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FINITE-DIMENSIONAL PSEUDOFINITE GROUPS OF SMALL DIMENSION, WITHOUT CFSG

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Abstract. Any simple pseudofinite group G is known to be isomorphic to a (twisted) Chevalley group over a pseudofinite field. This celebrated result mostly follows from the work of Wilson in 1995 and heavily relies on the classification of finite simple groups (CFSG). It easily follows that G is finite-dimensional with additive and fine dimension and, in particular, that if $\dim(G) = 3$ then G is isomorphic to $\mathrm{PSL}_2(F)$ for some pseudofinite field F. We describe pseudofinite groups of fine and additive dimension $\leqslant 3$ and, in particular, show that the classification $G \cong \mathrm{PSL}_2(F)$ is independent of CFSG.

Introduction

Pseudofinite groups (resp. fields) are infinite groups (fields) which satisfy every first-order property that is true in all finite groups (fields). In [3], Ax characterised pseudofinite fields in purely algebraic terms. Using Ax's result and the classification of finite simple groups (CFSG), Wilson proved that a simple pseudofinite group is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field [23].

A structure is *supersimple of finite* SU*-rank* if its definable sets (in some monster model) are equipped with a notion of dimension, called SU-rank, taking integer values. It is well known [7] that a pure pseudofinite field F is supersimple of SU-rank 1. Hence, using Wilson's classification, the theory of any simple pseudofinite group G is supersimple of finite SU-rank [16, Theorem 4.1]. In particular, it is easy to see [9, Proposition 6.1] that a simple pseudofinite group G with $\mathrm{SU}(G) = 3$ is isomorphic to $\mathrm{PSL}_2(F)$ for some pseudofinite field F. However, this observation heavily relies on CFSG.

A more general notion was introduced by the second author in [22, Definition 1.1]: a structure is *finite-dimensional* if there is a dimension function dim from the collection of all interpretable sets (again, in some monster model) to $\mathbb{N} \cup \{-\infty\}$. In particular, supersimple structures of finite SU-rank are finite-dimensional, where the dimension is SU-rank. In this case the dimension is *additive* and *fine* (see Section 1.3).

Our main result below describes pseudofinite groups of dimension 3 (when the dimension is additive and fine) without invoking CFSG. As an immediate corollary (Corollary 2.12), we obtain that the identification above, $G \cong \mathrm{PSL}_2(F)$, can be proven without using CFSG. Thus we solve the problem proposed in [9, p. 3, Question (3)] and in [16, p. 171].

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Theorem A (Without CFSG). — *Let G be a pseudofinite finite-dimensional group with additive and fine dimension. If $\dim(G) = 3$, then either G is soluble-by-finite, or $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ has a definable subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$ where F is a pseudofinite field.*

Let G be a finite-dimensional pseudofinite group and assume that the dimension is additive and fine. To prove Theorem A we need to understand the structure of G when its dimension is 1 or 2. It is easy to observe (see Section 1.3) that G satisfies the icc^0-*condition*: In G, any chain of intersections of uniformly definable subgroups, each having infinite index in its predecessor, has finite length, bounded by some n_φ depending only on the defining formula φ. In particular, G satisfies the icc^0-condition for centralisers, called the $\widetilde{\mathfrak{M}}_c$-*condition*, and therefore it follows from [22, Corollaries 4.14 and 5.2] that if $\dim(G) = 1$ then G is finite-by-abelian-by-finite, and if $\dim(G) = 2$ then G is soluble-by-finite. However, only the result in the case $\dim(G) = 1$ is proven independently of CFSG. Indeed, to the authors knowledge, there is no published result without CFSG which describes the structure of G when $\dim(G) = 2$. The second author has proven such result without CFSG and the pre-print where the proof appears is available at [21]. In this paper, we give a different proof. Namely, using similar arguments in dimension 2 and 3 (Section 2.2), we show that if G is not soluble-by-finite, then in dimension 2 we get a contradiction whereas in dimension 3 we may identify $\mathrm{PSL}_2(F)$ as in Theorem A. Thus, in dimension 2 we recover the following:

Theorem B (Without CFSG, cf. [21]). — *Let G be a finite-dimensional pseudofinite group with additive and fine dimension. If $\dim(G) = 2$, then G is soluble-by-finite. Moreover, if G is not finite-by-abelian-by-finite, then there is a definable normal subgroup N of G with $\dim(N) = 1$.*

Our results suggest the following question.

Question. — *Let G be a non-abelian definably simple pseudofinite group. Can one show, independently of CFSG, that G is finite-dimensional with additive and fine dimension?*

We prove our results in Section 2 and give all necessary background results in Section 1. In Section 1.1 we discuss which results from finite group theory are needed in our proofs.

1. Background results

Let G be a group. We use the following standard terminology and notation:

- If the largest soluble (resp. nilpotent) normal subgroup of G exists, then it is denoted by $\mathrm{Rad}(G)$ (resp. $\mathrm{Fitt}(G)$) and called the *soluble radical* (resp. *Fitting subgroup*) of G. Note that if $\mathrm{Rad}(G)$ or $\mathrm{Fitt}(G)$ exists, then it is definable [17, Theorem 1.1].
- The set of involutions of G is denoted by $I(G)$.
- If P and Q are properties of groups then G is called *P-by-Q* if G has a normal subgroup N so that N has the property P and G/N has the property Q. Note that P-by-(Q-by-R) implies (P-by-Q)-by-R, but may be strictly stronger. However, finite-by-abelian-by-finite is unambiguous:

if F is finite normal in A which is normal of finite index in G and A/F is abelian, then A' is finite characteristic in A, whence normal in G, so (finite-by-abelian)-by-finite implies finite-by-(abelian-by-finite). Moreover, $C_G(A')$ is a normal subgroup of finite index in G and $C_A(A')$ is nilpotent of class at most 2, so G is nilpotent-by-finite. Similarly, (finite-by-nilpotent)-by-finite equals nilpotent-by-finite, and (finite-by-soluble)-by-finite equals soluble-by-finite (the nilpotency class or the derived length may go up by 1).

1.1. **Needed results from finite group theory.** As one suspects, the Feit–Thompson theorem is used in our results.

Theorem 1.1 (Feit–Thompson [10]). — *A finite group without involutions is soluble.*

The *rank* of a finite group G is the smallest cardinality of a generating set, and the 2-*rank* of G, denoted by $m_2(G)$, is the largest rank of an elementary abelian 2-subgroup. By the Feit–Thompson theorem, if $m_2(G) = 0$, then G is solvable. If $m_2(G) = 1$ them, by Burnside's transfer theorem and the Brauer–Suzuki theorem [6], the group G is not simple.

Let then G be a finite simple group of $m_2(G) = 2$. Then the Sylow 2-subgroups of G are either dihedral groups, quasidihedral groups, wreathed groups, or isomorphic to a Sylow 2-subgroup of the projective special unitary group $\mathrm{PSU}_3(4)$ [2]. Gorenstein and Walter proved that a simple group with dihedral Sylow 2-subgroups is isomorphic either to $\mathrm{PSL}_2(q)$ for $q \geqslant 5$ or to the alternating group A_7 [12]. Alperin, Brauer and Gorenstein proved that a simple group with quasidiheral or wreathed Sylow 2-subgroups is isomorphic either to $\mathrm{PSL}_3(q)$ or $\mathrm{PSU}_3(q)$ for odd q or to the Mathieu group M_{11} [1, 2]. Finally, Lyons showed that if a finite simple group G has Sylow 2-subgroups isomorphic to Sylow 2-subgroups of $\mathrm{PSU}_3(4)$ then $G \cong \mathrm{PSU}_3(4)$ [15].

The results explained above are combined into a theorem in [11]:

Theorem 1.2 (See e.g. [11, p. 6]). — *Let G be a non-abelian finite simple group with $m_2(G) \leqslant 2$. Then $m_2(G) = 2$ and G is isomorphic to one of the following groups*

$$\mathrm{PSL}_2(q),\ \mathrm{PSL}_3(q),\ \mathrm{PSU}_3(q)\ \textit{ for odd } q,\ \mathrm{PSU}_3(4), A_7 \textit{ or } M_{11}.$$

Theorem 1.2 is used to recognise a simple pseudofinite group over a pseudofinite field F of characteristic $\neq 2$. For the case of $\mathrm{char}(F) = 2$ we use Bender's result below. A *strongly embedded* subgroup of a finite group G is a proper subgroup E of even order such that $E \cap E^g$ has odd order whenever $g \in G \setminus E$.

