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EQUIVARIANT COHOMOLOGY AND CURRENT ALGEBRAS

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This paper touches upon two big themes, equivariant cohomology and current algebras. Our first main result is as follows: we define a pair of current algebra functor which assigns Lie algebras (current algebras) $\mathcal{CA}(M, A)$ and $\mathcal{SA}(M, A)$ to a manifold M and a differential graded Lie algebra (DGLA) A. The functors $\mathcal{CA}$ and $\mathcal{SA}$ are contravariant with respect to M and covariant with respect to A. If $A = C\mathfrak{g}$, the cone of a Lie algebra $\mathfrak{g}$ spanned by Lie derivatives $L(x)$ and contractions $I(x)$ ($x \in \mathfrak{g}$) and satisfying the Cartan's magic formula $[d, I(x)] = L(x)$, the corresponding current algebras coincide, and they are equal to $\mathcal{CA}(M, C\mathfrak{g}) = \mathcal{SA}(M, C\mathfrak{g}) \cong C^\infty(M, \mathfrak{g})$, the space of smooth $\mathfrak{g}$-valued functions on M with the pointwise Lie bracket. Other examples include affine Lie algebras on the circle and Faddeev–Mickelsson–Shatashvili (FMS) extensions of higher-dimensional current algebras. The second set of results is related to the construction of a new DGLA $D\mathfrak{g}$ assigned to a Lie algebra $\mathfrak{g}$. It is generated by $L(x)$ and $I(x)$ (similar to $C\mathfrak{g}$) and by higher contractions $I(x^2), I(x^3)$ etc. Similar to $C\mathfrak{g}$, $D\mathfrak{g}$ can be used in building differential models of equivariant cohomology. In particular, twisted equivariant cohomology (including twists by 3-cocycles and higher odd cocycles) finds a natural place in this new framework. The DGLA $D\mathfrak{g}$ admits a family of central extensions $D_p\mathfrak{g}$ parametrized by homogeneous invariant polynomials $p \in (S\mathfrak{g}^*)^\mathfrak{g}$. There is a Lie homomorphism from $\mathcal{CA}(M, D_p\mathfrak{g})$ to the FMS current algebra defined by p. Let G be a Lie group integrating the Lie algebra $\mathfrak{g}$. The current algebras $\mathcal{SA}(M, D\mathfrak{g})$ and $\mathcal{SA}(M, D_p\mathfrak{g})$ integrate to groups $DG(M)$ and $D_pG(M)$. As a topological application, we consider principal G-bundles, and for every homogeneous polynomial $p \in (S\mathfrak{g}^*)^\mathfrak{g}$ we pose a lifting problem (defined in terms of $DG(M)$ and $D_pG(M)$) with the only obstruction the Chern–Weil class $\mathrm{cw}(p)$. When M is a sphere, we study integration of the current algebra $\mathcal{CA}(M, D_p\mathfrak{g})$. It turns out that the corresponding group is a central extension of $DG(M)$. Under certain conditions on the polynomial p, this is a central extension by a circle.

Keywords: Current algebras; equivariant cohomology.

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1. Introduction

This paper has two main themes: differential models of equivariant cohomology and current algebras.

Our first main result is the construction of the current algebra functor $\mathcal{CA}$ which associates a Lie algebra (a current algebra) $\mathcal{CA}(M, A)$ to a pair of a manifold M and a differential graded Lie algebra (DGLA) A. The functor $\mathcal{CA}$ is contravariant with respect to M and covariant with respect to A. As a vector space,

$$\mathcal{CA}(M, A) = (\Omega(M) \otimes A)^{-1}/(\Omega(M) \otimes A)^{-1}_{\text{exact}}.$$

Here $\Omega(M)$ stands for differential forms on M. The Lie bracket of $\mathcal{CA}(M, A)$ is defined by the derived bracket construction of [11].

Another natural current algebra functor is given by

$$\mathcal{SA}(M, A) = (\Omega(M) \otimes A)^{0}_{\text{closed}}.$$

If A is acyclic as a complex, $\mathcal{CA}(M, A)$ and $\mathcal{SA}(M, A)$ are naturally isomorphic to each other. In contrast to $\mathcal{CA}(M, A)$, the construction of $\mathcal{SA}(M, A)$ is local, hence it defines a sheaf of Lie algebras on M.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $C\mathfrak{g}$ be a DGLA spanned in degree 0 by Lie derivatives $L(x)$ and in degree -1 be contractions $I(x)$ for $x \in \mathfrak{g}$. The differential on $C\mathfrak{g}$ is defined by the Cartan's magic formula $[d, I(x)] = L(x)$. The corresponding current algebra $\mathcal{CA}(M, C\mathfrak{g})$ is isomorphic to the space of maps $C^\infty(M, \mathfrak{g})$ with the pointwise Lie bracket. Other examples of current algebras include affine Lie algebras over the circle and Faddeev–Mickelsson–Shatashvili (FMS) extensions of $C^\infty(M, \mathfrak{g})$ for $\dim(M) \geq 3$ (for the definition of FMS current algebras, see [6, 7, 13]).

The DGLA $C\mathfrak{g}$ is the basis of Cartan's construction of differential models of equivariant cohomology. In more detail, let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and M be a manifold acted by G. Then, the cochain complex $\Omega(M)$ carries a compatible $C\mathfrak{g}$-action (defined by Lie derivatives and contractions). By Cartan's theorem, the equivariant cohomology $H_G(M, \mathbb{R})$ coincides with the cohomology of $(W\mathfrak{g} \otimes \Omega(M))^{C\mathfrak{g}}$, where $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ is the Weil algebra. The $C\mathfrak{g}$-action and the differential on $W\mathfrak{g} \otimes \Omega(M)$ can be chosen in two different ways which are called Weil and Cartan differential models of equivariant cohomology (see [2, 3], and [9] for a modern review). The equivalence between the two models is established by the Kalkman map induced by a group-like element $\phi \in W\mathfrak{g} \otimes \mathcal{U}(C\mathfrak{g})$.

Our second observation is that for $W\mathfrak{g} \otimes \Omega(M)$ to carry a differential and a compatible $C\mathfrak{g}$-action it is not necessary for $\Omega(M)$ to be a $C\mathfrak{g}$-module. Instead, it may be a module under the action of a bigger DGLA $D\mathfrak{g}$ (see Theorem 2). The latter has generators $L(x)$ and $I(x)$ (similar to $C\mathfrak{g}$), and also contains "higher contractions" $I(x^2), I(x^3)$ etc. In general, for every $D\mathfrak{g}$-module V we obtain a differential and a compatible $C\mathfrak{g}$-action on $W\mathfrak{g} \otimes V$. One can define the equivariant cohomology of V as the cohomology of $(W\mathfrak{g} \otimes V)^{C\mathfrak{g}}$. Again, there are two different models of equivariant cohomology, and the equivalence is established by the Kalkman-type twist induced by a group-like element $\Phi \in W\mathfrak{g} \otimes \mathcal{U}(D\mathfrak{g})$ (see Theorem 3). Among other things, the element Φ contains the information about chains of transgression for all invariant polynomials $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ (a chain of transgression is an element $e \in W\mathfrak{g}$ such that $d_W e = p \otimes 1$, where d_W is the Weil differential on $W\mathfrak{g}$).

One set of examples is provided by the theory of twisted equivariant cohomology. In more detail, in the Cartan model of equivariant cohomology an equivariant cocycle on M is an element $\omega(t) \in (S\mathfrak{g}^* \otimes \Omega(M))^{\mathfrak{g}}$ which is closed under the Cartan differential $d_{\mathfrak{g}} = d - I(t)$. Generators of $S\mathfrak{g}^*$ have degree 2, and we view equivariant differential forms as polynomial functions on $\mathfrak{g}$ and put $t \in \mathfrak{g}$. If $\omega(t)$ is an equivariant 3-cocycle, it can be written in the form $\omega(t) = \omega_3 + \omega_1(t)$, where $\omega_3 \in \Omega^3(M)^{\mathfrak{g}}$ and $\omega_1 \in (\mathfrak{g}^* \otimes \Omega^1(M))^{\mathfrak{g}}$. This allows to twist the differential and the $C\mathfrak{g}$ action on $\Omega(M)$ in the following way:

$$\tilde{d} = d + \omega_3, \quad \tilde{I}(x) = I(x) + \omega_1(x), \quad \tilde{L}(x) = L(x).$$

This twisted action finds its use in the theory of group-valued moment maps (see, e.g. [1]). It is intimately related to twisting of the Cartan's differential

$$\tilde{d}_{\mathfrak{g}} = d_{\mathfrak{g}} + \omega(t), \tag{1}$$

and to the theory of twisted equivariant cohomology [10]. The Cartan differential can be twisted by odd equivariant cocycles of higher degree. But the twist of the $C\mathfrak{g}$ action on $\Omega(M)$ does not generalize to this case. Instead, one should consider a twist of a certain $D\mathfrak{g}$-action on $\Omega(M)$.

