to v_i in Γ . Thus, there is a vertex of the form $v_{i,\beta}^\ell$ in Γ_e for some $\beta \in \mathbb{Z}^N$ and $\ell \in \mathbb{N}$. We have $\varphi(b_{i,\beta}^\ell) = \varphi(q^\ell z^\beta b_i) = \lambda^{-\ell} t^\beta \pi_i$ so that $\pi_i = \frac{\varphi(b_{i,\beta}^\ell)}{\lambda^{-\ell} t^\beta} \in \mathbb{R}_+^*$ since $\lambda, t^\beta, \varphi(b_{i,\beta}^\ell)$ all lie in $\mathbb{R}_+^*$ (φ is positive on B_e). We note here that if $v_{i,\beta'}^{\ell'} \in V_e$ then

$$\varphi(q^{\ell+\ell'} z^{\beta+\beta'} b_i) = \varphi(b_{i,\beta}^\ell) \lambda^{-\ell'} t^{\beta'} = \varphi(b_{i,\beta'}^{\ell'}) \lambda^{-\ell} t^\beta$$

and

$$\frac{\varphi(b_{i,\beta}^\ell)}{\lambda^{-\ell} t^\beta} = \frac{\varphi(b_{i,\beta'}^{\ell'})}{\lambda^{-\ell'} t^{\beta'}} = \pi_i.$$

The coefficient of A_t (the matrix A in which each z_i is specialised at t_i) in position (i, j) is $\varphi(a_{i,j})$ (recall that $a_{i,j} \in \mathbb{K}[Z^{\pm 1}]$). Let $1 \leq j \leq n$. We have

$$qA_\Gamma b_j = q \sum_{i=1}^n a_{i,j} b_i.$$

It follows that $\varphi(qA_\Gamma) \pi_j = \varphi(q) \sum_{i=1}^n \varphi(a_{i,j}) \pi_i$ and $\sum_{i=1}^n \varphi(a_{i,j}) \pi_i = \lambda \pi_j$. Hence, this shows that λ is a left eigenvalue of A_t associated to the left eigenvector $(\pi_1, \dots, \pi_n)$. The matrix A_t is still irreducible (because the t_i 's are strictly positive), therefore π is equal to the unique left Perron–Frobenius vector of A_t and $\varphi(A_\Gamma) = \lambda$ is the associated Perron–Frobenius eigenvalue. The corresponding positive extremal function f is defined on Γ_e by $f(v_{i,\beta}^\ell) = \varphi(b_{i,\beta}^\ell) = \lambda^{-\ell} t^\beta \pi_i$.

Conversely, for any $t = (t_1, \dots, t_N) \in \mathbb{R}_{>0}^N$, let $(\pi_1 = 1, \pi_2, \dots, \pi_n)$ be the normalised Perron–Frobenius vector of A_t . We have a morphism $\varphi : \mathbb{K}[Z^{\pm 1}] \rightarrow \mathbb{R}$ defined by $\varphi(z_k) = t_k$ for any $k = 1, \dots, N$. By Lemma 2.4, we know that $\mathcal{A} = \mathbb{K}[Z^{\pm 1}][A_\Gamma]$ is integral over $\mathbb{K}[Z^{\pm 1}]$, it follows by [3, Section V.2.1, Corollary 4] that we can extend φ to a morphism (that we will also denote φ) from $\mathcal{A} = \mathbb{K}[Z^{\pm 1}][A_\Gamma]$ to $\mathbb{R}$ by setting $\varphi(A_\Gamma) = \lambda = \sum_{i=1}^n \varphi(a_{i,1}) \pi_i$. Therefore since $\mathcal{A}_e$ is a subalgebra of $\mathbb{K}[Z^{\pm 1}, q][A_\Gamma]$, we get by restriction a morphism $\varphi : \mathcal{A}_e \rightarrow \mathbb{R}$. Applying φ to the relation

$$qA_\Gamma b_j = q \sum_{i=1}^n a_{i,j} b_i$$

gives $\lambda \varphi(b_j) = \sum \varphi(a_{i,j}) \varphi(b_i)$. It follows that $(\varphi(b_1), \dots, \varphi(b_n))$ is the left eigenvector of A_t associated to λ . Further $b_1 = 1$ so $\varphi(b_1) = 1$. Since the normalised Perron–Frobenius vector is unique, we must have $\varphi(b_i) = \pi_i$ for all $i = 1, \dots, n$. This proves that we can indeed define from any $t = (t_1, \dots, t_m) \in \mathbb{R}_+^{*N}$ a morphism $\varphi : \mathcal{A}_e \rightarrow \mathbb{R}$ positive on the basis B_e and such that $\varphi(\gamma) = 1$. $\square$

Remark 6.15. — The arguments used in the proof of the previous theorem show that the morphisms obtained (and thus also the extremal harmonic functions) are essentially determined by their restrictions to $\mathbb{K}[Z^{\pm 1}]$ thanks to the Perron–Frobenius theorem. Nevertheless, in the previous construction, it can happen that two elements in $\mathbb{R}_+^{*N}$ give the same positive harmonic function so that we do not get a complete parametrisation of the extremal harmonic functions.

7. THE GROUP OF MAXIMAL INDICES

We shall assume in this section that $Z = \emptyset$ and $\mathbb{K}(Z^{\pm 1}) = \mathbb{C}$ (thus we only consider $\mathbb{C}$ -algebras). All the graphs in this section belong to $\mathbf{Graph}_n(\mathbb{R}_+)$. We have seen that in general, the property of a finite graph to be positively multiplicative depends on a choice of the vertex considered as a root. Let $\Gamma \in \mathbf{Graph}_n(\mathbb{R}_+)$ with vertices $\{v_1, \dots, v_n\}$ be a positively multiplicative graph at v_1 with respect to the matrix realisation $(\mathcal{A}, \mathbf{B})$ where $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. We further assume that $\mathcal{A}$ is commutative. Consider the cone

$$\mathcal{C}(\mathbf{B}) = \bigoplus_{j=1}^n \mathbb{R}_+ \mathbf{b}_j$$

in the algebra $\mathcal{A}$. In this section, we will consider the set of indices i such that

$$\mathcal{C}(\mathbf{B})(e_i) = \mathbb{R}_+^n \quad \text{where } (e_1, \dots, e_n) \text{ is the canonical basis of } \mathbb{C}^n.$$

We will prove in particular that it admits the structure of a commutative group. We start with PM algebras coming from PM graphs of maximal dimension and next consider the general situation of possibly infinite dimensional PM algebras.