Theorem 1.3 (Bender [4]). — *Let G be a finite group containing a strongly embedded subgroup. Assume that $m_2(G) > 1$ and $\mathrm{Rad}(G) = 1$. Then G has a normal subgroup which is isomorphic to one of the following groups*

$$\mathrm{PSL}_2(2^n),\ \mathrm{Sz}(2^{2n-1}),\ \textit{or } \mathrm{PSU}_3(2^n),\ \textit{for } n \geqslant 2.$$

1.2. **Commensurability and almost operators.** Let G be a group, and H, $K \leqslant G$. The group H is said to be *almost contained* in K, written $H \lesssim K$, if $H \cap K$ has finite index in H. The subgroups $H, K \leqslant G$ are *commensurable* if both $H \lesssim K$ and $K \lesssim H$. This is denoted by $H \simeq K$. If H and K are not commensurable then we write $H \not\simeq K$. A family $\mathcal{H}$ of subgroups of G is *uniformly commensurable* if there is $n \in \mathbb{N}$ so that $|H_1 : H_1 \cap H_2| < n$ for all $H_1, H_2 \in \mathcal{H}$. Likewise, $K \leqslant G$ is uniformly commensurable to $\mathcal{H}$ if and only if $\mathcal{H}$ is uniformly commensurable and K is commensurable to some (equiv. any) group in $\mathcal{H}$. The following result is due to Schlichting but the formulation we give here can be found in [20, Theorem 4.2.4].

Theorem 1.4 (Schlichting's Theorem). — *Let G be a group and $\mathcal{H}$ a uniformly commensurable family of subgroups of G. Then there is a subgroup N of G which is uniformly commensurable with all members of $\mathcal{H}$ and is invariant under all automorphisms of G which fix $\mathcal{H}$ setwise. In particular, if $\mathcal{H}$ consists of definable subgroups, then N is definable.*

Definition 1.5. — Let G be a group, and $H, K \leqslant G$. We define the following subgroups:

(1) $\widetilde{N}_K(H) = \{k \in K : H \simeq H^k\}$ is the *almost normaliser* of H in K.
(2) $\widetilde{C}_K(H) = \{k \in K : H \lesssim C_H(k)\}$ is the *almost centraliser* of H in K.
(3) $\widetilde{Z}(G) = \widetilde{C}_G(G)$ is the *almost centre* of G.

The commensurability is *uniform* in $\widetilde{N}_K(H)$ (resp. in $\widetilde{C}_K(H)$) if there is some $m \in \mathbb{N}$ so that if $H \simeq H^k$ then $|H : H \cap H^k| < m$ (resp. if $H \lesssim C_H(k)$ then $|H : C_H(k)| < m$).

Notice that given a group G and definable subgroups H, K of G so that the commensurability is uniform in $\widetilde{N}_K(H)$ (resp. in $\widetilde{C}_K(H)$), then $\widetilde{N}_K(H)$ (resp. $\widetilde{C}_K(H)$) is a definable subgroup of G.

If $\widetilde{N}_G(H) = G$ we say that H is *almost normal* in G.

Fact 1.6 (Hempel [13, Theorem 2.17]). — *Let G be a group and $H, K \leqslant G$ be definable subgroups so that the commensurability is uniform in $\widetilde{C}_K(H)$ and in $\widetilde{C}_H(K)$. Then $H \lesssim \widetilde{C}_H(K)$ if and only if $K \lesssim \widetilde{C}_K(H)$.*

1.3. **Finite-dimensional groups.** A group G is called *finite-dimensional* if there is a dimension function dim from the collection of all interpretable sets in models of $\mathrm{Th}(G)$ to $\mathbb{N} \cup \{-\infty\}$ such that, for any formula $\phi(x, y)$ and interpretable sets X and Y, the following hold:

(1) *Invariance:* if $a \equiv a'$ then $\dim(\phi(x, a)) = \dim(\phi(x, a'))$.
(2) *Algebraicity:* if $X \neq \emptyset$ is finite then $\dim(X) = 0$, and $\dim(\emptyset) = -\infty$.
(3) *Union:* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
(4) *Fibration:* if $f : X \to Y$ is an interpretable map such that $\dim(f^{-1}(y)) \geqslant d$ for all $y \in Y$ then $\dim(X) \geqslant \dim(Y) + d$.

The dimension of a tuple a of elements over a set B is defined as

$$\dim(a/B) := \inf\{\dim(\phi(x)) : \phi \in \mathrm{tp}(a/B)\}.$$

We say that the dimension is:

- *additive:* if $\dim(a, b/C) = \dim(a/b, C) + \dim(b/C)$ holds for any tuples a and b and for any set C; and
- *fine:* if $\dim(X) = 0$ implies that X is finite.

In this paper we work with a finite-dimensional group with additive and fine dimension. Below we list those properties of such groups which will be used repeatedly throughout the paper. For any further details on (finite-)dimensional groups we refer to [22].

FACT 1.7 (Lascar equality). — *Let G be a finite-dimensional group with additivite and fine dimension and $H \leqslant G$ be a definable subgroup. Then* $\dim(G) = \dim(H) + \dim(G/H)$.

Proof. — The map $G \to G/H$ has fibres of dimension $\dim(H)$. Hence, by additivity, we get $\dim(G) = \dim(H) + \dim(G/H)$ [22, Remark 1.4]. □

LEMMA 1.8. — *Let G be a finite-dimensional group with additive and fine dimension. Then G satisfies the chain condition on interscetions of uniformly definable subgroups,* icc^0*:*

> icc^0*: Given a family $\mathcal{H}$ of uniformly definable subgroups of G, there is $m < \omega$ so that there is no sequence $\{H_i : i \leqslant m\} \subset \mathcal{H}$ with $|\bigcap_{i<j} H_i : \bigcap_{i\leqslant j} H_i| \geqslant m$ for all $j \leqslant m$.*

(If $\mathcal{H}$ is the family of centralisers of elements, this is called the $\widetilde{\mathfrak{M}}_c$-condition.*)*

Proof. — Assume to the contrary that G does not satisfy the icc^0. Then, by compactness, the condition

$$\left|\bigcap_{i<m} H_i : \bigcap_{i\leqslant m} H_i\right| \geqslant m \quad \text{for all} \;\; m < \omega$$

is a consistent first-order condition on the parameters needed to define the groups $(H_i : i < \omega)$. So there is a family $\{H_i : i \leqslant \dim(G)\} \subset \mathcal{H}$ so that

$$G > H_0 > H_1 \cap H_0 > \cdots > \bigcap_{i\leqslant \dim(G)} H_i$$

is a descending chain of length $\dim(G)+2$ of definable subgroups of G, each having infinite index in its predecessor, contradicting Fact 1.7. □

COROLLARY 1.9. — *Let G be an* icc^0*-group. If $H \leqslant G$ is definable, then $\widetilde{N}_G(H)$ is also definable.*

Proof. — We need to show that the commensurability is uniform for conjugates of H. So suppose not. Let m be given by the icc^0-condition, and choose a maximal chain

$$H > H \cap H^{g_0} > \cdots > H \cap \bigcap_{i=0}^{k} H^{g_i}$$

with every group of finite index at least m in its predecessor. Then $k < m$. Now, if there were $g \in G$ so that $\infty > |H : H \cap H^g| > m\,|H : H \cap \bigcap_{i=0}^{k} H^{g_i}|$, then

$$\left|H \cap \bigcap_{i=0}^{k} H^{g_i} : H \cap \bigcap_{i=0}^{k} H^{g_i} \cap H^g\right| > m,$$

contradicting the maximality of k. Since $|H^g : H^g \cap H| = |H : H \cap H^{g^{-1}}|$, this finishes the proof. □

Fact 1.10 ([22, Lemma 4.4]). — *Let G be an $\widetilde{\mathfrak{M}}_c$-group. If $H \leqslant G$ is definable, then $\widetilde{C}_G(H)$ is also definable.*

Recall that if in a group G there is a finite bound on the size of its conjugacy classes, then the derived subgroup G' is finite. In particular, in any $\widetilde{\mathfrak{M}}_c$-group the almost centre $\widetilde{Z}(G)$ is finite-by-abelian.