In contrast to $C\mathfrak{g}$, $D\mathfrak{g}$ admits many central extensions. Under certain assumptions, these central extensions are classified by homogeneous invariant polynomials $p \in (S\mathfrak{g}^*)^{\mathfrak{g}}$ (see Theorem 4), and we use notation $D_p\mathfrak{g}$ for the central extension defined by the polynomial p. If p is of degree 2, the extension descends to $C\mathfrak{g}$, and the new Lie bracket of contractions is given by $[I(x), I(y)] = p(x, y)c$ (here c is the central generator). For p of degree 3 and higher, one has to use $D\mathfrak{g}$ to describe the corresponding central extension. One result relating the 2 parts of the paper is as follows: there is a Lie homomorphism from the current algebra $\mathcal{CA}(M, D_p\mathfrak{g})$ to the FMS current algebra on M defined by the invariant polynomial p. For p of degree 2 and $M = S^1$, $\mathcal{CA}(S^1, D_p\mathfrak{g})$ coincides with the standard central extension of the loop algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.

The DGLA $D\mathfrak{g}$ is acyclic (see Theorem 1). Hence, the current algebra functors $\mathcal{CA}$ and $\mathcal{SA}$ coincide, and they define a sheaf of Lie algebras $\mathcal{SA}(M, D\mathfrak{g})$. If G is a connected Lie group with Lie algebras $\mathfrak{g}$, one can integrate $\mathcal{SA}(M, D\mathfrak{g})$ to a sheaf of groups $DG(M)$. We define the gauge groupoid $\widehat{\mathcal{G}}(M)$ as the set of $\mathfrak{g}$-connections $A \in \mathcal{G}(M) = \Omega^1(M) \otimes \mathfrak{g}$ together with gauge transformations $A \mapsto \mathrm{Ad}_{g^{-1}} A + g^{-1} dg$. It turns out that the generalized Kalkman element Φ defines a morphism of groupoids $\mu : \widehat{\mathcal{G}}(M) \to DG(M)$ (see Theorem 6). Both $\widehat{\mathcal{G}}(M)$ and $DG(M)$ admit a family of central extensions by $\Omega(M)^{2n-2}_{\text{closed}}$ defined by an invariant homogeneous polynomial $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$. The morphism μ admits lifts to $\mu_p : \widehat{\mathcal{G}}_p(M) \to D_pG(M)$.

As a topological application, we consider the theory of $DG(M)$-torsors. Their isomorphism classes are classified by the isomorphism classes of underlying principal G-bundles. It turns out that a $DG(M)$-torsor lifts to a $D_pG(M)$-torsor if and only if the Chern–Weil class $\mathrm{cw}(p)$ vanishes (see Theorem 8).

The DGLA $D_p\mathfrak{g}$ is not acyclic. Let p be an invariant polynomial of degree n and $\eta_p \in H^{2n-1}(G, \mathbb{R})$ be its image under the transgression map. We integrate the current algebra $\mathcal{CA}(M, D_p\mathfrak{g})$ in the case of $M = S^{2n-3}$ a sphere of dimension $2n - 3$. The resulting group is a central extension of $DG(S^{2n-3})$ by $\mathbb{R}/\mathrm{im}(\Pi)$ (see Theorem 9), where $\Pi : \pi_{2n-1}(G, \mathbb{Z}) \to \mathbb{R}$ is the map defined by integration of the class η_p. If G is a compact connected and simply connected simple Lie group and p is a homogeneous generator of $(S^+\mathfrak{g})^\mathfrak{g}$ (with the exception of some special cases for $G = \mathrm{SO}(2k)$), then $\mathrm{im}(\Pi) \cong S^1$ and one obtains a central extension by a circle.

The structure of the paper is as follows. In Sec. 2, we recall Cartan and Weil differential models of equivariant cohomology. In Sec. 3, we define the DGLA $D\mathfrak{g}$, establish the fact that it is acyclic, and show that a structure of a $D\mathfrak{g}$-module on V gives rise to a structure of a $C\mathfrak{g}$-module on $W\mathfrak{g} \otimes V$. In Sec. 4, we discuss central extensions of $D\mathfrak{g}$ and construct homomorphisms from $D\mathfrak{g}$ to other DGLAs. In Sec. 5, we define and discuss properties of the current algebra functors $\mathcal{CA}$ and $\mathcal{SA}$. In Sec. 6, we discuss sheaves of groups $DG(M)$ and $D_pG(M)$ and study torsors over these sheaves of groups. In Sec. 7, we integrate to a group the Lie algebra $\mathcal{CA}(M, D_p\mathfrak{g})$ in the case of M being a sphere.

2. Differential Models of Equivariant Cohomology

In this section, we recall Cartan and Weil differential models of equivariant cohomology (for details, see [9]). For completeness, we include some proofs which resemble more difficult proofs in other sections.

Let $\mathfrak{g}$ be a Lie algebra. The cone of $\mathfrak{g}$ is the differential graded Lie algebra (DGLA) $C\mathfrak{g} = \mathfrak{g}[\varepsilon] = \mathfrak{g} \oplus \mathfrak{g}\varepsilon$, where ε is an auxiliary variable of degree -1. The Lie bracket of $C\mathfrak{g}$ is induced by the Lie bracket of $\mathfrak{g}$, and the differential given by $d/d\varepsilon$. For $x \in \mathfrak{g}$, we denote by $L(x)$ the element $x \in C\mathfrak{g}$ and by $I(x)$ the element $x\varepsilon \in C\mathfrak{g}$. They satisfy the standard relations $dI(x) = L(x)$, $[L(x), I(y)] = I([x, y])$, $[I(x), I(y)] = 0$, $[L(x), L(y)] = L([x, y])$.

In general, a module over a DGLA A is a cochain complex V equipped with a DGLA homomorphism $A \to \mathrm{End}(V)$. To define a $C\mathfrak{g}$-module, one needs $L_V(x) \in \mathrm{End}(V)^0$ and $I_V(x) \in \mathrm{End}(V)^{-1}$ verifying the defining relations of $C\mathfrak{g}$. A $C\mathfrak{g}$-module is also called a $\mathfrak{g}$-differential space. If V is a graded commutative differential algebra, and the action of $C\mathfrak{g}$ is by derivations, one says that V is a $\mathfrak{g}$-differential algebra.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and M be a manifold acted by G. Then, $C\mathfrak{g}$ acts by derivations on differential forms $\Omega(M)$, $L(x)$ acting by Lie derivatives and $I(x)$ acting by contractions. This action turns $\Omega(M)$ into a $\mathfrak{g}$-differential algebra. Another basic example of a $\mathfrak{g}$-differential algebra is the *Weil* *algebra* $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ which serves as a model for differential forms on the total space of the classifying G-bundle EG. The action of $L(x)$ is by the diagonal coadjoint action (extended to $S\mathfrak{g}^*$ and $\wedge\mathfrak{g}^*$), and the action of $I(x)$ is by contractions on $\wedge\mathfrak{g}^*$. Let us choose a basis e_a of $\mathfrak{g}$ with structure constants $[e_a, e_b] = f_{ab}^c e_c$. We shall denote the generators of $\wedge\mathfrak{g}^*$ by θ^a (this is a dual basis in $\mathfrak{g}^*$) and the generators of $S\mathfrak{g}^*$ by t^a. The Weil differential is the unique degree 1 derivation of $W\mathfrak{g}$ such that

$$d\theta^a = t^a - \frac{1}{2} f_{bc}^a \theta^b \theta^c.$$

One can also choose θ^a and $d\theta^a$ as generators of $W\mathfrak{g}$. Then, it is identified with the Koszul algebra of the graded vector space $\mathfrak{g}^*[-1]$. Thus, $H^0(W\mathfrak{g}) = \mathbb{R}$ and $H^i(W\mathfrak{g}) = 0$ for $i \geq 1$.

Sometimes it is convenient to consider a bigger DGLA $W\mathfrak{g} \otimes C\mathfrak{g}$ with Lie bracket induced by the one of $C\mathfrak{g}$ and the differential $d = d_{W\mathfrak{g}} + d_{C\mathfrak{g}}$. Consider elements $L(\theta) = \theta^a L(e_a), I(t) = t^a I(e_a) \in W\mathfrak{g} \otimes C\mathfrak{g}$.