7.1. generalised permutations and PM graphs of maximal dimension. Let $\Gamma \in \mathbf{Graph}_n(\mathbb{R}_+)$ with vertices $\{v_1, \dots, v_n\}$ be a PM graph at v_1 with respect to the matrix realisation $(\mathcal{A}, \mathbf{B})$ where $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. We will assume in this section that Γ is of maximal dimension so that $\mathcal{A} = \mathbb{C}[A_\Gamma]$. Also, we can apply Theorem 3.17 to Γ to see that the vector e_1 is cyclic and the map

$$\begin{aligned} \varphi : \mathbb{C}[A_\Gamma] &\longrightarrow \bigoplus_{i=1}^n \mathbb{C} e_i \\ P(A_\Gamma) &\longmapsto P(A_\Gamma) \cdot e_1 \end{aligned}$$

is an isomorphism of vector spaces. Moreover $\mathbf{b}_i$ is the unique element in $\mathcal{A}$ such that $\mathbf{b}_i(e_1) = e_i$ for all i .

DEFINITION 7.1. — We say that a $i_0 \in \{1, \dots, n\}$ is *maximal* if for any $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n$ with $\mu_1 + \dots + \mu_n = 1$ there exists $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}_+^n$ such that

$$\sum_{i=1}^n \nu_i \mathbf{b}_i(e_{i_0}) = \mu.$$

Equivalently, i_0 is maximal if $\mathcal{C}(\mathbf{B})(e_{i_0}) = \mathbb{R}_+^n$ but the definition above will be easier to extend to the case of infinite dimensional algebras. It is easy to show that 1 is maximal. Indeed, since $\mathbf{b}_i(e_1) = e_i$ for all $i = 1, \dots, n$, we have

$$\sum_{i=1}^n \mu_i \mathbf{b}_i(e_1) = \mu \quad \text{for all } \mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}_+^n.$$

Recall from Section 4.2 that $\mathcal{G}_n$ is the group of generalised permutations of size n and that we have defined the subgroup

$$G_\Gamma = \{U \in \mathcal{G}_n \mid UA_\Gamma = A_\Gamma U\}$$

of $\mathcal{G}_n$ which is the group of generalised automorphisms of the graph Γ . The group G_Γ is contained in the centralizer of A_Γ which is equal to $\mathbb{C}[A_\Gamma]$ (since Γ is of maximal dimension) and we have $G_\Gamma = \mathcal{G}_n \cap \mathcal{A}$. Consider U in G_Γ and set $U(e_i) = \lambda_i e_{\sigma(i)}$ with $\lambda_i \in \mathbb{R}_+^*$ for $i = 1, \dots, n$ and $\sigma \in \mathfrak{S}_n$. We get

$$\mathbf{b}_i U(e_1) = U \mathbf{b}_i(e_1) = U(e_i) = \lambda_i e_{\sigma(i)}.$$

Then

$$\sum_{i=1}^n \frac{\mu_{\sigma(i)}}{\lambda_i} \lambda_1 \mathbf{b}_i(e_{\sigma(1)}) = \sum_{i=1}^n \frac{\mu_{\sigma(i)}}{\lambda_i} \mathbf{b}_i U(e_1) = \mu.$$

This shows that $\sigma(1)$ is maximal and $\mathbf{b}_{\sigma(1)} = \frac{1}{\lambda_1} U \in G_\Gamma$ because U belongs to $\mathcal{A}$ and $\frac{1}{\lambda_1} U(e_1) = e_{\sigma(1)}$. Let us denote by $I_m \subset \{1, \dots, n\}$ the set of maximal indices for Γ . The results that we obtain in this section (working with countable infinite dimensional PM algebras) will imply the following statements:

- for all $i_0 \in I_m$, e_{i_0} is a cyclic vector for A_Γ . Thus there exists a matrix realisation $(A_\Gamma, \mathbf{B}^{(i_0)})$ of Γ where $\mathbf{B}^{(i_0)} = \{\mathbf{b}_1^{(i_0)}, \dots, \mathbf{b}_n^{(i_0)}\}$ and $\mathbf{b}_{i_0}^{(i_0)} = 1$,
- for all $i_0 \in I_m$, the basis $\mathbf{B}^{(i_0)}$ is contained in the group G_Γ ,
- the group G_Γ acts transitively on I_m ,
- the set I_m itself has the structure of an abelian group.

7.2. generalised adjacency algebra and positivity. Consider a positively multiplicative commutative algebra $\mathcal{A}$ with respect to the countable basis $\mathbf{B} = \{\mathbf{b}_i \mid i \in I\}$. Let $A_i = (c_{i,j}^k)_{j,k \in I}$ be the (possibly infinite) matrix of the multiplication $\mathbf{b}_i$ expressed in the basis $\mathbf{B}$ (that is $\mathbf{b}_i \mathbf{b}_j = \sum_k c_{i,j}^k \mathbf{b}_k$). Then A_i can be regarded as the adjacency matrix of an oriented weighted graph with set of vertices $\{x_i \mid i \in I\}$ and edge weight function ω defined by $\omega(x_j, x_k) = c_{i,j}^k$ for all j, k in I (recall our convention that $\omega(x, x') = 0$ when there is no edge from x to x'). When $n = \text{card}(I)$ is finite, the family of adjacency matrices $\{A_i, i \in I\}$ generates a subalgebra of $M_n(\mathbb{C})$ isomorphic to $\mathcal{A}$.

We can formalise this phenomenon by the notion of generalised adjacency algebra. Throughout this section, I is a countable set and V will denote the normed vector space $\ell^1(I)$ of complex sequences (v_i) indexed by I with the ℓ^1 -norm $\|v\|_1 = \sum_{i \in I} |v_i|$. The vector space V is complete, and we identify I with the Schauder³ basis $(e_i)_{i \in I}$ of V given by $e_i(j) = \delta_{ij}$ for $i, j \in I$. We write V_+ for the cone of V consisting of complex sequences taking non-negative real values, and $V_{+,1}$ the subset of V_+ having ℓ^1 -norm equal to one. From a measure theory point of view, V denotes the set of finite complex measures on the set I , while V_+ (resp. $V_{+,1}$) denotes the set of positive (resp. probability) measures on V .

³A Schauder basis S of a Banach space V is a subset of V such that any element $v \in V$ as a unique writing as a convergent sum $\sum_{s \in S} a_s s$ with $a_s \in \mathbb{C}$, $s \in S$. This is the natural extension of the definition of the basis of a vector space in the Banach space setting.