Fact 1.11 (Hempel [13, Theorem 4.7.]). — *Let G be a group in which all definable sections are $\widetilde{\mathfrak{M}}_c$-groups. Then the Fitting subgroup* $\mathrm{Fitt}(G)$ *exists and is therefore definable.*

For an arbitrary group H denote by $R(H)$ the group generated by all soluble normal subgroups of H; note that if $R(H)$ is soluble then $R(G) = \mathrm{Rad}(H)$. To prove our results we need the following fact.

Theorem 1.12. — *Let G be a finite-dimensional group with additive and fine dimension. Then the soluble radical* $\mathrm{Rad}(G)$ *exists and is therefore definable.*

Proof. — We prove the claim by induction on the dimension. If $\dim(G) = 0$ then G is finite and $R(G) = \mathrm{Rad}(G)$. Assume that the claim holds when $\dim(G) = n-1$ and let $\dim(G) = n$. Now, by Fact 1.11, the Fitting subgroup $\mathrm{Fitt}(G)$ exists. Note that $R(\mathrm{Fitt}(G)) = \mathrm{Rad}(\mathrm{Fitt}(G)) = \mathrm{Fitt}(G)$. If $\mathrm{Fitt}(G)$ is infinite then $\dim(G/\mathrm{Fitt}(G)) < n$ and $R(G/\mathrm{Fitt}(G)) = \mathrm{Rad}(G/\mathrm{Fitt}(G))$ by the inductive assumption. Since $\mathrm{Fitt}(G)$ is a normal and soluble subgroup of G, we get that $R(G) = \mathrm{Rad}(G)$.

Now, we may assume that $\mathrm{Fitt}(G)$ is finite. Set $G_0 := C_G(\mathrm{Fitt}(G))$ and $Z := Z(\mathrm{Fitt}(G))$. Then G_0 is a normal finite index subgroup of G. If $AZ/Z \leqslant G_0/Z$ is an abelian normal subgroup then $AZ \leqslant \mathrm{Fitt}(G_0) \leqslant \mathrm{Fitt}(G) \cap G_0 = Z$. So G_0/Z has no non-trivial abelian normal subgroups and hence $R(G_0) = \mathrm{Rad}(G_0) = Z$. Now $R(G) \cap G_0 = R(G_0)$ and since $|R(G) : R(G) \cap G_0| \leqslant |G : G_0|$ is finite, $R(G)$ is finite, whence soluble. □

1.4. Pseudofinite groups.

We denote by $\mathcal{L}_{\mathrm{gr}}$ the language of groups.

Definition 1.13. — A *pseudofinite* group is an infinite group which satisfies every first-order sentence of $\mathcal{L}_{\mathrm{gr}}$ that is true of all finite groups.

Note that by Łoś' Theorem, an infinite group (resp. $\mathcal{L}$-structure) is pseudofinite if and only if it is elementarily equivalent to a non-principal ultraproduct of finite groups ($\mathcal{L}$-structures) of increasing orders, see [16].

Typical examples of pseudofinite groups are torsion-free divisible abelian groups, infinite extraspecial groups of exponent $p > 2$ and rank n and (twisted) Chevalley groups over pseudofinite fields (recall that *using* CFSG, Wilson [23] proved that a simple pseudofinite group is elementarily equivalent to a (twisted) Chevalley group $X(F)$ over a pseudofinite field F. Further, as explained in [16], results by Point [18] allow one to conclude that a simple group is pseudofinite if and only if it is elementarily equivalent to $X(F)$. Ryten [19, Chapter 5] has generalised this by showing that "elementarily equivalent" can be strengthened to "isomorphic". Note that replacing $\equiv$ by $\cong$ is done without further use of CFSG).

We will use the following theorem repeatedly, often without referring to it.

Theorem 1.14 (Wagner [22, Corollary 4.14]). — *Let G be a finite-dimensional pseudofinite group with additive and fine dimension. If* $\dim(G) = 1$*, then G is finite-by-abelian-by-finite.*

Remark 1.15. — Let G be a pseudofinite finite-dimensional group with additive and fine dimension. If $\dim(G) = 1$, then, by the above, $\widetilde{Z}(G)$ is of finite index in G and the commutator group $\widetilde{Z}(G)'$ is finite. So G contains a definable infinite (finite and characteristic)-by-abelian characteristic subgroup.

One of the earliest (1955) results on centralisers of involutions in finite groups states that, up to isomorphism, there are only a finite number of finite simple groups with a given centraliser of an involution. This was shown by Brauer and Fowler [5]. The following result (which, like the result by Brauer and Fowler, is independent of CFSG) provides an alternative proof for this fact (see [14, Corollary 2.5]):

Fact 1.16 (Hempel–Palacín [14, Lemma 2.3]). — *Let G be a pseudofinite group and assume that $\widetilde{Z}(G) = 1$. Then the centraliser $C_G(i)$ is infinite for any $i \in I(G)$.*

2. Proofs of our results

Before starting our proofs, we still need some definitions and observations.

A group G is called *definably simple* if it has no proper non-trivial definable normal subgroup. It is *semisimple* if it has no non-trivial abelian normal subgroup. Note that if A is a non-trivial normal abelian subgroup and $1 \neq a \in A$, then $Z(C_G(a^G))$ is a definable non-trivial normal abelian subgroup, so semisimplicity is the same as definable semisimplicity. Clearly, a semisimple group has trivial soluble radical. Moreover, if G is semisimple and N is a normal subgroup of finite index, then N is also semisimple: if A were a normal abelian subgroup of N, the finitely many G-conjugates of A would generate a nilpotent subgroup normal in G, whose centre would be a non-trivial abelian normal subgroup of G.

The *socle* $\mathrm{Soc}(G)$ of a finite group G is the subgroup generated by all minimal normal non-trivial subgroups.

A definable subgroup H of G is *strongly embedded* if H has involutions, but $H \cap H^g$ does not for any $g \in G \setminus H$. By Łoś' Theorem, if $G = \prod_{i\in I} G_i/\mathcal{U}$ is pseudofinite, then H is strongly embedded if and only if H_i is strongly embedded in G_i for almost all i.

2.1. Useful lemmas.

Useful lemmas. We shall call an infinite group *almost simple* if it is not abelian-by-finite and has no definable normal subgroup of infinite index.

Lemma 2.1. — *An almost simple* icc^0*-group G is semisimple. Moreover, $\widetilde{Z}(G)$ is trivial, and if H is an infinite definable subgroup of infinite index, then $\widetilde{N}_G(H) < G$. If H is a definable infinite soluble-by-finite subgroup, then $\widetilde{N}_G(H)$ has infinite index in G.*

Proof. — If A is an abelian normal subgroup of G and $1 \neq a \in A$, then $Z(C_G(a^G))$ is a definable abelian normal subgroup. It has either finite index or smaller dimension than G, contradicting almost simplicity in both cases.

Now $\widetilde{Z}(G)'$ is finite normal in G, whence trivial, so $\widetilde{Z}(G)$ is abelian normal, and must be trivial as well.

If an infinite definable infinite index subgroup H is almost normal in G, then by Theorem 1.4 it is commensurable with a definable normal subgroup N of G, again contradicting almost simplicity as commensurability implies that N is also of infinite index in G.

Finally suppose H is definable infinite and soluble-by-finite. If $\widetilde{N}_G(H)$ is of finite index in G then

$$N = \bigcap_{g \in G} \widetilde{N}_G(H)^g$$

is a normal subgroup of finite index in G. By Theorem 1.4 there is $\overline{H}$ normal in $\widetilde{N}_G(H)$ and commensurable with the soluble-by-finite group H. Now $\overline{H}$ is again soluble-by finite, as is $\overline{H} \cap N$. The product of the finitely many G-conjugates of $\overline{H} \cap N$ is a definable subgroup S which is again soluble-by-finite, and normal in G. Hence $\mathrm{Rad}(S)$ is a non-trivial definable normal soluble subgroup of G, contradicting semisimplicity. □

We shall in particular apply Lemma 2.1 in the case when G is finite-dimensional and $\dim(H) = 1$.