Proposition 1. *$I(t) - L(\theta) \in W\mathfrak{g} \otimes C\mathfrak{g}$ is a Maurer–Cartan element.*

Proof. On the one hand,

$$\begin{aligned} d_{C\mathfrak{g}}(I(t) - L(\theta)) &= d_{C\mathfrak{g}}(t^a I(e_a) - \theta^a L(e_a)) \\ &= t^a L(e_a), \\ d_{W\mathfrak{g}}(I(t) - L(\theta)) &= d_{W\mathfrak{g}}(t^a I(e_a) - \theta^a L(e_a)) \\ &= -f_{bc}^a \theta^b t^c I(e_a) - t^a L(e_a) + \frac{1}{2} f_{bc}^a \theta^b \theta^c L(e_a). \end{aligned}$$

Hence,

$$d(I(t) - L(\theta)) = -f_{bc}^a \theta^b t^c I(e_a) + \frac{1}{2} f_{bc}^a \theta^b \theta^c L(e_a).$$

On the other hand,

$$\begin{aligned} [I(t) - L(\theta), I(t) - L(\theta)] &= [t^b I(e_b) - \theta^b L(e_b), t^c I(e_c) - \theta^c L(e_c)] \\ &= f_{bc}^a \theta^b \theta^c L(e_a) - 2 f_{bc}^a \theta^b t^c I(e_a). \end{aligned}$$

In conclusion, we obtain

$$d(I(t) - L(\theta)) = \frac{1}{2}[I(t) - L(\theta), I(t) - L(\theta)],$$

as required. □

Let $\mathcal{U}(C\mathfrak{g})$ be the degree completed universal enveloping algebra of $C\mathfrak{g}$ equipped with the standard coproduct. Then, $W\mathfrak{g} \otimes \mathcal{U}(C\mathfrak{g})$ is a Hopf algebra over $W\mathfrak{g}$. Consider a degree 0 group-like element $\phi = \exp(-I(\theta)) \in (W\mathfrak{g} \otimes \mathcal{U}(C\mathfrak{g}))^{\mathfrak{g}}$.

Proposition 2. $\phi^{-1}d\phi = -I(t) + L(\theta)$.

Proof. We denote $\phi = \exp(a)$, where $a = -I(\theta) = -\theta^a I(e_a)$. We have

$$\phi^{-1}d\phi = \frac{1-\exp(-\mathrm{ad}_a)}{\mathrm{ad}_a}\,da = da - \frac{1}{2}[a, da],$$

where we have used the standard formula for the derivative of the exponential map and have taken into account that $[a,[a,da]] = 0$. We compute

$$\begin{aligned} da &= -d(\theta^a I(e_a)) \\ &= -d_{W\mathfrak{g}}(\theta^a I(e_a)) - d_{C\mathfrak{g}}(\theta^a I(e_a)) \\ &= -t^a I(e_a) + \frac{1}{2} f^a_{bc}\theta^b\theta^c I(e_a) + \theta^a L(e_a), \\ -\frac{1}{2}[a, da] &= \frac{1}{2}[\theta^b I(e_b), \theta^c L(e_c)] \\ &= -\frac{1}{2} f^a_{bc}\theta^b\theta^c I(e_a). \end{aligned}$$

Hence,

$$\phi^{-1}d\phi = -t^a I(e_a) + \theta^a L(e_a) = -I(t) + L(\theta). \qquad \square$$

For a $C\mathfrak{g}$-module V, the basic subcomplex is defined as $V^{C\mathfrak{g}}$. If $M \to B$ is a principal G-bundle, the basic subcomplex is isomorphic to $\Omega(B)$. For the Weil algebra $W\mathfrak{g}$, the basic subcomplex is equal to $(S\mathfrak{g}^*)^{\mathfrak{g}}$ with vanishing differential. By definition, the equivariant cohomology of a $C\mathfrak{g}$-module V is

$$H_{\mathfrak{g}}(V) := H((W\mathfrak{g} \otimes V)^{C\mathfrak{g}}, d_{W\mathfrak{g}} + d_V).$$

This construction is usually referred to as the Weil model of equivariant cohomology. If V admits the structure of a $W\mathfrak{g}$-module compatible with the $C\mathfrak{g}$-action then $H_{\mathfrak{g}}(V) \cong H(V^{C\mathfrak{g}}, d_V)$. In particular, $H_{\mathfrak{g}}(W\mathfrak{g}) \cong (S\mathfrak{g}^*)^{\mathfrak{g}}$. If G is a compact connected Lie group and M is a G-manifold, Cartan's theorem states that $H_{\mathfrak{g}}(\Omega(M))$ is isomorphic to the equivariant cohomology $H_G(M, \mathbb{R}) = H((EG \times M)/G, \mathbb{R})$ defined by the Borel construction (here EG is the total space of the classifying G-bundle).

If $M \to B$ is a principal G-bundle, then every principal connection gives rise to a homomorphism of $\mathfrak{g}$-differential algebras $j: W\mathfrak{g} \to \Omega(M)$. The image of $\theta^a e_a \in W\mathfrak{g} \otimes \mathfrak{g}$ is the connection 1-form and the image of $t^a e_a$ is the corresponding curvature 2-form. Since j makes $\Omega(M)$ to a $W\mathfrak{g}$-module, we have an isomorphism

$$H_{\mathfrak{g}}(\Omega(M)) \cong H(\Omega(M)^{C\mathfrak{g}}) \cong H(B, \mathbb{R}).$$

This is in accordance with Cartan's theorem, as $(EG \times M)/G \cong EG \times B$ is homotopy equivalent to B.

The Kalkman map $\phi_V = \exp(-\theta_a I_V(e_a))$ is a natural automorphism of $W\mathfrak{g}\otimes V$. It transforms the differential and the action of I's in the following way:

$$I_{\text{new}}(x) = \phi_V^{-1}(I_{W\mathfrak{g}}(x) + I_V(x))\phi_V = I_V(x),$$

$$d_{\text{new}} = \phi_V^{-1}(d_{W\mathfrak{g}} + d_V)\phi_V = d_{W\mathfrak{g}} + d_V - I_V(t) + L_V(\theta).$$

Here in computing d_{new} we have used Proposition 2. In this way, one obtains the Cartan model of equivariant cohomology. In this model, $(W\mathfrak{g}\otimes V)^{C\mathfrak{g}} \cong (S\mathfrak{g}^*\otimes V)^{\mathfrak{g}}$ and the differential takes the form

$$d_{\mathfrak{g}} = d_V - I_V(t).$$

Remark. Since the $C\mathfrak{g}$-action on the Weil algebra $W\mathfrak{g}$ is free, we have $H_{\mathfrak{g}}(W\mathfrak{g}) \cong H((W\mathfrak{g})^{C\mathfrak{g}}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$. In the Cartan model, we obtain $H_{\mathfrak{g}}(W\mathfrak{g}) = H(S\mathfrak{g}^*\otimes W\mathfrak{g}, d_{\mathfrak{g}})$. Let $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$. Then, the cocycle $p\otimes 1 - 1\otimes p \in (S\mathfrak{g}^*\otimes W\mathfrak{g})^{\mathfrak{g}}$ belongs to the trivial cohomology class (since $p\otimes 1 - 1\otimes p \mapsto 1\cdot p - p\cdot 1 = 0$ under the product map). One can therefore find an element $e \in (S\mathfrak{g}^*\otimes W\mathfrak{g})^{\mathfrak{g}}$ such that $d_{\mathfrak{g}}e = p\otimes 1 - 1\otimes p$.

In this example, we denote the generators of $S\mathfrak{g}^*$ by t^a, and the generators of $W\mathfrak{g}$ by θ^a and f^a. For $n = 2$, one can choose an element e in the form

$$e = -\left(p(t+f,\theta) - \frac{1}{6}p(\theta,[\theta,\theta])\right),$$

where $t = t^a e_a, f = f^a e_a$ and $\theta = \theta^a e_a$. Putting $t = 0$ yields $e_{t=0} = -p(f,\theta) + \frac{1}{6}p(\theta,[\theta,\theta])$ which is a primitive of $-p(f,f) \in W\mathfrak{g}$.

3. The DGLA $D\mathfrak{g}$

Let $\mathfrak{g}$ be a Lie algebra. In this section, we define a new DGLA $D\mathfrak{g}$ which can be used instead of the $C\mathfrak{g}$ in differential models of equivariant cohomology. Roughly speaking, we are replacing $\mathfrak{g}\epsilon = (C\mathfrak{g})^{-1}$ by its canonical free resolution.