We denote by $B(V)$ the set of endomorphisms of V which are bounded with respect to the ℓ^1 -norm, namely the set of linear maps $T : V \rightarrow V$ such that

$$\|T\|_{1,1} = \sup_{v \in V, \|v\|_1=1} \|Tv\|_1 < +\infty.$$

The map $T \mapsto \|T\|_{1,1}$ defines a norm on $B(V)$. If $T \in B(V)$, we write T_{ij} for the coefficient of Te_j along e_i for $i, j \in I$. Given T_1 and T_2 in $B(V)$, we have

$$\|T_1 T_2\|_{1,1} \leq \|T_1\|_{1,1} \|T_2\|_{1,1}$$

so that $T_1 T_2$ also belongs to $B(V)$. This shows that $B(V)$ is stable by composition of operators.

Given any normed space E and any subset $S \subset E$, write

$$\mathcal{C}(S) = \overline{\left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \geq 0, s \in S \right\}} \quad \text{and} \quad \mathcal{C}_1(S) = \mathcal{C}(S) \cap B(0, 1)$$

where the closure is taken with respect to the norm on E and $B(0, 1)$ is the unit ball.

DEFINITION 7.2.

- An adjacency operator on I is an operator $T \in B(V)$ such that

$$T_{ij} \geq 0$$

for all $i, j \in I$.

- An adjacency algebra on I is a closed commutative unital subalgebra of $B(V)$ generated by adjacency operators on I .

For an adjacency algebra $\mathcal{A}$ (regarded as a normed space as previously), define the adjacency cone of $\mathcal{A}$ as the cone

$$\mathcal{A}_+ = \mathcal{C}(\{T \in \mathcal{A}, T \text{ adjacency operator on } I\}).$$

By definition, $\mathcal{A}$ is then generated by $\mathcal{A}_+$. Since $\mathcal{A}_+ \cdot \mathcal{A}_+ \subset \mathcal{A}_+$, this implies that

$$\mathcal{A} = (\mathcal{A}_+). \tag{7.1}$$

Example 7.3. — Let $I = \{1, \dots, n\}$ and let V be n -dimensional $\mathbb{C}$ -vector space with basis $\{e_1, \dots, e_n\}$. Let A_Γ is the adjacency matrix of an oriented graph $\Gamma \in \text{Graph}_n(\mathbb{R}_+)$. Then $\mathbb{C}[A_\Gamma]$ is an adjacency algebra on I . We always have $\mathcal{C}(A^k, k \geq 0) \subset \mathbb{C}[A]_+$, but in general the inclusion is strict. Consider for example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{where } A - A^2/2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}[A]_+ \setminus \mathcal{C}(A^k, k \geq 0).$$

DEFINITION 7.4. — Two adjacency algebras $\mathcal{A}, \mathcal{A}'$ on I are isomorphic if there exists an invertible adjacency operator $U : V \rightarrow V$ such that U^{-1} is again an adjacency operator and such that $\mathcal{A}' = \{UAU^{-1} \mid A \in \mathcal{A}\}$.

It is then straightforward to prove that for two isomorphic adjacency algebras $\mathcal{A}$ and $\mathcal{A}'$ we have $\mathcal{A}'_+ = U\mathcal{A}_+U^{-1}$. As the following lemma shows, isomorphisms between adjacency algebras are infinite-dimensional analogues of the generalised permutations of the set I (i.e. permutations and diagonal rescaling of the basis $(e_i)_{i \in I}$).

LEMMA 7.5. — *If T is an invertible adjacency operator such that T^{-1} is also an adjacency operator, then T is a permutation up to multiplication by a diagonal operator with positive coefficients, meaning that there exists a permutation σ of I and a family $\{\lambda_i\}_{i \in I}$ of positive reals such that*

$$Te_i = \lambda_i e_{\sigma(i)}.$$

Proof. — Suppose by contradiction the existence of $i_0 \in I$ and $i_1, i_2 \in I$ with $i_1 \neq i_2$ such that

$$Te_{i_0} = \lambda e_{i_1} + \mu e_{i_2} + \sum_{i \in I \setminus \{i_1, i_2\}} l_i e_i$$

with $\lambda, \mu > 0$ and $l_i \geq 0$ for all $i \in I \setminus \{i_1, i_2\}$. Then, since $T^{-1}Te_{i_0} = e_{i_0}$ and T^{-1} has non-negative entries, the latter equality implies that both $T^{-1}e_{i_1}$ and $T^{-1}e_{i_2}$ belong to $\mathbb{R}_{\geq 0}e_{i_0}$. But this contradicts the fact that T^{-1} is invertible. $\square$

For $i \in I$ and an adjacency algebra $\mathcal{A}$, denote by $\text{ev}_i : \mathcal{A} \rightarrow V$ the evaluation map $A \mapsto Ae_i$. In the following definition, recall that $\mathcal{A}$ is a Banach space as a closed subspace of $B(V)$ and that a linear map $f : E \rightarrow F$ between Banach spaces is *coercive* if $\inf_{x \in E, \|x\|_E=1} \|f(x)\|_F > 0$. Remark in particular that coercivity implies injectivity for a linear map. Both notions coincide when I is finite because $\mathcal{A}$ is then finite-dimensional and its unit ball is compact.

DEFINITION 7.6. — Let $\mathcal{A}$ be an adjacency algebra $\mathcal{A}$ on I and consider an element $i \in I$.

- The element i is *nondegenerate* if ev_i is coercive (and thus injective) as a linear map from $\mathcal{A}$ to V .
- The element i is *maximal* if ev_i is surjective from $\mathcal{A}_+$ to V_+ .

In the latter definition, coercivity can be replaced by injectivity when I is finite. Observe also that when i is maximal, by (7.1) the vector e_i is cyclic for the algebra $\mathcal{A}$, that is we have $\mathcal{A}e_i = V$.

We first state a result proving that we mostly have to establish nondegeneracy at one maximal element to get nondegeneracy at any maximal element.

LEMMA 7.7. — *Suppose that $i_0 \in I$ is nondegenerate and maximal. Then, any $i \in I$ which is maximal is also nondegenerate. If I is finite and $i_0 \in I$ is maximal for $\mathcal{A}$, then i_0 is also nondegenerate and $\dim \mathcal{A} = |I|$.*

Proof. — Suppose that $i_0 \in I$ is maximal and nondegenerate for $\mathcal{A}$, and set

$$\alpha = \inf_{T \in \mathcal{A}, \|T\|_{1,1}=1} \|\text{ev}_{i_0}(T)\|_1.$$

We have $\alpha > 0$ by coercivity of ev_{i_0} . Let $i \in I$ be another element which is also maximal. Let $T \in \mathcal{A}_+$ be such that $\|T\|_{1,1} = 1$. Since e_i and e_{i_0} are maximal, there exist $U \in \mathcal{A}_+$ such that $Ue_i = e_{i_0}$ and $U' \in \mathcal{A}_+$ such that $U'e_{i_0} = e_i$. Hence, $UU'e_{i_0} = e_{i_0}$, and by nondegeneracy of e_{i_0} we have $UU' = Id = U'U$ (recall that $\mathcal{A}$ is commutative). In particular, $\|TU'\|_{1,1} \geq \|U\|_{1,1}^{-1}\|T\|_{1,1}$. Hence,

$$\|\text{ev}_i(T)\|_1 = \|\text{ev}_{i_0}(TU')\|_1 \geq \alpha\|TU'\|_{1,1} \geq \alpha\|U\|_{1,1}^{-1}\|T\|_{1,1},$$

so that $\inf_{T \in \mathcal{A}, \|T\|_{1,1}=1} \|\text{ev}_i(T)\|_1 \geq \alpha\|U\|_{1,1}^{-1} > 0$, and ev_i is also coercive. Hence, i is also nondegenerate for $\mathcal{A}$.