Lemma 2.2. — *Let $G \equiv \prod_{i \in I} G_i/\mathcal{U}$ be a non-abelian semisimple pseudofinite finite-dimensional group with additive and fine dimension. Then G has involutions. Assume further that $\dim(G) \leqslant 3$ and that at least one of the following holds.*

(1) *$m_2(G) \leqslant 2$.*
(2) *G contains a definable strongly embedded subgroup.*

Then $\dim(G) = 3$ and G has a definable normal subgroup of finite index isomorphic to $\mathrm{PSL}(F)$ where F is a pseudofinite field.

Proof. — Assume first that $G = \prod_{i \in I} G_i/\mathcal{U}$. We shall argue modulo $\mathcal{U}$ even if only implicitly.

By semisimplicity $C_{G_i}(\mathrm{Soc}(G_i)) = 1$; thus $G_i \hookrightarrow \mathrm{Aut}(\mathrm{Soc}(G_i))$. Now the non-trivial socle $\mathrm{Soc}(G_i)$ of the finite group G_i is a direct product of non-abelian simple finite groups; the Feit–Thompson theorem then implies that $\mathrm{Soc}(G_i)$ has involutions for almost all i, and so does G.

Assume now that $\dim(G) \leqslant 3$. If (1) holds, then, by Theorem 1.2, each simple factor in the direct product $\mathrm{Soc}(G_i)$ is of 2-rank 2 and hence $\mathrm{Soc}(G_i)$ is simple. Further, again by Theorem 1.2, $\mathrm{Soc}(G_i) = X(q_i)$ where $X \in \{\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3\}$ and q_i is odd. If (1) does not hold and (2) holds, then Theorem 1.3 implies that G_i has a normal subgroup $X(q_i)$ where $X \in \{\mathrm{PSL}_2, \mathrm{PSU}_3, \mathrm{Sz}\}$ and q_i is a power of 2. Note that since $|G_i| \leqslant |\mathrm{Aut}(X(q_i))|$, the q_i grow without a bound when i varies. Therefore, in either of the two cases, by [8], there is $x_i \in X(q_i)$ so that $X(q_i) = x_i^{X(q_i)} x_i^{X(q_i)}$. (The result in [8] states that Thompson's conjecture holds for finite simple (twisted) Chevalley groups $X(q)$, provided that $q > 8$. This does not use CFSG. Note however that while Thompson's conjecture is known to hold for "almost all" finite simple groups (see [8, Introduction]), this more general result uses a case-by-case analysis provided by CFSG.) Since $X(q_i) \trianglelefteq G_i$ we see that $X(q_i) = x_i^{G_i} x_i^{G_i}$, and there is a definable normal subgroup $N = \prod_{i \in I} X(q_i)/\mathcal{U} = X(F)$, where $F = \prod_{i \in I} \mathbb{F}_{q_i}/\mathcal{U}$ is a pseudofinite field and $X \in \{\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3, \mathrm{Sz}\}$. As $\dim(N) \leqslant 3$, we have $\dim(N) = 3$ and $X = \mathrm{PSL}_2$ by [9, Proposition 6.1].

Now the field F and the linear structure $X(F)$ are definable in G. It follows that the result also holds for any group elementarily equivalent to G (see [19, Chapter 5]). $\square$

Lemma 2.3. — *Let G be a pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) \leqslant 3$, and that there is a definable subgroup $H < G$ with $\dim(H) = 2$ which is not almost normalised by G. Then either $\widetilde{Z}(H)$ is finite, or $\dim(\widetilde{Z}(G)) \geqslant 1$.*

Proof. — Consider $x \in G \setminus \widetilde{N}_G(H)$. Then $\dim(H \cap H^x) \geqslant 1$, as otherwise $\dim(HH^x) = 4 > \dim(G)$. Also $\dim(H \cap H^x) < 2$ by the choice of x. So $\dim(H \cap H^x) = 1$ and $H \cap H^x$ is finite-by-abelian-by-finite. Thus $\widetilde{Z}(H \cap H^x) \simeq H \cap H^x$.

Now assume that $\widetilde{Z}(H)$ is infinite, and put $L = \widetilde{Z}(H \cap H^x) \cap \widetilde{C}_H(\widetilde{Z}(H))$. Since $H \lesssim \widetilde{C}_H(\widetilde{Z}(H))$ (Fact 1.6), we have $L \simeq \widetilde{Z}(H \cap H^x)$.

Suppose first that $\widetilde{Z}(H) \cap \widetilde{Z}(H \cap H^x)$ is finite. If $y \in L$, then $C_H(y)$ almost contains both $\widetilde{Z}(H)$ and $\widetilde{Z}(H \cap H^x)$, so $\dim(C_H(y)) = 2$ and $y \in \widetilde{Z}(H)$. Thus $\widetilde{Z}(H \cap H^x) \lesssim L \lesssim \widetilde{Z}(H)$.

On the other hand, if $\widetilde{Z}(H) \cap \widetilde{Z}(H \cap H^x)$ is infinite, we again get $\widetilde{Z}(H \cap H^x) \lesssim \widetilde{Z}(H)$ since $\dim(\widetilde{Z}(H \cap H^x)) = 1$.

Similarly $\widetilde{Z}(H \cap H^x) \lesssim \widetilde{Z}(H^x)$. So $\widetilde{Z}(H \cap H^x) \lesssim \widetilde{Z}(H) \cap \widetilde{Z}(H^x) \leqslant \widetilde{Z}(G)$, and $\dim(\widetilde{Z}(G)) \geqslant 1$. $\square$

2.2. **The identification lemma.** In this section we prove Lemma 2.6 which plays a key role in the proof of both Theorem A and Theorem B. We use similar arguments in dimension 2 and 3, but in dimension 2 this will lead to a contradiction, while in dimension 3 we shall identify $\mathrm{PSL}_2(F)$. However, the case of dimension 3 is more complicated due to the possible existence of definable proper subgroups of dimension 2. So we start with the analysis of such subgroups.

Lemma 2.4. — *Let G be an almost simple pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) = 3$ and that all definable subgroups of dimension 2 are soluble-by-finite. If L is a definable subgroup of dimension 2, then:*

(1) *$B = \widetilde{N}_G(L)$ is a maximal definable subgroup of dimension 2. Moreover, $B = N_G(B) = \widetilde{N}_G(B)$; if $Z = \widetilde{Z}(B)$ then Z is finite, and B/Z is a Frobenius group with Frobenius kernel U/Z and Frobenius complement T/Z, where $U = \widetilde{C}_G(U)$ and $T = \widetilde{C}_B(T)$ have dimension 1 and are finite-by-abelian.*
(2) *There is a pseudofinite field F such that $U/Z \cong F^+$ and T/Z embeds into $F^\times$ as a subgroup of finite index. In particular T/Z is abelian, and T has only finitely many elements of any given order.*
(3) *If $C = U_C T_C$ is another (maximal) subgroup of dimension 2 and $Z_C := \widetilde{Z}(C) \neq 1$, then $U^g \not\lesssim T_C$ for any $g \in G$.*
(4) *If $g \in G \setminus B$ then $U \cap U^g = 1$. If B contains involutions, then either B is strongly embedded, or there is an involution in $T \setminus Z$ and no involution in $U \setminus Z$.*
(5) *Suppose $Z \neq 1$. If $g \in G \setminus B$ then $B \cap B^g$ is a finite index subgroup of T^u for some $u \in U$. Moreover, if $x \in B$ with $\dim(C_G(x)) = 2$ then $x \in Z$.*

Proof. — Put $N = \widetilde{N}_G(L)$. Then $\dim(N) = 2$ by Lemma 2.1. If K is any group commensurable with L then $K \leqslant \widetilde{N}_G(K) = \widetilde{N}_G(L)$. This shows maximality of N, as well as $N = \widetilde{N}_G(N) = N_G(N)$.