3.1. *Definition and basic properties of $D\mathfrak{g}$*

Let V be a negatively graded vector space $V = \bigoplus_{n<0} V^n$ with finite dimensional graded components V^n. We denote by $\mathcal{L}(V)$ the graded free Lie algebra generated by V. The graded components of $\mathcal{L}(V)$ are also finite dimensional.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $S^+\mathfrak{g}$ be the graded vector space $\bigoplus_{n\geq 1} S^n\mathfrak{g}$ with the degree defined by formula $\deg S^n\mathfrak{g} = 1-2n$. We define the graded Lie algebra $D\mathfrak{g}$ as a semi-direct sum $\mathfrak{g}\ltimes\mathcal{L}(S^+\mathfrak{g})$, where elements of $\mathfrak{g}$ have degree 0, and the action of $\mathfrak{g}$ on $\mathcal{L}(S^+\mathfrak{g})$ is induced by the adjoint action.

One can also view $D\mathfrak{g}$ as a graded Lie algebra defined by generators $l(x)$ for $x\in\mathfrak{g}$ and $\mathcal{I}(u)$ for $u\in S^+\mathfrak{g}$, and relations $[l(x),l(y)] = l([x,y])$ and $[l(x),\mathcal{I}(u)] = \mathcal{I}(\mathrm{ad}_x(u))$. For $x\in\mathfrak{g}$, it is convenient to introduce the generating function

$$i(x) = \sum_{k=1}^{\infty}\mathcal{I}(x^k).$$

Remark. Low degree graded components of $D\mathfrak{g}$ are as follows: $D\mathfrak{g}^0 \cong \mathfrak{g}$ with generators $l(x)$, $D\mathfrak{g}^{-1} \cong \mathfrak{g}$ with generators $\mathcal{I}(x)$, $D\mathfrak{g}^{-2} \cong S^2\mathfrak{g}$ spanned by $[\mathcal{I}(x), \mathcal{I}(y)]$, $D\mathfrak{g}^{-3} \cong S^2\mathfrak{g} \oplus \ker(\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g})$, where $S^2\mathfrak{g}$ is spanned by $\mathcal{I}(xy)$, the map $\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g}$ is the symmetrization, and $\ker(\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g})$ is spanned by $[\mathcal{I}(x), [\mathcal{I}(y), \mathcal{I}(z)]]$ (subject to the Jacobi identity).

Proposition 3. *The operator $d \in \mathrm{End}^1(D\mathfrak{g})$ defined by equations $dl(x) = 0$ and*

$$di(x) = [i(x), i(x)]/2 + l(x) \tag{2}$$

makes $D\mathfrak{g}$ into a differential graded Lie algebra.

Proof. The defining relations of $D\mathfrak{g}$ express the invariance of the definition under the adjoint $\mathfrak{g}$-action. Since Eq. (2) is invariant under this action, it defines a derivation of $D\mathfrak{g}$

Next, we need to verify that $d^2 = 0$. Indeed,

$$\begin{aligned} d^2 i(x) &= d([i(x), i(x)]/2 + l(x)) = [di(x), i(x)] \\ &= [[i(x), i(x)]/2 + l(x), i(x)] = [[i(x), i(x)], i(x)]/2 = 0, \end{aligned}$$

where the last equality follows from the Jacobi identity. □

Remark. In low degrees, the differential is defined by the Cartan's magic formula $d\mathcal{I}(x) = l(x)$ for $x \in \mathfrak{g}$, and by its higher analogues such as

$$d\mathcal{I}(xy) = \frac{1}{2}[\mathcal{I}(x), \mathcal{I}(y)], \quad d\mathcal{I}(x^3) = [\mathcal{I}(x), \mathcal{I}(x^2)].$$

More generally, for $k \geq 2$ we have

$$d\mathcal{I}(x^k) = \frac{1}{2}\sum_{s=1}^{k-1}[\mathcal{I}(x^s), \mathcal{I}(x^{k-s})].$$

Proposition 4. *There is a canonical projection of DGLAs $\pi : D\mathfrak{g} \to C\mathfrak{g}$ defined on generators by $l(x) \mapsto L(x), \mathcal{I}(x) \mapsto I(x)$ and $\mathcal{I}(x^k) \mapsto 0$ for $k \geq 2$.*

Proof. The defining relations of $D\mathfrak{g}$ is a subset of the defining relations of $C\mathfrak{g}$. Hence, π is a homomorphism of graded Lie algebras. Then, we have $\pi(i(x)) = I(x)$, and

$$\pi(di(x)) = \pi\left(\frac{1}{2}[i(x), i(x)] + l(x)\right) = \frac{1}{2}[I(x), I(x)] + L(x) = L(x) = d\pi(i(x)),$$

as required. □

Note that i can be also viewed as a formal map $\mathfrak{g}[2] \to D\mathfrak{g}$ of degree 1. For a DGLA A, defining a DGLA homomorphism $D\mathfrak{g} \to A$ is equivalent to giving a Lie homomorphism $\tilde{l} : \mathfrak{g} \to A$ and a formal map $\tilde{i} : \mathfrak{g}[2] \to A$ of degree 1 such that

$\tilde{i}(0) = 0$ and such that the identity (2) is satisfied. The maps l and i define the tautological isomorphism $D\mathfrak{g} \to D\mathfrak{g}$. Another example is given by the canonical projection $\pi : D\mathfrak{g} \to C\mathfrak{g}$ with $\tilde{i}(x) = I(x)$ and $\tilde{l}(x) = L(x)$.

Theorem 1. *As a complex*, $D\mathfrak{g}$ *is acyclic.*

We need the following auxiliary statement.

Proposition 5. *The cohomology of the DGLA* $\mathcal{L}(S^+\mathfrak{g})$ *with differential defined by formula* $di(x) = [i(x), i(x)]/2$ *is equal to* $\mathfrak{g} = S^1\mathfrak{g} \subset S^+\mathfrak{g}$.

Indeed, $\mathcal{L}(S^+\mathfrak{g})$ is the canonical free resolution of $\mathfrak{g}[1]$ (for the standard reference, see [8]). For convenience of the reader, we include the proof due to Drinfeld [5]. We assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra.

Proof of Proposition 5. Let us consider the differential graded associative algebra $\mathcal{U}(\mathcal{L}(S^+\mathfrak{g})) = T(S^+\mathfrak{g})$. We will be using the natural grading on $S^+\mathfrak{g}$ defined by formula $\deg S^l\mathfrak{g} = l$. With respect to this grading, the differential is of degree 0, and we thus have $T(S^+\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T_n$ as a direct sum of complexes.

Let I^n be the standard n-dimensional cube, and consider the following simplicial complex representing I^n modulo the boundary. Degree k simplices are labeled by surjective maps

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, k\}.$$

The geometric simplex labeled by σ is singled out by conditions $(x_1, \ldots, x_n) \in I^n$, $x_i = x_j$ if $\sigma(i) = \sigma(j)$, and $x_i \leq x_j$ if $\sigma(i) < \sigma(j)$. Informally, one can represent it by inequalities

$$0 \leq x_{\sigma^{-1}(1)} \leq x_{\sigma^{-1}(2)} \leq \cdots \leq x_{\sigma^{-1}(k)} \leq 1,$$

where $x_{\sigma^{-1}}(i)$ stands for all x_j with $\sigma(j) = i$ (they are all equal to each other). We denote by $[\sigma]$ the simplex associated to σ. The standard boundary operator (modulo ∂I^n) has the form $\partial[\sigma] = \sum_{i=1}^{k-1}(-1)^{i-1}[\sigma_i]$, where $\sigma_i(l) = \sigma(l)$ if $\sigma(l) \leq i$ and $\sigma_i(l) = \sigma(l) - 1$ if $\sigma(l) > i$ (that is, σ_i is gluing together the preimages of i and $i + 1$).

Denote by C^n the corresponding simplicial cochain complex. We consider the basis of simplicial cochains dual to the basis of chains formed by $[\sigma]$, and denote the basis element dual to $[\sigma]$ by $[\hat{\sigma}]$. The differential of a degree k cochain has the form $d[\hat{\sigma}] = \sum_{i=1}^{k}(-1)^{i-1}[\hat{\sigma}_i]$, where σ_i stands for the sum of all maps obtained from σ by splitting the preimage of i into two non-empty subsets (the new preimages of i and $i + 1$). The cohomology of C^n is one-dimensional, and it is concentrated in degree n, $H(C^n) \cong H(I^n, \partial I^n) \cong \mathbb{R}[-n]$. The permutation group S_n acts on I^n preserving its boundary and the simplicial decomposition. The induced action on the cohomology $H(C^n)$ is given by the signature representation.