Suppose that I is finite of cardinal n , so that V is finite dimensional. Then, $\mathcal{A}$ is a subalgebra of $M_n(\mathbb{C})$. By maximality at i_0 , the map ev_{i_0} is surjective from $\mathcal{A}$ to V , which yields $\dim(\mathcal{A}) \geq \dim V = n$. Suppose that $T \in \mathcal{A}$ is such that $Te_{i_0} = 0$. By surjectivity of ev_{i_0} , for all $v \in V$ there exists $T_v \in \mathcal{A}$ such that $T_v e_{i_0} = v$. Then, by commutativity of $\mathcal{A}$,

$$Tv = TT_v e_{i_0} = T_v T e_{i_0} = 0.$$

Hence, $Tv = 0$ for all $v \in V$, and so $T = 0$, which implies that ev_{i_0} is injective and $\dim(\mathcal{A}) \geq n$. Thus $\dim \mathcal{A} = n$ and ev_{i_0} is a linear isomorphism, in particular it is coercive. $\square$

Remark 7.8. — In general, a commutative subalgebra of $M_n(\mathbb{C})$ can have a dimension much bigger than n : by a theorem of Schur, such an algebra can have dimension at most $\lfloor n^2/4 \rfloor + 1$, the bound being sharp. Hence, by the latter lemma, a finite-dimensional adjacency algebra for which there exists a maximal element is much closer to the case of a commutative subalgebra of $M_n(\mathbb{C})$ stable by the adjoint involution (see Example 2.2(4)). Beware however that an adjacency algebra is not necessarily diagonalisable (i.e isomorphic to $\mathbb{C}^n$), even if there exists a maximal element. This is for example the case for the algebra of upper triangular Toeplitz matrices

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ 0 & \lambda_1 & \ddots & \vdots \\ \vdots & & \ddots & \lambda_2 \\ & & \dots & \lambda_1 \end{pmatrix}, \lambda_1, \dots, \lambda_n \in \mathbb{C} \right\},$$

for which e_n is maximal.

PROPOSITION 7.9. — *Suppose that $\mathcal{A}$ is an adjacency algebra for which i_0 is nondegenerate and maximal. Then, there exists a unique basis $\mathbf{B} = \{\mathbf{b}_i \mid i \in I\} \subset \mathcal{A}_+$ of $\mathcal{A}$ such that $\mathcal{A}_+ = \mathcal{C}(\mathbf{B})$ and $\mathbf{b}_i e_{i_0} = e_i$ for all $i \in I$. In particular, we have $b_{i_0} = 1$ and $\mathcal{A}$ is positively multiplicative.*

Proof. — Suppose that i_0 is nondegenerate and maximal for $\mathcal{A}$, and denote by $\mathbf{b}_i$ the element of $\mathcal{A}_+$ such that $\mathbf{b}_i e_{i_0} = e_i$. Remark that $\{\mathbf{b}_i \mid i \in I\}$ is a free family, since it acts freely on e_{i_0} by nondegeneracy. Let $T \in \mathcal{A}_+$. Then, $Te_{i_0} = \sum_{i \in I} \lambda_i e_i$, with $\lambda_i \geq 0$ because T is an adjacency operator, and

$$\sum_{i \in I} \lambda_i = \|Te_{i_0}\|_1 \leq \|T\|_{1,1} \|e_{i_0}\|_1 = \|T\|_{1,1} < +\infty.$$

By nondegeneracy of i_0 , the map ev_{i_0} is coercive, implying that

$$\inf_{T \in \mathcal{A}, T \neq 0} \frac{\|Te_{i_0}\|_1}{\|T\|_{1,1}} = \alpha > 0.$$

Hence, for $i \in I$ we have $\|\mathbf{b}_i e_{i_0}\|_1 \geq \alpha \|\mathbf{b}_i\|_{1,1}$. On the other hand by construction, $\|\mathbf{b}_i e_{i_0}\|_1 = \|e_i\|_1 = 1$ for $i \in I$. Hence, for all $i \in I$ we have $\|\mathbf{b}_i\|_{1,1} \leq \frac{1}{\alpha}$.

Hence, $T' := \sum_{i \in I} \lambda_i \mathbf{b}_i$ satisfies $\|T'\|_{1,1} \leq \frac{1}{\alpha} \sum_{i \in I} \lambda_i < +\infty$ and thus is a well-defined element of $\mathcal{A}_+$ which satisfies $T' e_{i_0} = Te_{i_0}$. By nondegeneracy, $T' = T$, and thus $T \in \mathcal{C}(\mathbf{b}_i, i \in I)$. Therefore, $\mathcal{A}_+ = \mathcal{C}(\mathbf{b}_i, i \in I)$. For $i, i' \in I$, we have $\mathbf{b}_i \mathbf{b}_{i'} \in \mathcal{A}_+$ and by the previous result $\mathbf{b}_i \mathbf{b}_{i'} = \sum_{i'' \in I} \lambda_{ii'}^{i''} \mathbf{b}_{i''}$ for some nonnegative coefficients $\lambda_{ii'}^{i''}$. Therefore the basis $\{\mathbf{b}_i\}_{i \in I}$ is positively multiplicative. We deduce

that $\mathcal{A}$ is positively multiplicative. Finally, remark that the basis $\{\mathbf{b}_i\}_{i \in I}$ is uniquely defined due to the injectivity of ev_{i_0} . $\square$

The proposition above yields an infinite dimensional version of Corollary 3.9.