Since $\widetilde{Z}(G)$ is trivial and N is not almost normal in G, Lemma 2.3 yields that $Z := \widetilde{Z}(N)$ is finite. But $\widetilde{Z}(N)$ contains all finite normal subgroups of N. As N is soluble-by-finite by assumption, there is $a \in N \setminus \widetilde{Z}(N)$ such that $\langle a^N\rangle/\widetilde{Z}(N)$ is abelian; it is infinite since $a \notin \widetilde{Z}(N)$. Then $C_N(a^N/\widetilde{Z}(N))$ is an infinite definable subgroup containing a^N; if it were of dimension 2 then $a^N \subseteq Z$, a contradiction. Thus $\dim(C_N(a^N/\widetilde{Z}(N))) = 1$. Define:

$$(1)\ U := \left\{x \in N : C_N(x) \gtrsim C_N\left(a^N/\widetilde{Z}(N)\right)\right\} = \widetilde{C}_N\left(C_N\left(a^N/\widetilde{Z}(N)\right)\right).$$

Note that U only depends on the commensurability class of $C_N(a^N/\widetilde{Z}(N))$. If $\dim(U) = 2$, then $N \lesssim U = \widetilde{C}_N(C_N(a^N/\widetilde{Z}(N)))$, so $C_N(a^N/\widetilde{Z}(N)) \lesssim \widetilde{C}_N(N) = Z$ which is finite, a contradiction. Hence $U \simeq C_N(a^N/\widetilde{Z}(N))$, and $U = \widetilde{Z}(U)$ is finite-by-abelian. Clearly $N \leqslant N_G(U) \leqslant \widetilde{N}_G(U)$, and we have equality by maximality and almost simplicity of G.

Now N/U is 1-dimensional, whence finite-by-abelian-by-finite. For any $x_0 \in \widetilde{C}_N(N/U) \setminus U$ the orbit $(x_0U)^N$ in N/U is finite. Then $\dim(x_0^N) = \dim(U) = 1$ and $C_N(x_0)$ has dimension 1. Note that $x_0 \notin U$ implies finiteness of $C_N(x_0) \cap U$. So we can define the following:

(2) $T := \{x \in N : C_N(x) \gtrsim C_N(x_0)\} = \widetilde{C}_N(C_N(x_0))$.
(3) $B := UT$.

Note that $\widetilde{N}_G(B) = \widetilde{N}_G(N) = N$, as B has finite index in N. As $\dim(C_N(x_0)U) = 2$, the index of $C_N(x_0)U$ in N is finite, so

$$Z = \widetilde{Z}(N) = \widetilde{C}_N(C_N(x_0)U) = \widetilde{C}_N(C_N(x_0)) \cap \widetilde{C}_N\left(C_N\left(a^N/\widetilde{Z}(N)\right)\right) = T \cap U.$$

Thus $\dim(T) = 1$; it follows that $T \simeq C_N(x_0)$ and $T = \widetilde{C}_N(T)$ is finite-by-abelian. Moreover, any N-conjugate of T which is commensurable with T must be equal to T, and $\widetilde{N}_N(T) = N_N(T)$. For any $g \in N \setminus N_N(T)$ we have $T \cap T^g = Z$, since $C_N(h)$ has dimension 2 for any $h \in T \cap T^g$. Moreover $N_B(T) = N_U(T) \times_Z T$; it cannot have dimension 2 as otherwise it would be finite-by-abelian-by-finite, contradicting finiteness of $\widetilde{Z}(N)$. Hence $N_U(T)$ is finite and $N_U(T) \leqslant \widetilde{C}_U(T) = Z$. Thus $N_B(T) = T$.

Put $A := U/Z$, $H := T/Z$ and $\widehat{B} := B/Z$. Clearly $A \cap H = 1$ and $\widetilde{Z}(\widehat{B}) = 1$, so $\widehat{B} = A \rtimes H$ and $H \cap H^g = 1$ for any $g \in \widehat{B} \setminus H$. This means that $\widehat{B}$ is a pseudofinite Frobenius group with a definable Frobenius kernel A. Therefore, by the structure of finite Frobenius groups, the conjugates of H cover $\widehat{B} \setminus A$, and for any definable subgroup K of $\widehat{B}$ with $K \cap A = 1$ and $AK = \widehat{B}$ there is $a \in A$ with $K = H^a$.

Now, let $M \leqslant N$ be definable of dimension 1 with $M \cap U$ finite. As above $T_1 := \widetilde{C}_N(M)$ is of dimension 1, commensurable with M, satisfies $T_1 \cap U = Z$, and $N_{UT_1}(T_1) = T_1$. Then $\widehat{B}_1 = UT_1/Z \leqslant \widehat{B}$ is also a Frobenius group with Frobenius kernel A. Since $H_1 = H \cap \widehat{B}_1$ satisfies $H_1 \cap A = 1$ and $AH_1 = \widehat{B}_1$, there is $u \in U$ with $H_1^u = T_1/Z$. Thus $T^u \simeq (T \cap B_1)^u = T_1 \simeq M$, so $T_1 = \widetilde{C}_N(M) = \widetilde{C}_N(T^u) = \widetilde{C}_N(T)^u = T^u$. In particular, any definable subgroup of N of dimension 1 is commensurable either with U or with a U-conjugate of T. Moreover, for any $g \in N$ there is $u \in U$ with $T^g = T^u$, so $N = U\, N_N(T)$. This shows (1), once we know that $N_N(T) = T$ (which is proven below), since that implies $N = B$.

We say that a subset X of A is *almost H-invariant* if there is some finite subset $H_0 \subseteq H$ so that for any $h \in H$ we have $X^h \subseteq \bigcup_{h_0 \in H_0} X^{h_0}$. Any finite almost H-invariant subset of A is almost central in $\widehat{B}$ and thus trivial. So, as A is (finite and characteristic)-by-abelian, it is abelian.

Now, let $R := \mathrm{End}_H(A)$ be the ring of endomorphisms of A generated by H. Let $r \in R \setminus \{0\}$. We show that r is an automorphism of A: since H is finite-by-abelian, both $\ker(r)$ and $\mathrm{im}(r)$ are almost H-invariant definable subgroups of A and hence either finite, and thus trivial, or finite index subgroups of the 1-dimensional group A. Now $\ker(r)$ cannot have finite index in A as otherwise $\mathrm{im}(r)$ is finite, whence trivial, so we would get $r = 0$. So any $r \in R \setminus \{0\}$ is injective; since a definable injective map from a pseudofinite group to itself must be surjective, R acts on A by automorphisms. In particular it is invertible. By [22, Proposition 3.6 or Corollary 3.10] the field of fractions F of R is an interpretable skew field; $A \cong F^+$ and $H \hookrightarrow F^\times$. Since $\dim(H) = \dim(A) = \dim(F)$, the image of H has finite index in $F^\times$. As F is pseudofinite and any finite skew field is commutative by Wedderburn's little theorem, $F^\times$ is commutative and contains only finitely many elements in each finite order; clearly the same holds for H.

Finally, suppose $g \in N_N(T)$. Then g induces an automorphism σ of F of finite order via conjugation: If $r \in R$ is given by $x \mapsto \prod_i x^{h_i}$, then $\sigma(r)$ is given by $x \mapsto \prod_i x^{h_i^g}$, and extends to F by $\sigma(r^{-1}r') = \sigma(r)^{-1}\sigma(r')$. Let F_0 be the fixed field of σ. Then $[F : F_0] = o(\sigma)$ and $1 = \dim(F) = [F : F_0]\dim(F_0) \geqslant [F : F_0]$, as $\dim(F_0) \geqslant 1$. Hence σ fixes F, so $g \in C_N(T/Z) \leqslant \widetilde{C}_N(T) = T$. This shows (2), as well as $N_N(T) = T$ and $N = B$. So we have proven (1) and (2).

From now on, we think of U and T as the "unipotent" and the "semisimple" parts of B, respectively.

In order to show (3), assume that there is $g \in G$ so that $U^g \simeq T_C$. Then $\widetilde{N}_G(T_C) = \widetilde{N}_G(U^g) = B^g$. Therefore $T_C \leqslant B^g$; since T_C is finite-by-abelian, $T_C \leqslant \{x \in B^g : U^g \lesssim C_{B^g}(x)\} = U^g$. By part (2), T_C/Z_C contains only finitely many elements of any given order. But T_C has finite index in U^g, and U^g is finite-by-abelian. Therefore U^g only contains finitely many elements of any given order.