Define a map $\zeta : (C^n \otimes \mathfrak{g}^{\otimes n})[2n] \to T_n$ by formula

$$\zeta([\hat{\sigma}] \otimes (a_1 \otimes \cdots \otimes a_n)) = \frac{n_1! \cdots n_k!}{n!} \prod_{i_1 \in \sigma^{-1}(1)} a_{i_1} \otimes \cdots \otimes \prod_{i_k \in \sigma^{-1}(k)} a_{i_k},$$

where $n_i = |\sigma^{-1}(i)|$. Under the grading where $\deg S^l \mathfrak{g} = 1 - 2l$, this map is degree preserving (both sides have degree $k - 2n$). Furthermore, it is invariant under the diagonal action of S_n on C^n and on $\mathfrak{g}^{\otimes n}$. It is easy to see that on the S_n-invariant subspace it restricts to an isomorphism $(C^n \otimes \mathfrak{g}^{\otimes n})^{S_n}[2n] \cong T_n$. Moreover, this is an isomorphism of complexes. We illustrate this statement by the following example: let $n = 2$, and let $\sigma : \{1, 2\} \to \{1\}$ be the map gluing 1 and 2. Then, $d[\hat{\sigma}] = [e] + [s]$, where e is the neutral element of S_2 and s is the transposition of 1 and 2. Choosing $a_1 = a_2 = x$, we compute

$$\zeta(d[\hat{\sigma}] \otimes (x \otimes x)) = \zeta(([e] + [s]) \otimes (x \otimes x)) = \frac{1}{2}(x \otimes x + x \otimes x) = x \otimes x,$$

and

$$d\zeta([\hat{\sigma}] \otimes (x \otimes x)) = dx^2 = x \otimes x.$$

Thus, for the cohomology of T_n we obtain

$$H(T_n) \cong H((C^n \otimes \mathfrak{g}^{\otimes n})^{S_n}[2n]) \cong (H(C^n) \otimes \mathfrak{g}^{\otimes n})^{S_n}[2n] \cong \wedge^n \mathfrak{g}[n].$$

Here we used the fact that $H(C^n)$ carries the signature representation in degree n. Note that the cohomology of $T(S^+\mathfrak{g})$ is isomorphic to $\bigoplus_{n=0}^{\infty} \wedge^n \mathfrak{g}[n] = S(\mathfrak{g}[1])$. Since the symmetrization map $\mathrm{Sym} : S(\mathcal{L}(S^+\mathfrak{g})) \to T(S^+\mathfrak{g})$ is an isomorphism of complexes, we have $S(H(\mathcal{L}(S^+\mathfrak{g}))) \cong H(T(S^+\mathfrak{g}))$. By comparing with $H(T(S^+\mathfrak{g})) \cong S(\mathfrak{g}[1])$, we infer that $H^{-1}(\mathcal{L}(S^+\mathfrak{g})) \cong \mathfrak{g}$. For dimensional reasons, $H^{-k}(\mathcal{L}(S^+\mathfrak{g}))$ vanishes for $k \geq 2$ (here we are using the fact that the dimension of $\mathfrak{g}$ is finite). □

Proof of Theorem 1. Let us consider the DGLA $D\mathfrak{g}_s$ ($s \in \mathbb{R}$), which is isomorphic to $D\mathfrak{g}$ as a graded Lie algebra, and the differential is modified as follows:

$$di(x) = [i(x), i(x)]/2 + sl(x), \quad dl(x) = 0.$$

By Proposition 5, we have $H^0(D\mathfrak{g}_0) \simeq H^{-1}(D\mathfrak{g}_0) \simeq \mathfrak{g}$, $H^i(D\mathfrak{g}_0) = 0$ otherwise.

Notice now that $D\mathfrak{g} \simeq D\mathfrak{g}_s$ whenever $s \neq 0$ (the isomorphism is given by redefining $i(t)$ to be $i(st)$, i.e. by multiplying each $S^n\mathfrak{g}$ by s^n). Since the cohomology cannot increase under a small deformation, we only need to check what happens in degrees -1 and 0. The differential $\mathfrak{g} = (D\mathfrak{g}_s)^{-1} \to \mathfrak{g} = (D\mathfrak{g}_s)^0$ is given by multiplication by s, hence the cohomology vanishes for $s \neq 0$. □

Remark. Since $D\mathfrak{g}$ is acyclic, it can be represented (as a complex) as a cone of some graded vector space, $D\mathfrak{g} \cong CV$, where the low degree graded components of V are of the form $V^0 \cong \mathfrak{g}, V^1 = 0, V^2 \cong S^2\mathfrak{g}, V^3 \cong \ker(\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g})$ etc.

For the future use, we shall consider a bigger DGLA, $W\mathfrak{g} \otimes D\mathfrak{g}$ with differential $d_{W\mathfrak{g}} + d_{D\mathfrak{g}}$ and the Lie bracket induced by the Lie bracket of $D\mathfrak{g}$. Let $i(t) \in W\mathfrak{g} \otimes D\mathfrak{g}$ denote the element

$$i(t) = t^a \otimes \mathcal{I}(e_a) + t^a t^b \otimes \mathcal{I}(e_a e_b) + t^a t^b t^c \otimes \mathcal{I}(e_a e_b e_c) + \cdots .$$

Proposition 6. *$i(t) - l(\theta) \in W\mathfrak{g} \otimes D\mathfrak{g}$ is a Maurer–Cartan element.*

Proof. The proof is similar to the one of Proposition 1. We compute directly in shorthand notation,

$$d_{D\mathfrak{g}}(i(t) - l(\theta)) = \frac{1}{2}[i(t), i(t)] + l(t),$$

$$d_{W\mathfrak{g}}(i(t) - l(\theta)) = -[l(\theta), i(t)] - l(t) + \frac{1}{2} l([\theta, \theta]),$$

where $l([\theta, \theta]) = f^c_{ab} \theta^a \theta^b l(e_c)$. Adding up these two expressions we obtain:

$$d(i(t) - l(\theta)) = \frac{1}{2}[i(t), i(t)] - [l(\theta), i(t)] + \frac{1}{2} l([\theta, \theta]) = \frac{1}{2}[i(t) - l(\theta), i(t) - l(\theta)],$$

as required. □

3.2. *Modules over the Weil algebra and $D\mathfrak{g}$-modules*

A module over the DGLA $D\mathfrak{g}$ is a cochain complex V and a DGLA homomorphism $D\mathfrak{g} \to \mathrm{End}(V)$. We shall denote the corresponding generating functions by $i_V(t)$ and $l_V(t)$ (they stand for $\tilde{i}$ and $\tilde{l}$ of the previous section). The following proposition is our motivation for introducing $D\mathfrak{g}$.

Theorem 2. (1) *Let V be a $D\mathfrak{g}$-module. Then, the free $W\mathfrak{g}$-module $U = W\mathfrak{g} \otimes V$ endowed with the differential*

$$d_U = d_{W\mathfrak{g}} + d_V - i_V(t) + l_V(\theta) \tag{3}$$

carries a compatible $C\mathfrak{g}$-action given by formulas

$$I_U(x) = I_{W\mathfrak{g}}(x), \tag{4a}$$

$$L_U(x) = L_{W\mathfrak{g}}(x) + l_V(x) \tag{4b}$$

for $x \in \mathfrak{g}$.
(2) *Let U be a free (that is, isomorphic to $W\mathfrak{g} \otimes V$ for some graded vector space V) $W\mathfrak{g}$-module with a compatible $C\mathfrak{g}$-action. Suppose that one can choose an isomorphism $S\mathfrak{g}^* \otimes_{W\mathfrak{g}} U \cong S\mathfrak{g}^* \otimes V$ in such a way that the action of $\mathfrak{g}$ splits into the standard action on $S\mathfrak{g}^*$ and an action l_V on V. Then, V is naturally a $D\mathfrak{g}$-module and U is naturally isomorphic to the $W\mathfrak{g}$-module described in part* 1.

Proof. For the first statement, note that by Proposition 6 the combination $i_V(t) - l_V(\theta)$ is a Maurer–Cartan element in $W\mathfrak{g} \otimes \mathrm{End}(V)$. Hence, d_U defined by Eq. (3) squares to zero, $d_U^2 = 0$. We shall also check Cartan's formula:

$$\begin{aligned}[d_U, I_U(x)] &= [d_{W\mathfrak{g}} + d_V - i_V(t) + l_V(\theta), I_{W\mathfrak{g}}(x)] \\ &= [d_{W\mathfrak{g}}, I_{W\mathfrak{g}}(x)] + [l_V(\theta), I_{W\mathfrak{g}}(x)] \\ &= L_{W\mathfrak{g}}(x) + l_V(x),\end{aligned}$$

as required. It is easy to see that other relation of $C\mathfrak{g}$ are also verified.