Example 7.10. — Given a discrete commutative group G , define

$$\ell^1(G) = \left\{ f : G \longrightarrow \mathbb{C} \left| \sum_{g \in G} |f(g)| < +\infty \right. \right\},$$

and consider the group algebra $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}g$ with multiplication given by the group structure. Consider the left-regular representation ρ of G on the basis $\{\delta_g, g \in G\}$ of $\ell^1(G)$ such that

$$\rho(g)\delta_{g'} = \delta_{gg'}.$$

Let $\mathcal{A}^G$ be the closure of $\rho(\mathbb{C}[G])$ in $B(\ell^1(G))$ with respect to the $\|\cdot\|_{1,1}$ norm. The algebra $\mathcal{A}^G$ is then an adjacency algebra G , and the adjacency cone is $\mathcal{A}_+^G = \bigoplus_{g \in G} \mathbb{R}_{>0} \rho(g)$. Indeed, each operator $\rho(g)$ acts by permutation and thus is an adjacency operator. Moreover, since $\rho(g)\rho(g') = \rho(gg')$ for $g, g' \in G$, we indeed have $\text{span}(\mathcal{A}_+^G) = \mathcal{A}_G$. Remark that every element δ_g with $g \in G$ is maximal and nondegenerate for $\mathcal{A}_G$.

We will see below that the example above is actually the only example of adjacency algebra on a countable set I for which every element of I is nondegenerate and maximal, up to a normalization. We will first prove a more general result in Theorem 7.11.

Since I is countable, any permutation σ of I extend linearly to an operator of $B(V)$ (also denoted by σ) with the formula

$$\sigma(e_i) = e_{\sigma(i)}$$

and we have $\|\sigma\|_{1,1} = 1$.

So, let $\mathcal{A}$ be an adjacency algebra on I , and denote by $I_m \subset I$ the set of elements which are *maximal and nondegenerate*. We will assume that I_m is not empty (thus contains all the maximal elements by Lemma 7.7). Also denote by $\mathcal{G}$ the group of linear automorphisms $T : V \rightarrow V$ such that $T(e_i) = \lambda_i e_{\sigma(i)}$ with $\lambda_i \in \mathbb{R}_{>0}$ and σ any permutation of I . Recall that $G_{\mathcal{A}}$ is the subgroup $\mathcal{G} \cap \mathcal{A}$.

THEOREM 7.11. — *Under the previous hypotheses, the following assertions hold.*

- (1) Each $i_0 \in I_m$ defines a multiplicative basis $\mathbf{B}^{(i_0)} = \{\mathbf{b}_i^{(i_0)} \mid i \in I\}$ for the algebra $\mathcal{A}$.
- (2) For any $i_0 \in I_m$, the set $\mathbf{B}^{(i_0)} = \{\mathbf{b}_i^{(i_0)} \mid i \in I\}$ is contained in the group $G_{\mathcal{A}}$.
- (3) The group $G_{\mathcal{A}}$ acts on I_m transitively.
- (4) The set I_m has the structure of a commutative group. Moreover, there exists a representation ρ of I_m on V such that $\{\rho(i) \mid i \in I_m\} \subset \mathcal{A}_+$ and $\rho(i)e_j \in \mathbb{R}_{>0}e_{i \cdot j}$ for $i, j \in I_m$, where $(i, j) \mapsto i \cdot j$ denotes the product structure on I_m .

Example 7.12. — We refer to Section 5.5 for the example of the fusion rule for the affine group $\widehat{su}(2)$. The set I_m and its associated group structure can be easily

made explicit. Indeed, the fusion rules shows that $\mathbf{b}_0 \mathbf{b}_j = 1$ and $\mathbf{b}_l \mathbf{b}_j = \mathbf{b}_{l-j}$ for any $j = 0, \dots, l$. Also when $i = 1, \dots, l-1$ the action of $\mathbf{b}_i$ does not yield a generalised permutation of the basis $\mathbf{B}$. Thus $I_m = \{0, l\}$ and the group structure is given by

$$l \cdot l = 0, \quad l \cdot 0 = 0 \cdot l = l, \quad 0 \cdot 0 = 0.$$

The theorem above says in particular that, up to a nondegeneracy condition, the set of maximal elements I_m has the structure of an abelian group and, up to a scaling, the action of the group algebra $\mathbb{C}[I_m]$ on $\ell^1(I_m)$ (as introduced in Example 7.10) extends to a representation on all $\ell^1(I)$ whose image forms an adjacency subalgebra of $\mathcal{A}$. Beware that the group structure on I_m is not uniquely defined and depend on the choice of an element in I_m .

Proof of Theorem 7.11. — As already observed, the existence of a maximal nondegenerate element and Lemma 7.7 imply that I_m is the set of maximal elements for $\mathcal{A}$. For $i \in I_m, i' \in I$, let $\mathbf{b}_{i,i'} \in \mathcal{A}_+$ be such that $\mathbf{b}_{i,i'} e_i = e_{i'}$. Then, if $i' \in I_m$, there exists $\mathbf{b}_{i',i} \in \mathcal{A}_+$ such that $\mathbf{b}_{i',i} e_{i'} = e_i$. This implies that $\mathbf{b}_{i',i} \mathbf{b}_{i,i'} e_i = e_i$, and by nondegeneracy of e_i and the fact that $\text{Id} \in \mathcal{A}_+$, this in turn implies that $\mathbf{b}_{i',i} \mathbf{b}_{i,i'} = \mathbf{b}_{i,i'} \mathbf{b}_{i',i} = \text{Id}$. Since $\mathbf{b}_{i,i'}$ and $\mathbf{b}_{i',i}$ are adjacency operators, Lemma 7.5 yields that $\mathbf{b}_{i,i'}$ and $\mathbf{b}_{i',i}$ belong to $G_{\mathcal{A}}$. Hence, for each $i, i' \in I_m$, there exists a infinite permutation matrix $\Sigma_{i,i'}$ and an infinite diagonal matrix $D_{i,i'}$ with positive diagonal entries such that

$$\mathbf{b}_{i,i'} = \Sigma_{i,i'} D_{i,i'}. \quad (7.2)$$

Fix an element $i_0 \in I_m$. We get the basis $\mathbf{B}^{(i_0)}$ by Proposition 7.9 such that $\mathbf{b}_i(e_{i_0}) = e_i$ for any $i \in I$. When i belongs to I_m , we have by the previous arguments $\mathbf{b}_{i_0,i} = \Sigma_{i_0,i} D_{i_0,i}$ and we can set $\mathbf{b}_i = \mathbf{b}_{i_0,i}$, $\Sigma_i = \Sigma_{i_0,i}$ and $D_i = D_{i_0,i}$. We can write

$$\mathbf{b}_i \mathbf{b}_{i'} = \sum_{i'' \in I} \lambda_{i''} \mathbf{b}_{i''}, \quad (7.3)$$