Let $1 \neq z \in Z_C$ be of order ℓ and put $\Omega_\ell = \{t \in U^g : o(t) = \ell\}$. Then Ω_ℓ is finite and contains z. So $C_G(\Omega_\ell) \simeq N_G(\Omega_\ell) \geqslant B^g$. This implies $z \in Z_C \cap Z^g$, whence $C \simeq C_G(z) \simeq B^g$ and $U^g \simeq T_C = \widetilde{N}_C(T_C) \simeq \widetilde{N}_{B^g}(U^g) = B^g$, a contradiction.

To show (4), consider $g \in G \setminus B$. Suppose that there is non-trivial $y \in U \cap U^g$, and put $C = C_G(y)$. Then C almost contains U and U^g and must be of dimension 2. Clearly $y \in Z_C = \widetilde{Z}(C)$, so neither U nor U^g can be commensurable with T_C (the semisimple part of C) by part (3). Hence both are commensurable with U_C, so $U \simeq U^g$ and $g \in \widetilde{N}_G(U) = B$, a contradiction. In particular $Z \cap Z^g \leqslant U \cap U^g = 1$.

Next, note that if $\mathrm{char}(F) \neq 2$ then $I(U \setminus Z) = \emptyset$, and if $\mathrm{char}(F) = 2$ then $I(T \setminus Z) = \emptyset$. In the latter case, since the U-conjugates of $T \setminus Z$ cover $B \setminus U$, we have $I(B) = I(U)$, so $I(B \cap B^g) \leqslant U \cap U^g = 1$, and B is strongly embedded.

To show (5), assume $Z \neq 1$ and consider $g \in G \setminus B$. As $\dim(B) = \dim(B^g) = 2$, $\dim(G) = 3$ and $B \not\simeq B^g$, we have $\dim(B \cap B^g) = 1$. Hence $B \cap B^g$ is commensurable with U or a U-conjugate of T, and with U^g or a U^g-conjugate of T^g. Clearly $U \simeq B \cap B^g \simeq U^g$ is impossible, as $g \notin \widetilde{N}_G(U) = B$. Part (3) excludes the mixed case, so $B \cap B^g \simeq T^u$ for some $u \in U$. Hence

$$B \cap B^g \leqslant \widetilde{N}_B(B \cap B^g) = \widetilde{N}_B(T^u) = \widetilde{N}_B(T)^u = T^u.$$

Finally, consider $x \in B$ with $\dim(C_G(x)) = 2$. Note that $x \in \widetilde{Z}(C_G(x))$, which is finite by Lemma 2.3. So the order $o(x)$ is finite. If $x \in U$ then $x \in U \cap U^c$ for any $c \in C_G(x)$, so $c \in B$ and $C_G(x) \leqslant B$. Thus $x \in Z$.

Now suppose $x \in T \setminus Z$. Then $T \simeq C_B(x)$; by parts (1) and (3) we have $C_G(x) \simeq U_x T_x$, where the unipotent part U_x is not almost contained in B and the semisimple part T_x is commensurable to T. For any $u_x \in C_{U_x}(x) \setminus B$ we have $x \in B \cap B^{u_x}$ and $x \in (T \cap T^{u_x}) \setminus (Z \cup Z^{u_x})$. But the U-conjugates of $T \setminus Z$ are disjoint and there are $u, u' \in U$ with $T^u \geqslant B \cap B^{u_x} \leqslant T^{u' u_x}$. It follows that $u, u' \in N_U(T) = Z$, so $T \simeq T^{u_x}$ and $u_x \in \widetilde{N}_G(T)$. Thus $\widetilde{N}_{U_x}(T_x) = \widetilde{N}_{U_x}(T)$ is infinite, a contradiction. As the conjugates of T cover $B \setminus U$, we are done.

This finishes the proof of the Lemma 2.4. □

From now on, we use the following notation.

Notation. — *Let G be a group. For $g \in G$ we denote $C_g := \widetilde{C}_G(C_G(g))$. Note that if G is finite-dimensional with additive and fine dimension and if $\dim(C_G(g)) = 1$, then $C_G(g) \lesssim C_g$ by Theorem 1.14.*

Lemma 2.5. — *Let G be an almost simple pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) \leqslant 3$. If $g \in G$ satisfies $\dim(C_G(g)) = 1$, then $C_g \simeq C_G(g)$ and $\widetilde{N}_G(C_g) = N_G(C_g)$. Moreover, if $i, j \in N_G(C_g) \setminus C_g$ are distinct involutions with $\dim(C_G(i)) = \dim(C_G(j)) = 1$, then $j \in iC_g$.*

Proof. — Note that $C_G(g)$ is finite-by-abelian-by-finite, so $C_G(g) \lesssim C_g$. Suppose that C_g had finite index in G. Since $C_g \lesssim \widetilde{C}_G(C_G(g)))$, by Fact 1.6 we have $C_G(g) \lesssim \widetilde{C}_G(C_g) = \widetilde{Z}(G)$, contradicting triviality of $\widetilde{Z}(G)$. So $1 \leqslant \dim(C_g) < \dim(G) \leqslant 3$. Suppose $\dim(C_g) = 2$. As $C_G(g) \lesssim \widetilde{Z}(C_g)$ by Fact 1.6, we get $\widetilde{N}_G(C_g) = G$ by Lemma 2.3, contradicting almost normality. So $\dim(C_g) = 1$ and $C_g \simeq C_G(g)$, whence $\widetilde{N}_G(C_g) = N_G(C_g)$.

Now, let $i, j \in N_G(C_g) \setminus C_g$ be distinct involutions with

$$\dim(C_G(i)) = \dim(C_G(j)) = 1.$$

Let $Z \leqslant A$ be definable characteristic subgroups of C_g so that Z is finite and A/Z is infinite and abelian (exist by Remark 1.15). By the above and our assumptions, $\dim(C_g) = 1$, $\dim(C_G(i)) = 1$ and $C_G(i) \not\simeq C_G(g)$; hence $C_A(i)$ is finite, as is $C_{A/Z}(i)$. Then the subgroup of A/Z inverted by iZ, say B_i/Z, contains $[r, A]Z/Z$ and thus is 1-dimensional. Similar observations can be made for the group B_j/Z of elements of A/Z inverted by jZ; so $(B_i \cap B_j)/Z$ is infinite. Now ijZ centralises $(B_i \cap B_j)/Z$; as Z is finite there is $z \in Z$ such that $ijx_\ell ji = x_\ell z$ for infinitely many $x_\ell \in B_i \cap B_j$. Hence there is $1 \neq x_0 \in B_i \cap B_j$ so that ij centralises $x_\ell x_0^{-1}$ and $C_{B_i \cap B_j}(ij)$ is infinite. Thus $\dim(C_{C_G(g)}(ij)) = 1$ and $ij \in C_g$, yielding $j \in iC_g$. □

Lemma 2.6. — *Let G be an almost simple pseudofinite finite-dimensional group with additive and fine dimension. Assume that $\dim(G) \leqslant 3$, and that, if $\dim(G) = 3$,* *all definable subgroups of dimension 2 are soluble-by-finite. Then* $\dim(G) = 3$ *and* G *has a definable subgroup of finite index isomorphic to* $\mathrm{PSL}_2(F)$*, where* F *is a pseudofinite field.*

Proof. — Note first that G is semisimple and has trivial almost centre by Lemma 2.1; moreover it has dimension at least 2. If $\dim(G) = 2$ then all centralisers of non-trivial elements have dimension at most 1 by Theorem 1.4. Note that we may assume that $m_2(G) > 2$, that G has no strongly embedded definable subgroup and that G has involutions (Lemma 2.2). Further, by Fact 1.16 all involutions of G have infinite centralisers.