For the second statement, let us start with the isomorphism $S\mathfrak{g}^* \otimes_{W\mathfrak{g}} U \cong S\mathfrak{g}^* \otimes V$ for which the $\mathfrak{g}$-action splits, and denote the action of $\mathfrak{g}$ on V by l_V. Since $I_{W\mathfrak{g}}(e_a) = \partial_{\theta^a}$ (e_a is a basis of $\mathfrak{g}$), there is a unique extension of this isomorphism to $U \cong W\mathfrak{g} \otimes V$ so that $I_U(x) = I_{W\mathfrak{g}}(x)$ (this is the so-called Kalkman trick). We thus have

$$d_U = d_{W\mathfrak{g}} + \delta(\theta, t), \tag{5a}$$

$$I_U(x) = I_{W\mathfrak{g}}(x) = x^a \frac{\partial}{\partial \theta^a}, \tag{5b}$$

$$L_U(x) = [d_U, I_U(x)] = L_{W\mathfrak{g}}(x) + x^a \frac{\partial \delta}{\partial \theta^a} \tag{5c}$$

for some $\delta \in (W\mathfrak{g} \otimes \mathrm{End}(V))^1$. This implies

$$[L_U(x), I_U(y)] = I_U([x, y]) + x^a y^b \frac{\partial^2 \delta}{\partial \theta^a \partial \theta^b}.$$

Hence, δ is at most linear in θ's. Moreover, from (5c) we see that the θ-linear part of δ is in fact equal to $l_V(\theta)$.

Let us now write δ as $\delta(\theta, t) = d_V - i(t) + l(\theta)$, where $d_V = \delta(0, 0)$ and $i(t) = \delta(0, 0) - \delta(0, t)$. Then, the condition $d_U^2 = 0$ reads (using the action of $d_{W\mathfrak{g}}$ on t and θ)

$$0 = d_U^2 = d_V^2 - [d_V, i_V(t)] + \frac{1}{2}[i_V(t), i_V(t)] + l_V(t).$$

Putting $t = 0$ yields $d_V^2 = 0$, and the remaining part of the equation gives $[d_V, i(t)] = [i_V(t), i(t)]/2 + l_V(t)$. Hence, i_V and l_V define a DGLA homomorphism $D\mathfrak{g} \to \mathrm{End}(V)$, as required. □

Remark. Notice that the differential (3) and the action (4) of $C\mathfrak{g}$ resemble the differential and the $C\mathfrak{g}$-action in the Cartan model of equivariant cohomology. Later in this section, we shall find a natural endomorphism of U which transforms d_U into $d_{W\mathfrak{g}} + d_V$ and thus gives an analogue of the Weil model.

Since U is a $W\mathfrak{g}$-module, we have $H_\mathfrak{g}(U) \cong H(U^{C\mathfrak{g}})$. Here $U^{C\mathfrak{g}} \cong (S\mathfrak{g}^* \otimes V)^\mathfrak{g}$ with differential $d_\mathfrak{g} = d_V - i_V(t)$. Assume that the $D\mathfrak{g}$-action on V is induced by a $C\mathfrak{g}$-action via the canonical projection $\pi : D\mathfrak{g} \to C\mathfrak{g}$. Then, $i_V(t) = I_V(t)$ and $(S\mathfrak{g}^* \otimes V)^{\mathfrak{g}}$ with differential $d_{\mathfrak{g}} = d_V - I_V(t)$ is the Cartan model of $H_{\mathfrak{g}}(V)$.

We shall now transform the differential (3) on $U = W\mathfrak{g} \otimes V$ into $d_{W\mathfrak{g}} + d_V$. Such a construction follows easily from the fact that $W\mathfrak{g}$ is $\mathfrak{g}$-equivariantly contractible. Note that $W\mathfrak{g} \otimes \mathcal{U}(\mathcal{L}(S^+\mathfrak{g}))$ is a graded Hopf algebra over $W\mathfrak{g}$ with the coproduct induced by the canonical coproduct of the enveloping algebra $\mathcal{U}(\mathcal{L}(S^+\mathfrak{g}))$.

Theorem 3. *There exists a* $\mathfrak{g}$*-invariant group-like element of degree* 0

$$\Phi \in W\mathfrak{g} \otimes \mathcal{U}(\mathcal{L}(S^+\mathfrak{g})) \subset W\mathfrak{g} \otimes \mathcal{U}(D\mathfrak{g})$$

such that

$$\Phi^{-1} d\Phi = -i(t) + l(\theta). \tag{6}$$

Proof. Recall the following fact: let A be a DGLA and α be a Maurer–Cartan element in $A \otimes \Omega(I)$, where $I = [0, 1]$ is the unit interval. We have $\alpha = a(s) + b(s)\, ds$, where $a(s)$ is a family of Maurer–Cartan elements in A (parametrized by s) and $b(s) \in A^0$. Let Φ be the holonomy from 0 to 1 of the A^0-connection $b(s)\, ds$ on I. Then, Φ transforms $a(0)$ to $a(1)$, i.e. $a(1) = \Phi^{-1} a(0) \Phi - \Phi^{-1} d\Phi$.

By Proposition 6, $i(t) - l(\theta) \in W\mathfrak{g} \otimes D\mathfrak{g}$ is a Maurer–Cartan element. Consider the morphism of dg algebras $W\mathfrak{g} \to W\mathfrak{g} \otimes \Omega(I)$ given by $\theta^a \mapsto \theta^a \otimes s$, where s is the coordinate on I. It gives rise to a morphism of DGLAs

$$W\mathfrak{g} \otimes D\mathfrak{g} \to W\mathfrak{g} \otimes D\mathfrak{g} \otimes \Omega(I).$$

Let $\alpha = a(s) + b(s)\, ds$ be the image of $i(t) - l(\theta)$ under this morphism. Then, we have $a(0) = 0$ and $a(1) = i(t) - l(\theta)$. The element $b(s)$ takes values in the pronilpotent subalgebra $W\mathfrak{g} \otimes \mathcal{L}(S^+\mathfrak{g})$, so the holonomy Φ is well defined. This implies, $i(t) - l(\theta) = -\Phi^{-1} d\Phi$, as required. ◻

Remark. Theorem 3 should be compared to Proposition 2. In contrast to equation $\phi = \exp(-I(\theta))$, the explicit formula for the element Φ is more involved. For $x, y \in \mathfrak{g}$ let $\langle \partial i(x), y \rangle$ be defined by

$$\langle \partial i(x), y \rangle = \frac{d}{dr} i(x + ry) \bigg|_{r=0}.$$

Then, Φ is the parallel transport from $s = 0$ to $s = 1$ of the connection

$$- \left\langle \partial i \left(s\, t + \frac{s^2 - s}{2} [\theta, \theta] \right), \theta \right\rangle ds.$$

Computing the contributions up to degree 3 yields

$$\Phi = \exp(-\mathcal{I}(\theta)) - \mathcal{I}\left(t\theta - \frac{1}{6}[\theta, \theta]\theta \right) + (\text{terms of degree} \geq 4).$$

Let us define a $\mathfrak{g}$-equivariant linear map $Y : \mathfrak{g} \to (W\mathfrak{g} \otimes D\mathfrak{g})^{-1}$ by formula

$$Y(x) = -(I_{W\mathfrak{g}}(x)\Phi)\Phi^{-1}.$$

For a $D\mathfrak{g}$-module V, one can define $\Phi_V \in (W\mathfrak{g} \otimes \mathrm{End}(V))^0$ and $Y_V(x) \in (W\mathfrak{g} \otimes \mathrm{End}(V))^{-1}$ as images of Φ and $Y(x)$ under the action map.

Proposition 7. *Under the natural transformation defined by Φ_V^{-1}, the $C\mathfrak{g}$-module $U = W\mathfrak{g} \otimes V$ given by (3) and (4) is naturally isomorphic to $U' = W\mathfrak{g} \otimes V$ with differential and $C\mathfrak{g}$-action given by*

$$d_{U'} = d_{W\mathfrak{g}} + d_V,$$

$$I_{U'} = I_{W\mathfrak{g}} + Y_V,$$

$$L_{U'} = L_{W\mathfrak{g}} + l_V.$$

Proof. Since Φ is $\mathfrak{g}$-equivariant, we have

$$L_{U'}(x) = \Phi_V L_U(x)\Phi_V^{-1} = L_U(x) = L_{W\mathfrak{g}}(x) + l_V(x).$$

For contractions, we obtain

$$I_{U'}(x) = \Phi_V I_U(x)\Phi_V^{-1} = \Phi_V I_{W\mathfrak{g}}(x)\Phi_V^{-1} = I_{W\mathfrak{g}}(x) + Y_V(x),$$

as required. Finally, note that

$$\Phi_V^{-1}(d_{W\mathfrak{g}} + d_V)\Phi_V = d_{W\mathfrak{g}} + d_V - i_V(t) + l_V(t) = d_U.$$

Hence,

$$\Phi_V d_U \Phi_V^{-1} = d_{W\mathfrak{g}} + d_V = d_{U'}.$$ □

Remark. Again, one can replace $D\mathfrak{g}$ by $C\mathfrak{g}$ in Proposition 7. Then, the map Y takes the form $Y(x) = -(I_{W\mathfrak{g}}(x)\phi)\phi^{-1} = I(x)$, and $I_{U'}(x) = I_{W\mathfrak{g}}(x) + I_V(x)$. That is, we obtain the Weil model of equivariant cohomology of the $\mathfrak{g}$-differential space V.