for all $i, i' \in I_m$ with $\lambda_{i''} \in \mathbb{R}_{\geq 0}$ for any $i'' \in I$. Since $\mathbf{b}_i$ and $\mathbf{b}_{i'}$ belong to the group $G_{\mathcal{A}}$, we also have $\mathbf{b}_i \mathbf{b}_{i'}$ in $G_{\mathcal{A}}$. Therefore, by looking to the expression on the right hand side of the equality, there should exist $j \in I$ such that $\mathbf{b}_i \mathbf{b}_{i'} = \mu \mathbf{b}_j$ for some $\mu > 0$. Indeed $\mathbf{b}_i \mathbf{b}_{i'}$ must send the vector e_{i_0} on a scalar multiple of a vector e_j with $j \in I$ because $\mathbf{b}_i$ and $\mathbf{b}_{i'}$ belong to $G_{\mathcal{A}}$. This is only possible when all but one of the coefficients $\lambda_{i''}$ are equal to zero. Recall that $\mathbf{b}_j(e_{i_0}) = e_j$ and $i_0 \in I_m$. It follows that we also have $j \in I_m$ since $\mathcal{A}_+ \mathbf{b}_j \subset \mathcal{A}_+$ and we can set $i \cdot i' := j \in I_m$ and $\lambda_{i,i'} = \lambda_j > 0$ to get

$$\mathbf{b}_i \mathbf{b}_{i'} = \lambda_{i,i'} \mathbf{b}_{i \cdot i'}.$$

From the commutativity and the associativity of the product in $\mathcal{A}$, we deduce that the product $(i, i') \mapsto i \cdot i'$ is commutative and associative. There is also a neutral element given by i_0 , since $\mathbf{b}_{i_0} = \text{Id}$. Likewise, expending $\mathbf{b}_{i,i_0}$ on the basis $\{\mathbf{b}_i \mid i \in I\}$ and using a similar argument as before yields an element $i' \in I$ such that

$$\mathbf{b}_{i,i_0} = \mathbf{b}_i^{-1} = \lambda \mathbf{b}_{i'}.$$

Hence, this implies

$$\mathbf{b}_{i_0} = \text{Id} = \lambda \mathbf{b}_{i'} \mathbf{b}_i = \lambda \lambda_{i',i} \mathbf{b}_{i' \cdot i},$$

and $i' \cdot i = i_0$. Finally, I_m has a commutative group structure given by the product $(i, i') \mapsto i \cdot i'$. Let us denote by G_{I_m} this group.

Set $H = \{\lambda \mathbf{b}_i \mid \lambda > 0, i \in I\} \subset \mathcal{A}$. By the above reasoning, H is a subgroup of the group of invertible elements of $\mathcal{A}$ with multiplication $(\lambda \mathbf{b}_i) \cdot (\mu \mathbf{b}_{i'}) = (\lambda \mu \lambda_{i,i'}) \mathbf{b}_{i \cdot i'}$. Moreover, $\mathbb{R}_{>0} \mathbf{b}_{i_0}$ is a divisible subgroup of H , which means that for any $n \in \mathbb{N}$, the endomorphism $\lambda \mathbf{b}_{i_0} \mapsto (\lambda \mathbf{b}_{i_0})^n$ is surjective. Hence, by [7, Theorem 21.2], there exists a subgroup $F \subset H$ such that

$$H = (\mathbb{R}_{>0} \mathbf{b}_{i_0}) \times F.$$

For $i \in I$, let $\bar{\mathbf{b}}_i \in F, \mu_i \in \mathbb{R}_{>0}$ be such that $\mathbf{b}_i = (\mu_i \mathbf{b}_{i_0}) \bar{\mathbf{b}}_i = \mu_i \bar{\mathbf{b}}_i$. For $i, i' \in I$, on the one hand, $\bar{\mathbf{b}}_i \bar{\mathbf{b}}_{i'} \in F$, and on the other hand

$$\bar{\mathbf{b}}_i \bar{\mathbf{b}}_{i'} = \frac{1}{\mu_i \mu_{i'}} \mathbf{b}_i \mathbf{b}_{i'} = \frac{\lambda_{i,i'}}{\mu_i \mu_{i'}} \mathbf{b}_{i \cdot i'} = \frac{\lambda_{i,i'} \mu_{i,i'}}{\mu_i \mu_{i'}} \bar{\mathbf{b}}_{i \cdot i'} = \left(\frac{\lambda_{i,i'} \mu_{i,i'}}{\mu_i \mu_{i'}} \mathbf{b}_{i_0} \right) \bar{\mathbf{b}}_{i \cdot i'}.$$

Since $H = (\mathbb{R}_{>0} \mathbf{b}_{i_0}) \times F$ and $\bar{\mathbf{b}}_i \bar{\mathbf{b}}_{i'} \in F$ we must have $\frac{\lambda_{i,i'} \mu_{i,i'}}{\mu_i \mu_{i'}} = 1$ and thus

$$\bar{\mathbf{b}}_i \bar{\mathbf{b}}_{i'} = \bar{\mathbf{b}}_{i \cdot i'}.$$

Hence, the map $\rho : i \mapsto \bar{\mathbf{b}}_i$ yields a representation of G_{I_m} on V such that $\rho(i) \in G_{\mathcal{A}}$ for all $i \in I_m$. Moreover, for $i, j \in I_m$,

$$\bar{\mathbf{b}}_i e_j = \bar{\mathbf{b}}_i \mathbf{b}_j e_{i_0} = \mu_j \bar{\mathbf{b}}_i \bar{\mathbf{b}}_j e_{i_0} = \frac{\mu_j}{\mu_{i \cdot j}} \mathbf{b}_{i \cdot j} e_{i_0} = \frac{\mu_j}{\mu_{i \cdot j}} e_{i \cdot j} \in \mathbb{R}_{>0} e_{i \cdot j},$$

which implies that ρ yields a free and transitive permutation representation on the set of half-lines $\{\mathbb{R}_{>0} e_i, i \in I_m\}$. $\square$

As a corollary of Theorem 7.11, we can consider the case where all elements of I are maximal for $\mathcal{A}$.