CLAIM 2.7. — *There is an involution* $i \in G$ *with* $\dim(C_G(i)) = 1$*, and for any such involution there is an involution* $k \in N_G(C_i) \setminus C_i$*. Moreover, if there is an involution* $j \in G$ *with* $\dim(C_G(j)) = 2$*, then we can choose* $k \in C_G(i) \setminus C_i$*.*

Proof of Claim. — Note first that if $i \in G$ is an involution with $\dim(C_G(i)) = 1$, then $\dim(\widetilde{N}_G(C_i)) = 1$. For otherwise, setting $H = \widetilde{N}_G(C_i)$ and $B = \widetilde{N}_G(H)$ we have $\dim(H) = \dim(B) = 2$. As $\dim(\widetilde{N}_B(C_i)) = 2$ we must have $C_i \simeq U$ in the notation of Lemma 2.4, so $i \in C_i = \widetilde{C}_G(C_i) = \widetilde{C}_G(U) = U$. Moreover $\dim(C_G(i)) = 1$ implies $i \notin \widetilde{Z}(B)$. But then B is strongly embedded by Lemma 2.4 (4), a contradiction.

Next, suppose that $\dim(C_G(i)) = 1$ for all involutions $i \in G$, and fix some involution i. Put $H = \widetilde{N}_G(C_G(i))$. Then $C_G(i) \lesssim C_i \leqslant H$ are all commensurable, so $H = \widetilde{N}_G(H)$. Since H is not strongly embedded, there is $g \in G \setminus H$ and an involution $k \in H \cap H^g$; if $k \in C_i \cap C_i^g$ then $C_G(k)$ almost contains C_i and C_i^g; as $g \notin H = \widetilde{N}_G(C_i)$ the intersection $C_i \cap C_i^g$ is finite, so $C_G(k)$ has dimension 2, a contradiction. If $k \notin C_i$ we are done, otherwise $k \notin C_i^g$ and $k^{g^{-1}} \notin C_i$ is as required.

Now suppose that $j \in G$ is an involution with $\dim(C_G(j)) = 2$. Note that this implies $\dim(G) = 3$, as $\widetilde{Z}(G) = 1$. We take $L = C_G(j)$ in Lemma 2.4, so $B = \widetilde{N}_G(L)$ and $Z = \widetilde{Z}(B)$. Then $j \in Z \leqslant B$, so by Lemma 2.4 (4) there is an involution $i \in T \setminus Z$, and we have $\dim(C_G(i)) = 1$ by Lemma 2.4 (5). This shows the first part of the claim.

Let $i \in G$ be any involution with $\dim(C_G(i)) = 1$ and put $N = \widetilde{N}_G(C_i)$. Consider the conjugacy class j^G, which has dimension 1. Recall that in any finite group two involutions are either conjugate or there is an involution commuting with both of them; by Łoś' Theorem this also holds in G. Clearly i and j^g are not conjugate for any $g \in G$, so for each $g \in G$ there is an involution $k_g \in C_G(i) \cap C_G(j^g)$.

We want to show that $k_g \notin C_i$ for some $g \in G$. So suppose otherwise. Now either $\dim(C_G(k_1)) = 2$, and we can take $B = \widetilde{N}_G(C_G(k_1))$. Then $i \in C_G(k_1) \leqslant B$, so $C_B(i)$ is infinite, as is $C_i \cap B$. If $C_i \cap B \simeq U$, then $U = \widetilde{C}_G(U) = \widetilde{C}_G(C_i) = C_i$ contains an involution i with $\dim(C_G(i)) = 1$, so $i \notin Z$ and B is strongly embedded, a contradiction. Hence $C_i \cap B = \widetilde{C}_G(C_i) \cap B = \widetilde{C}_G(T) \cap B = T$.

Otherwise $\dim(C_G(k_1)) = 1$, so $C_i = C_{k_1}$ and we can take $B = \widetilde{N}_G(C_G(j))$. The same argument (with k_1 and j instead of i and k_1) yields that $C_i \cap B = T$. But C_i is finite-by-abelian, T has finite index in C_i and T contains only finitely many involutions (Lemma 2.4 (2)). Hence C_i only contains finitely many involutions, and there is an involution $k \in C_i$ which commutes with infinitely many distinct conjugates j^g.

Fix $j' \in C_G(k) \cap j^G$, put $B' = \widetilde{N}_G(C_G(j'))$ and $Z' = \widetilde{Z}(B') \ni j'$, and let U' and T' denote the unipotent and a semisimple subgroups of B'. Then $k \in C_G(j') \leqslant B'$. Put $X = \{x \in G : k \in B'^x\}$. Then $X \cap j^G \supseteq C_G(k) \cap j^G$ is infinite; since $B' \cap j^G \subseteq Z'$ is finite, there is $x_0 \in X \setminus B'$, and we may choose $T' \leqslant B'$ so that $k \in B' \cap B'^{x_0} \leqslant T'$ (recall that $\widetilde{N}_G(B') = B'$). As $Z' \cap B' \cap B'^{x_0} = 1$ we have $k \notin Z'$, and T' is the unique U'-conjugate of B' containing k. Hence T' does not depend on x_0. But for any $x \in X$ there is $u \in U'$ such that $B' \cap B'^x \leqslant T'^u$, so clearly $T'^u = T'$. As k is in some U'^x-conjugate of T'^x, there is $u' \in U'$ with $k \in T'^{u'x} \leqslant B'^x$, so $T' \geqslant B' \cap B'^x \leqslant T'^{u'x}$, whence $T' \simeq T'^{u'x}$ and $u'x \in \widetilde{N}_G(T')$. Moreover, if $x \in C_G(k)$ then we can take $u' = 1$. Hence $C_G(k) \cap j^G \subseteq \widetilde{N}_G(T')$. It follows that $\widetilde{N}_G(T')$ cannot have dimension 2, since this would imply $C_G(k) \cap j^G \subseteq \widetilde{Z}(\widetilde{N}_G(T'))$ which is finite. Therefore $\dim(\widetilde{N}_G(T')) = 1$.

Put $B^\sharp = \bigcap_{g \in \widetilde{N}_G(T')} (T')^g$, a normal subgroup of $\widetilde{N}_G(T')$ of finite index contained in all B'^x for $x \in X$. Now $k \notin Z'$, so $C_G(k) \simeq T'$ and $C_G(k) \lesssim B^\sharp$. Hence there is fixed $j'^{x'} \in C_G(k)$ and infinitely many distinct $j'^{x_\ell} \in C_G(k)$ so that $j'^{x'} = j'^{x_\ell} t_\ell$ for some $t_\ell \in B^\sharp$. Now $j'^{x'} \in B'^{x'}$, whence $k \in C_G(j'^{x'}) \leqslant B'^{x'}$ and $x' \in X$. Also $t_\ell \in B^\sharp \leqslant B'^{x'}$, whence $j'^{x_\ell} \in B'^{x'}$. So $j'^{x_\ell} \in \widetilde{Z}(B'^{x'})$ by Lemma 2.4(5), contradicting finiteness of $\widetilde{Z}(B'^{x'})$. This shows that there is an involution $k \in C_G(i) \setminus C_i$ commuting with i. □

We now fix an involution $i \in G$ with $\dim(C_G(i)) = 1$ and put $N = N_G(C_i)$.

Claim 2.8. — *An involution $k \in N \setminus C_i$ centralises a unique involution $\ell \in C_i$, and* $\dim(C_G(k)) = \dim(C_G(\ell)) = 1$.

Proof of Claim. — In a finite group N, if C is normal in N and both C and $N \setminus C$ contain involutions, then for any involution $k \in N \setminus C$ a Sylow 2-subgroup P containing k must intersect C non-trivially, as all Sylow 2-subgroups are conjugate. Then $Z(P) \cap C$ is non-trivial by nilpotency of P, and C contains an involution j commuting with k. By Łoś' Theorem, the same holds for $N > C_i$, and there is an involution $\ell \in C_i$ commuting with k.

Next we show $\dim(C_G(k)) = \dim(C_G(\ell)) = 1$. This is clear if $\dim(G) = 2$, so suppose $\dim(G) = 3$.

Assume first that $\dim(C_G(k)) = 2$. Then $\dim(\widetilde{N}_G(C_G(k))) = 2$ by almost simplicity of G. If $\dim(C_G(\ell)) = 2$, then, as $\ell \in C_G(k) \leqslant \widetilde{N}_G(C_G(k))$, we get $\ell \in \widetilde{Z}(N_G(C_G(k)))$. Hence $C_G(i) \lesssim C_G(\ell) \simeq C_G(k)$, contradicting $k \notin C_i$. If $\dim(C_G(\ell)) = 1$, then $C_G(i) \simeq C_G(\ell) \lesssim C_G(k)$, again a contradiction. Thus $\dim(C_G(k)) = 1$.