4. Central Extensions and DGLA Homomorphisms of $D\mathfrak{g}$

In this section we study further properties of $D\mathfrak{g}$ including central extensions and homomorphisms from $D\mathfrak{g}$ to other DGLAs.

4.1. *Central extensions of $C\mathfrak{g}$*

We start with an easier problem of central extensions of $C\mathfrak{g}$. Let $C \to A \to C\mathfrak{g}$ be a central extension of $C\mathfrak{g}$ by a graded vector space C. Assume that the extension A is split over $\mathfrak{g}$, and that there is a $\mathfrak{g}$-equivariant injective map $\tilde{I} : C\mathfrak{g}^{-1} = \mathfrak{g}\varepsilon \to A^{-1}$ such that the composition $\mathfrak{g}\varepsilon \to A^{-1} \to \mathfrak{g}\varepsilon$ is the identity map. For instance, for $\mathfrak{g}$ reductive these assumptions are always satisfied.

In general, central extensions are classified by the second cohomology group of $C\mathfrak{g}$ with values in C. A 2-cocycle consists of a degree 0 map $\omega\colon \wedge^2 C\mathfrak{g} \to C$ and a degree 1 map $\partial\colon C\mathfrak{g} \to C$. For the map ω, note that $[L(x), L(y)] = L([x,y])$ (the extension is split over $\mathfrak{g}$), and $[L(x), \tilde{I}(y)] = \tilde{I}([x,y])$ (the map $\tilde{I}$ is $\mathfrak{g}$-equivariant). Hence, the only non-trivial part of ω is the map $\omega\colon \wedge^2 \mathfrak{g}\epsilon \to C^{-2}$. It is easy to see that the only condition on ω is $\mathfrak{g}$-invariance. For instance, if $C^{-2} = \mathbb{R}c$, ω is defined by a degree 2 invariant polynomial $p \in (S^2\mathfrak{g}^*)^{\mathfrak{g}}$. Then, $[\tilde{I}(x), \tilde{I}(y)] = -2p(x,y)c$, where the normalization is chosen to match the natural normalization of the next section. We shall denote this central extension by $C_p\mathfrak{g}$. For the map ∂, it is completely defined by a character $\chi\colon \mathfrak{g} \to C^0$. We have $d\tilde{I}(x) = L(x) + \chi(x)$ and $dL(x) = -d\chi(x)$. This extension is trivial since $\tilde{I}(x)$ and $\tilde{L}(x) = L(x) + \chi(x)$ define a DGLA homomorphism $C\mathfrak{g} \to A$.

4.2. *Central extensions of $D\mathfrak{g}$*

Again, let

$$C \to A \to D\mathfrak{g} \tag{7}$$

be a central extension of DGLAs split over $\mathfrak{g} \subset D\mathfrak{g}$. Similar to the previous section, we assume that the map $i\colon S^+\mathfrak{g} \to D\mathfrak{g}$ can be lifted to a $\mathfrak{g}$-*equivariant* (grading-preserving) map $\tilde{\imath}\colon S^+\mathfrak{g} \to A$. For instance, this is always true if $\mathfrak{g}$ is reductive. Together with the splitting over $\mathfrak{g}$, the lift $\tilde{\imath}$ defines a morphism of graded Lie algebras $s\colon D\mathfrak{g} \to A$ which is a splitting of the extension (7). Let $\mathcal{J} = [D\mathfrak{g}, D\mathfrak{g}] + \mathfrak{g} \subset D\mathfrak{g}$. Notice that $s|_{\mathcal{J}}$ does not depend on the choice of $\tilde{\imath}$ and is a morphism of DGLAs (unlike s). Since $D\mathfrak{g}/\mathcal{J} = (S^+\mathfrak{g})_{\mathfrak{g}}$ with vanishing bracket and differential, central extensions of $D\mathfrak{g}$ by C are in one-to-one correspondence with extensions of complexes

$$C \to A' \to (S^+\mathfrak{g})_{\mathfrak{g}}.$$

We have thus proved.

Theorem 4. *The category of central extensions of $D\mathfrak{g}$ which are split over $\mathfrak{g} \subset D\mathfrak{g}$ and admit a $\mathfrak{g}$-equivariant lift of the map i is equivalent to the category of extensions of the complex $(S^+\mathfrak{g})_{\mathfrak{g}}$. In particular, extensions by a complex C are classified by maps*

$$(S^+\mathfrak{g})_{\mathfrak{g}} \to H(C)[1].$$

Let us also describe these extensions at the level of cochains. Since the lift $\tilde{\imath}$ defines a splitting s of the extension (7), we have $A \cong D\mathfrak{g} \oplus C$ as a graded Lie algebra. The differential on A satisfies

$$d\tilde{\imath}(t) = [\tilde{\imath}(t), \tilde{\imath}(t)]/2 + l(t) + q(t) \tag{8}$$

for some $\mathfrak{g}$-invariant map $q\colon S^+\mathfrak{g} \to C[1]$ (which can be seen as a power series $q\colon \mathfrak{g}[2] \to C$ of total degree 2, such that $q(0) = 0$). The formula (8) makes $D\mathfrak{g} \oplus C$

to a DGLA if and only if

$$dq(t) = 0.$$

We shall denote this DGLA by $D\mathfrak{g} \oplus_q C$. In particular, for $C = \mathbb{R}[2n-2]$ and the map q defined by an invariant degree n polynomial $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$ we shall denote $D\mathfrak{g} \oplus_q \mathbb{R}[2n-2]$ simply by $D_p\mathfrak{g}$.

Remark. For $n = 2$, $p \in (S^2\mathfrak{g}^*)^{\mathfrak{g}}$ defines an invariant symmetric bilinear form on $\mathfrak{g}$. At the level of generators, the differential of $\tilde{\mathcal{I}}(xy)$ is modified as follows:

$$d\tilde{\mathcal{I}}(xy) = \frac{1}{2}[\tilde{\mathcal{I}}(x), \tilde{\mathcal{I}}(y)] + p(x,y)c.$$

Note that this central extension descends to $C\mathfrak{g}$ (by putting all higher generators including $\tilde{\mathcal{I}}(xy)$ equal zero). The corresponding equation reads $[I(x), I(y)] = -2p(x,y)c$ giving rise to the extension $C_p\mathfrak{g}$ (see the previous section).

For $n = 3$, we choose $p \in (S^3\mathfrak{g}^*)^{\mathfrak{g}}$ and modify the differential of $\tilde{I}(x^3)$,

$$d\tilde{\mathcal{I}}(x^3) = [\tilde{\mathcal{I}}(x), \tilde{\mathcal{I}}(x^2)] + p(x^3)c.$$

This (and higher) extensions do not descend to $C\mathfrak{g}$.

One interesting property of $D_p\mathfrak{g}$ is as follows. Denote by s the injection $D\mathfrak{g} \to D_p\mathfrak{g} = D\mathfrak{g} \oplus_q \mathbb{R}[2n-2]$. Then, one can define a group-like element $\Phi_p = (\mathrm{id} \otimes s)\Phi \in W\mathfrak{g} \otimes \mathcal{U}(D_p\mathfrak{g})$ (here $\Phi \in W\mathfrak{g} \otimes \mathcal{U}(D_{\mathfrak{g}})$ is defined in Theorem 3).

Proposition 8. $\Phi_p^{-1} d\Phi_p = \tilde{\imath}(t) - l(\theta) - e \otimes c$, *where* $c \in D_p\mathfrak{g}$ *is the generator of the central line* $\mathbb{R}[2n-2]$, *and* $e \in (W\mathfrak{g})^{\mathfrak{g}}$ *is such that* $de = p$.

Proof. Since Φ_p is a group-like element, $\Phi_p^{-1} d\Phi_p$ is an element of $W\mathfrak{g} \otimes D_p\mathfrak{g}$, and its projection to $W\mathfrak{g} \otimes D\mathfrak{g}$ is equal to $\tilde{\imath}(t) - l(\theta)$ (see Theorem 3). Hence, $\Phi_p^{-1} d\Phi_p = \tilde{\imath}(t) - l(\theta) - e \otimes c$ for some $e \in W\mathfrak{g}$.