COROLLARY 7.13. — *Suppose that $\mathcal{A}$ is an adjacency algebra for which each $i \in I$ is maximal and nondegenerate. Then, there exists a commutative group structure on I , and a positive diagonal operator Λ such that $\Lambda \mathcal{A} \Lambda^{-1} = \mathcal{A}_I$, where as in Example 7.10 the algebra $\mathcal{A}_I$ is the norm-closure of the left-regular representation of $\mathbb{C}[I]$ on $\ell^1(I)$ in the Banach algebra $B(\ell^1(I))$.*

Proof. — By Proposition 7.11, I has a group structure and there is an action $\rho : I \rightarrow B(V)$ such that $\rho(i) \in \mathcal{A}$ for all $i \in I$ and $\rho(i) e_j \in \mathbb{R}_{>0} e_{i \cdot j}$ for all $i, j \in I$. Let i_0 be the neutral element of I and set $\mathbf{b}_i = \rho(i)$. By the arguments used in the previous proof, for any $i \in I$ there exists $\lambda_i > 0$ such that $\mathbf{b}_i e_{i_0} = \lambda_i e_{i \cdot i_0} = \lambda_i e_i$, with the particular case $\lambda_{i_0} = 1$. Hence, defining the diagonal operator Λ by $\Lambda e_i = \lambda_i^{-1} e_i$, we have $\Lambda \mathbf{b}_i \Lambda^{-1} e_{i_0} = e_i$. Hence, for $i, j \in I$,

$$\Lambda \mathbf{b}_i \Lambda^{-1} e_j = \Lambda \rho(i) \Lambda^{-1} \Lambda \rho(j) \Lambda^{-1} e_{i_0} = \Lambda \rho(i) \rho(j) \Lambda^{-1} e_{i_0} = \Lambda \rho(i \cdot j) \Lambda^{-1} e_{i_0} = e_{i \cdot j}.$$

Let $\mathcal{A}_I$ be the algebra generated by $\Lambda \mathbf{b}_i \Lambda^{-1}$, $i \in I$. By the latter results, we have $\mathcal{A}_I = \rho(\mathbb{C}[I])$, where ρ is the left regular representation introduced in Example 7.10. By Proposition 7.9, $\mathcal{A}$ is the norm closure of $\sum_{i \in I} \mathbb{C} \mathbf{b}_i$. Hence, $\Lambda \mathcal{A} \Lambda^{-1}$ is the norm closure of $\mathcal{A}_I$, and thus we have

$$\Lambda \mathcal{A} \Lambda^{-1} = \mathcal{A}_I. \quad \square$$

Example 7.14. — One can use Corollary 7.13 to construct finite PM-graphs with $I = I_m$. Consider for example the group algebra

$$\mathcal{A} = \mathbb{C} e_{(0,0)} \oplus \mathbb{C} e_{(1,0)} \oplus \mathbb{C} e_{(0,1)} \oplus \mathbb{C} e_{(1,1)}$$

of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let T and T' be the matrices of the multiplication by $e_{(1,0)}$ and $e_{(0,1)}$ expressed in the previous basis and $(a, b) \in \mathbb{R}_{>0}^2$. Then

$$U = aT + bT' = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ b & 0 & 0 & a \\ 0 & b & a & 0 \end{pmatrix}.$$

Now choose $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}_{>0}^4$ and set $V = DUD^{-1}$ where D is the diagonal matrix defined by λ . The matrix

$$V = \begin{pmatrix} 0 & a\frac{\lambda_1}{\lambda_2} & b\frac{\lambda_1}{\lambda_3} & 0 \\ \frac{a}{\lambda_1}\lambda_2 & 0 & 0 & b\frac{\lambda_2}{\lambda_4} \\ \frac{b}{\lambda_1}\lambda_3 & 0 & 0 & a\frac{\lambda_3}{\lambda_4} \\ 0 & \frac{b}{\lambda_2}\lambda_4 & \frac{a}{\lambda_3}\lambda_4 & 0 \end{pmatrix}$$

is the adjacency matrix of a PM-graph such that $I = I_m$.

We end this section by showing that the set of positive roots in finite PM-graphs (See Definition 4.1) coincides with the set I_m of maximal indices in its associated basis $\mathbf{B}$.

PROPOSITION 7.15. — *Assume $k = \mathbb{C}$ and let Γ be a finite PM graph rooted at v_{i_0} with basis $\mathbf{B}_{i_0}$ whose adjacency matrix has maximal dimension.*

- (1) *The set of maximal indices label the positive roots of Γ , that is $\mathcal{R}_\Gamma^+ = \{v_i \mid i \in I_m\}$.*
- (2) *Set $\mathbf{B}_{m,i_0} = \{\mathbf{b}_i \in \mathbf{B}_{i_0} \mid i \in I_m\}$. Then, we have $\mathbf{B}_{m,i_0} = \mathbf{B}_{i_0} \cap G_\Gamma$: the maximal indices label the generalised permutations appearing in $\mathbf{B}_{i_0}$.*

Proof. — Let us prove Assertion (1). The graph Γ is multiplicative at v_i for the basis $\mathbf{b}_i^{-1}\mathbf{B}_{i_0}$. Moreover since Γ has maximal dimension, this multiplicative basis is unique. Therefore v_i belongs to $\mathcal{R}_\Gamma^+$ if and only if $\mathbf{b}_i^{-1}\mathbf{B}_{i_0}$ has nonnegative structure constants. This is equivalent to say that $\mathbf{b}_i^{-1}\mathbf{B}_{i_0} \times \mathbf{b}_i^{-1}\mathbf{B}_{i_0} \subset \mathbf{b}_i^{-1}\mathcal{A}_+$ because $\mathcal{A}_+$ coincides with $\mathcal{C}(\mathbf{B}_{i_0})$ by Proposition 7.9 because $i_0 \in I_m$.

For any $i \in I_m$, it follows from Theorem 7.11 that $\mathbf{b}_i$ is a generalised permutation. Thus $\mathbf{b}_i^{-1} \in \mathcal{A}_+$ and also $\mathbf{b}_i^{-1}\mathbf{B}_{i_0} \subset \mathcal{A}_+$. Now we have

$$\mathbf{b}_i^{-1}\mathbf{B}_{i_0} \times \mathbf{b}_i^{-1}\mathbf{B}_{i_0} = \mathbf{b}_i^{-1}(\mathbf{b}_i^{-1}\mathbf{B}_{i_0} \times \mathbf{B}_{i_0}) \subset \mathbf{b}_i^{-1}(\mathcal{A}_+ \times \mathcal{A}_+) \subset \mathbf{b}_i^{-1}\mathcal{A}_+$$

because $\mathcal{A}_+$ is stable by product. This shows that $\{v_i \mid i \in I_m\} \subset \mathcal{R}_\Gamma^+$.

Conversely, assume that $i \in I$ is such that $v_i \in \mathcal{R}_\Gamma^+$. Then $\mathbf{b}_i$ is invertible and $\mathbf{b}_i^{-1} \in \mathbf{b}_i^{-1}\mathbf{B}_{i_0}$ since $1 \in \mathbf{B}_{i_0}$. Thus, by the previous arguments, we get $\mathbf{b}_i^{-1} \times \mathbf{b}_i^{-1} = \mathbf{b}_i^{-1}c$ with $c \in \mathcal{A}_+$. This proves that $\mathbf{b}_i^{-1} \in \mathcal{A}_+$. Since the cone $\mathcal{A}_+$ is stable by multiplication, it follows that the map $c \rightarrow \mathbf{b}_i c$ is a bijection from $\mathcal{A}_+$ on itself (with inverse the multiplication by $\mathbf{b}_i^{-1}$). This implies that

$$\mathcal{A}_+(e_i) = \mathcal{A}_+\mathbf{b}_i(e_{i_0}) = \mathbf{b}_i\mathcal{A}_+(e_{i_0}) = \mathcal{A}_+(e_{i_0}) = V_+.$$

We get that i is maximal. But it is also nondegenerate by Lemma 7.7 because Γ is finite. Thus, we have $\mathcal{R}_\Gamma^+ = \{v_i \mid i \in I_m\}$ as desired.