Next, suppose $\dim(C_G(\ell)) = 2$ and put $B'' = \widetilde{N}_G(C_G(\ell))$. Then $i, k \in C_\ell \leqslant B''$, so $C_G(i)$ and $C_G(k)$ are commensurable to two different B''-conjugates of the semisimple part T'' of B''. As $k \notin \widetilde{Z}(B'')$, this implies $k \notin N$, since $\widetilde{N}_{B''}(T'') = T''$. Hence $\dim(C_G(\ell)) = 1$.

Finally, we show uniqueness of ℓ. So assume that k fixes another involution $\ell' \in C_i$. Then $\dim(C_G(\ell')) = 1$; by Lemma 2.5 we have $\ell' \in \ell C_k$, so $\ell\ell' \in C_k$ and $C_G(k) \lesssim C_G(\ell\ell')$. Moreover $C_G(i) \lesssim C_G(\ell\ell')$. So $C_G(\ell\ell')$ is 2-dimensional. If

$\dim(G) = 2$ this is a contradiction; if $\dim(G) = 3$ we put $B''' = \widetilde{N}_G(C_G(\ell\ell'))$. Then $C_G(k)$ and $C_G(\ell) \simeq C_G(i)$ are commensurable to two different B'''-conjugates of the semisimple part T''' of B''', which again implies $k \notin N$. Thus k centralises a unique involution in C_i. □

Claim 2.9. — *Any two involutions x, y in C_i with $\dim(C_G(x)) = \dim(C_G(y)) = 1$ commute.*

Proof of Claim. — Let $k_x, k_y \in N \setminus C_i$ be involutions commuting with x, y respectively. Now C_i is normal in N, so y and k_x are not N-conjugate, and there is an involution $z \in N$ commuting with both. If $z \in C_i$ then, as it is centralised by k_x, we have $z = x$, so x and y commute. If $z \in N \setminus C_i$ then either $z = k_x$, or zk_x is an involution in C_i centralised by k_x, whence $zk_x = x$. In either case z commutes with x and with y, whence $x = y$. So x and y commute. □

Claim 2.10. — *If $x, y, z \in C_i$ are three distinct involutions with*

$$\dim(C_G(x)) = \dim(C_G(y)) = \dim(C_G(z)) = 1,$$

then $xyz = 1$.

Proof of Claim. — Let $k \in N \setminus C_i$ be an involution commuting with x. Then y and y^k are distinct commuting involutions in C_i. Therefore $yy^k = (yk)^2$ is an involution in C_i fixed by k, whence $(yk)^2 = x$. Similarly, $(zk)^2 = x$, and $k^y = k^z = xk$. So yz is an involution in C_i commuting with k, whence $yz = x$ and $xyz = 1$. □

Claim 2.11. — $\dim(C_G(j)) = 1$ *for any involution $j \in G$.*

Proof of Claim. — Suppose, as in the proof of Claim 2.7, that j is an involution with $\dim(C_G(j)) = 2$. Note that we may assume that $i, j \in B$ commute (where B is again as in Claim 2.7, whence as in Lemma 2.4). If not, then there is a third involution $z \in C_G(j) \cap C_G(i)$. If $z \in \widetilde{Z}(B)$ we can replace j by z. Otherwise $z \in T \setminus Z$, and we can replace i by z. Note that $ij \in T \setminus \widetilde{Z}(B)$, so $\dim(C_G(ij)) = 1$.

Consider $k \in C_G(i) \setminus C_i$ (exists by Claim 2.7). Then $B \cap B^k \simeq T$, and $\ell = j^k \in (C_i \cap C_G(i)) \setminus B$. Then i, ij and $i\ell$ are distinct involutions in C_i with $\dim(C_G(i)) = \dim(C_G(ij)) = \dim(C_G(i\ell)) = 1$. Then, by the above, $iiji\ell = 1$ whence $j = i\ell$, but $C_G(j)$ is 2-dimensional, a contradiction. □

Now, since $m_2(G) > 2$ there are three pairwise commuting distinct involutions x, y, z with $xyz \neq 1$. By Claim 2.10 and 2.11 we cannot have $C_x = C_y = C_z$. If $C_x = C_y \neq C_z$, then $z \in N_G(C_x) \setminus C_x$ centralises two distinct involutions $x, y \in C_x$, contradicting Claim 2.8. If $C_x \neq C_y \neq C_z \neq C_x$, then $y, z \in N_G(C_x)$, so $y \in zC_x$ by Lemma 2.5, and $yz \in C_x$ is an involution centralised by y, whence $yz = x$ again by Claim 2.8, contradicting our hypothesis.

This finishes the proof of Lemma 2.6. □

2.3. **Proof of Theorem B.**

Theorem B. — *Let G be a finite-dimensional pseudofinite group with additive and fine dimension. If $\dim(G) = 2$ then G is soluble-by-finite. Moreover, if G is not finite-by-abelian-by-finite, then there is a definable normal subgroup N of G with $\dim(N) = 1$.*

Proof. — Suppose first that G has a definable normal subgroup N of dimension 1. Then N is finite-by-abelian-by-finite. In fact $\widetilde{Z}(N)'$ is finite normal in G, and $C_N(\widetilde{Z}(N)')$ is a normal 2-nilpotent subgroup of G of dimension 1. The quotient $\overline{G} = G/C_N(\widetilde{Z}(N)')$ is also of dimension 1, whence finite-by-abelian-by-finite. Put $\overline{S} = C_{\overline{G}}(\widetilde{Z}(\overline{G})')$. Then $\overline{S}$ is 2-nilpotent of finite index in $\overline{G}$, and its preimage S is soluble of finite index in G. Thus G is soluble-by-finite and has a definable normal subgroup of dimension 1.

Now suppose that there is no definable normal subgroup of dimension 1. Every finite normal subgroup of G is contained in $\widetilde{Z}(G)$, which is finite-by-abelian.

If $\dim(\widetilde{Z}(G)) = 2$ we are done and G is finite-by-abelian-by-finite. So suppose that $\widetilde{Z}(G)$ is finite. Then $\overline{G} = G/\widetilde{Z}(G)$ has no non-trivial finite normal subgroup. Replacing G by $\overline{G}$, we may thus assume that G is almost simple. So we may apply Lemma 2.6, which implies that G has dimension 3, a contradiction. □

2.4. **Proof of Theorem A.**

Theorem A (Without CFSG). — *Let G be a pseudofinite finite-dimensional group with additive and fine dimension. If $\dim(G) = 3$, then either G is soluble-by-finite, or $\widetilde{Z}(G)$ is finite and $G/\widetilde{Z}(G)$ has a definable subgroup of finite index isomorphic to $\mathrm{PSL}_2(F)$ where F is a pseudofinite field.*

Proof. — Suppose first that G has a definable normal subgroup N with $1 \leqslant \dim(N) \leqslant 2$. Then $1 \leqslant \dim(G/N) \leqslant 2$, so by Theorem B both N and G/N are soluble-by-finite. Put $S = \mathrm{Rad}(N)$, which is characteristic in N and hence normal in G. Then $C_G(N/S)$ is a definable normal soluble subgroup of finite index in G.

So suppose that G has no definable normal subgroup of dimension 1 or 2. If $\dim(\widetilde{Z}(G)) = 3$, then G is finite-by-abelian-by-finite. Otherwise $\widetilde{Z}(G)$ is finite and contains all normal finite subgroups. Replacing G by $G/\widetilde{Z}(G)$, we may assume that G is almost simple. By Theorem B all definable subgroups of dimension 2 are soluble-by-finite. So we finish by Lemma 2.6. □

We finish the paper by stating the immediate corollary of Theorem A.

Corollary 2.12 (Without CFSG). — *A pseudofinite non-abelian definably simple finite-dimensional group with additive and fine dimension and of dimension 3 is isomorphic to $\mathrm{PSL}_2(F)$ for some pseudofinite field F.*

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