To prove Assertion (2), observe first that we have $\mathbf{B}_{m,i_0} \subset \mathbf{B}_{i_0} \cap G_\Gamma$ by Assertion (2) of Theorem 7.11. Now, if $\mathbf{b}_i$ belongs to $\mathbf{B}_{i_0} \cap G_\Gamma$, its inverse is an adjacency matrix in $\mathcal{A}$ and we have $\mathbf{b}_i^{-1} \in \mathcal{A}_+$. By using the previous arguments, we get

$\mathbf{b}_i \mathcal{A}_+ = \mathcal{A}_+$ and therefore $\mathcal{A}_+ v_i = \mathcal{A}_+ \mathbf{b}_i v_{i_0} = \mathcal{A}_+ v_{i_0} = V_+$ which shows that $i \in I_m$. Thus, $\mathbf{B}_{m, i_0} \supset \mathbf{B}_{i_0} \cap G_\Gamma$ as desired. $\square$

Remark 7.16. — It also follows from the preceding results of this section that (for $k = \mathbb{C}$) a graph Γ with n -vertices will be positively multiplicative as soon as its adjacency algebra $\mathbb{C}[A]$ can be embedded in a n -dimensional $\mathbb{C}$ -algebra $\mathcal{A}$ generated by matrices with nonnegative entries for which there exists an index $1 \leq i \leq n$ with $\mathcal{A}_+ e_i = \bigoplus_{1 \leq j \leq n} \mathbb{R}_{>0} e_j$.

REFERENCES

- [1] Arnaud Beauville. Conformal blocks, fusion rules and the Verlinde formula. In *Proceedings of the Hirzebruch 65 conference on algebraic geometry, Bar-Ilan University, Ramat Gan, Israel, May 2-7, 1993*, volume 9 of *Israel Mathematical Conference Proceedings*, pages 75–96. Bar-Ilan University, 1996.
- [2] Aaron Bertram, Ionuț Ciocan-Fontanine, and William Fulton. Quantum multiplication of Schur polynomials. *J. Algebra*, 219(2):728–746, 1999.
- [3] Nicolas Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitres 5 à 7*. Springer, reprint of the 1985 original edition, 2006.
- [4] Daniel Bump and Anne Schilling. *Crystal bases. Representations and combinatorics*. World Scientific, 2017.
- [5] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Graduate Texts in Contemporary Physics. Springer, 1997.
- [6] Jürgen Fuchs. *Affine Lie algebras and quantum groups. An introduction, with applications in conformal field theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1999. 1st paperback ed. with corrections, digital reprinting.
- [7] László Fuchs. *Infinite abelian groups. Vol. I*, volume 36 of *Pure and Applied Mathematics*. Academic Press Inc., 1970.
- [8] Augusto Gamba and Luigi A. Radicati. Sopra un teorema per la riduzione di talune rappresentazioni del gruppo simmetrico. *Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., Sez. I, VIII. Ser.*, 14:632–634, 1953.
- [9] Jérémie Guilhot, Cédric Lecouvey, and Pierre Tarrago. Structure constants of the homology rings of affine Grassmannians in type G_2 . Unpublished notes. http://www.lmpt.univ-tours.fr/~guilhot/Miscellaneous/G2_matrix.pdf.
- [10] Jérémie Guilhot, Cédric Lecouvey, and Pierre Tarrago. Quantum cohomology of the Grassmannian and unitary Dyson Brownian motion. <https://arxiv.org/abs/2211.12836v1>, 2022.
- [11] Jérémie Guilhot, Cédric Lecouvey, and Pierre Tarrago. Homology rings of affine grassmannians and positively multiplicative graphs. <https://arxiv.org/abs/2306.16105>, to appear in *Comb. Theory*, 2026.
- [12] Morton Hamermesh. *Group theory and its application to physical problems*. Addison-Wesley Series in Physics. Addison-Wesley Publishing Group, 1962.
- [13] Masaki Kashiwara. On crystal bases. In *Representations of groups. Canadian Mathematical Society annual seminar, June 15-24, 1994, Banff, Alberta, Canada*, volume 16 of *CMS Conference Proceedings*, pages 155–197. American Mathematical Society, 1995.
- [14] Sergei V. Kerov. *Asymptotic representation theory of the symmetric group and its applications in analysis*, volume 219 of *Translations of Mathematical Monographs*. American Mathematical Society, 2003.
- [15] Thomas Lam, Luc Lapointe, Jennifer Morse, Anne Schilling, Mark Shimozono, and Mike Zabrocki. *k-Schur functions and affine Schubert calculus*, volume 33 of *Fields Institute Monographs*. Springer; Fields Institute for Research in the Mathematical Sciences, 2014.
- [16] Luc Lapointe and Jennifer Morse. Quantum cohomology and the k -Schur basis. *Trans. Am. Math. Soc.*, 360(4):2021–2040, 2008.
- [17] Cédric Lecouvey, Emmanuel Lesigne, and Marc Peigné. Conditioned random walks from Kac–Moody root systems. *Trans. Am. Math. Soc.*, 368(5):3177–3210, 2016.
- [18] Cédric Lecouvey and Pierre Tarrago. Central measures on multiplicative graphs, representations of Lie algebras and weight polytopes. *Ann. Inst. Fourier*, 70(6):2361–2407, 2020.

- [19] Cédric Lecouvey and Pierre Tarrago. Alcove random walks, k -Schur functions and the minimal boundary of the k -bounded partition poset. *Algebr. Comb.*, 4(2):241–272, 2021.
- [20] Peter Littelmann. A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras. *Invent. Math.*, 116(1-3):329–346, 1994.

Manuscript received 28th September 2023,
revised 3rd January 2025,
accepted 22nd April 2026.

Jérémie GUILHOT

Institut Denis Poisson, Université de Tours, Tours, France

Cédric LECOUEVEY

Institut Denis Poisson, Université de Tours, Tours, France

Cedric.Lecouvey@math.cnrs.fr

Pierre TARRAGO

Laboratoire de Probabilités, Statistique et Modélisation, Sorbonne Université, Paris, France

pierre.tarrago@sorbonne-universite.fr