Note that the expression $\Phi_p^{-1} d\Phi_p$ is automatically a Maurer–Cartan element. This implies,

$$\begin{aligned} 0 &= d(\Phi_p^{-1} d\Phi_p) - \frac{1}{2}[\Phi_p^{-1} d\Phi_p, \Phi_p^{-1} d\Phi_p] \\ &= d(\tilde{\imath}(t) - l(\theta) - e \otimes c) - \frac{1}{2}[\tilde{\imath}(t) - l(\theta) - e \otimes c, \tilde{\imath}(t) - l(\theta) - e \otimes c] \\ &= p \otimes c - (de) \otimes c. \end{aligned}$$

We conclude that $de = p$, as required. □

4.3. *Deformations of a DGLA homomorphism*

Let A be a DGLA and $e \in A^1$ be an element of degree 1. Recall that the operator $d' = d - [e, \cdot]$ defines a new differential on A (that is, d' is a derivation of the Lie bracket and $d'^2 = 0$) if and only if the element $z = de - [e,e]/2$ lies in the center of A. The Jacobi identity implies that z is closed, $dz = 0$. Indeed,

$$dz = d\left(de - \frac{1}{2}[e,e]\right) = \frac{1}{2}\left([e,de] - [de,e]\right) = \left[e, z + \frac{1}{2}[e,e]\right] = \frac{1}{2}[e,[e,e]] = 0.$$

We shall denote the graded Lie algebra A equipped with this new differential by $A_{(e)}$.

A DGLA homomorphism $D\mathfrak{g} \to A$ consists of a Lie algebra homomorphism $l_A : \mathfrak{g} \to A$ and a $\mathfrak{g}$-equivariant formal power series $i_A : \mathfrak{g}[2] \to A$ of degree 1, satisfying $i_A(0) = 0$ and Eq. (2). Consider simultaneous deformations of the differential on A and of the map i_A. Choose a $\mathfrak{g}$-equivariant power series of total degree 1, $e : \mathfrak{g}[2] \to A$, and set

$$d' = d - [e(0), \cdot],$$
$$i'_A(t) = i_A(t) + e(t) - e(0).$$

These formulas define a DGLA homomorphism $D\mathfrak{g} \to A_{(e(0))}$ if and only if

$$d_{\mathfrak{g}} e(t) := de(t) - [i_A(t), e(t)] = \frac{1}{2}[e(t), e(t)] + z, \tag{9}$$

where $z \in A$ lies in the center of A. Note that z is invariant under the $\mathfrak{g}$-action defined by the Lie homomorphism $l_A : \mathfrak{g} \to A$. Then, the Jacobi identity implies that z is equivariantly closed, $d_{\mathfrak{g}} z = 0$.

More generally, assume that

$$d_{\mathfrak{g}} e(t) = [e(t), e(t)]/2 + z(t), \tag{10}$$

where $z(t)$ takes values in the center of A, and $q(t) = z(t) - z(0)$ takes values in the subspace $C \subset A$. Then, we obtain a DGLA homomorphism $D\mathfrak{g} \oplus_q C \to A_{(e(0))}$.

A natural framework for constructing examples is as follows: let B be a $C\mathfrak{g}$-module with a chosen $C\mathfrak{g}$-invariant element $1_B \in B^0_{\text{closed}}$. For example, B may be a unital (graded) commutative $\mathfrak{g}$-differential algebra. Consider the semi-direct sum $A = C\mathfrak{g} \ltimes B[n]$, where $B[n]$ is viewed as an abelian DGLA. Solutions of equation (10) are provided by the following construction:

Theorem 5. *Let $c \in B^{2k}$ be a basic cocycle, and $p \in (S^k\mathfrak{g}^*)^{\mathfrak{g}}$ be an invariant polynomial of degree k. Assume that the cohomology class of the element $p \otimes 1_B + 1 \otimes c \in (S\mathfrak{g}^* \otimes B)^{\mathfrak{g}}$ vanishes in $H_{\mathfrak{g}}(B)$, and let $e \in (S\mathfrak{g}^* \otimes B)^{\mathfrak{g}}$ be such that $d_{\mathfrak{g}} e = p \otimes 1_B + 1 \otimes c$. Consider $A = C\mathfrak{g} \ltimes B[2k-2]$. Then e is a solution of Eq.* (10) *with $z(t) = p(t)1_B + c, z(0) = c, q(t) = p(t)1_B$ and $C = \mathbb{R}[2k-2]1_B$.*

Proof. Since $B[2k-2] \subset A$ is an abelian DGLA, equation $d_{\mathfrak{g}} e = p \otimes 1 + 1 \otimes c$ implies Eq. (10) with $z(t) = p(t)1_B + c$. Putting $t = 0$ yields $z(0) = c$. This element is central in A since $c \in B^{2k}$ is basic. Finally, $q(t) = z(t) - z(0) = p(t)1_B$. ◻

Remark. There is a natural DGLA homomorphism $D\mathfrak{g} \to A$ induced by the canonical projection $D\mathfrak{g} \to C\mathfrak{g}$. Elements e described in Theorem 5 define DGLA homomorphisms $D_p\mathfrak{g} \to A$, where the central line $R[2k-2] \subset D_p\mathfrak{g}$ maps to $\mathbb{R}[2k-2]1_B \subset B[2k-2] \subset A$.

In the case of $p = 0$, we obtain a DGLA homomorphism $D\mathfrak{g} \to A$. Then, for the construction to work one only needs a structure of a $\mathfrak{g}$-differential vector space on B.

4.4. *Examples*

In this section, we consider two examples of the construction described above.

4.4.1. *Extensions of $C\mathfrak{g}$ by differential forms*

Let $\mathfrak{g}$ act on a manifold M. We choose $B = \Omega(M)$ with the natural structure of a $\mathfrak{g}$-differential algebra. Let $A = C\mathfrak{g} \ltimes \Omega(M)[m]$. If $\phi(t) \in (S\mathfrak{g}^* \otimes \Omega(M))^{\mathfrak{g}}$ is a degree $m+1$ equivariant cocycle, then the construction of the previous section defines a DGLA homomorphism $D\mathfrak{g} \to A_{(\phi(0))}$. If $m = 2n-2$ and ϕ verifies equation $d_{\mathfrak{g}}\phi = p \otimes 1$ for $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$, we obtain a DGLA homomorphism $D_p\mathfrak{g} \to A_{(\phi(0))}$.

In more detail, write $\phi(t) = \sum_{l=0}^{n-1} \phi_l(t)$, where $\phi_l(t)$ is a homogeneous polynomial of degree l with values in degree $2n-2l-1$ forms on M. Then, $\mathcal{I}(t)$ maps to $I_M(t) + \phi_1(t)$, $\mathcal{I}(t^k)$ map to $\phi_k(t)$ for $2 \le k \le n-1$, and $\mathcal{I}(t^k)$ map to zero for $k > n-1$. For $n = 2$, the images of higher contractions (with $k \ge 2$) vanish and the DGLA homomorphism $D_p\mathfrak{g} \to A_{(\phi(0))}$ descends to a homomorphism $C_p\mathfrak{g} \to A_{(\phi(0))}$. If $n > 2$, higher contractions map to non-vanishing differential forms $\phi_k(t)$, and the DGLA homomorphism does not descend to $C_p\mathfrak{g}$.

The DGLA $A_{(\phi(0))}$ acts on the differential graded algebra $\Omega(M)[[s]]$, where s is a formal variable of degree $2n-2$ satisfying $ds = -\phi(0)$: the action of $C\mathfrak{g}$ is the standard action on $\Omega(M)$ (that is, the action is trivial on s) and the action of $\alpha \in \Omega(M)[m]$ is via $\alpha\,\partial_s$. A homomorphism $D\mathfrak{g} \to A_{(\phi(0))}$ therefore gives rise to an action of $D\mathfrak{g}$ on $\Omega(M)[s]$. This action can be seen as a twist by ϕ of the standard action of $C\mathfrak{g}$ on $\Omega(M)$. One can get rid of the variable s using the embedding $\Omega(M)$ to $\Omega(M)[\![s]\!]$ by $\alpha \mapsto \alpha e^s$. The new differential on $\Omega(M)$ is then $d - \phi(0)$. ($\Omega(M)$ is only mod-2 graded and the differential and the action of $D\mathfrak{g}$ are no longer derivations).

4.4.2. *Extensions of $C\mathfrak{g}$ by the Weil algebra*

One can choose B equal to the Weil algebra $W\mathfrak{g}$. Recall that for $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$ one can choose an element $e \in (S\mathfrak{g}^* \otimes W\mathfrak{g})^{\mathfrak{g}}$ such that $d_{\mathfrak{g}}e = p \otimes 1 - 1 \otimes p$. Denote the generators of $S\mathfrak{g}^*$ by t^a and the generators of $W\mathfrak{g}$ by θ^a and f^a. The element $e(t)$ (here $t \in \mathfrak{g}$ refers to the first factor in $(S\mathfrak{g}^* \otimes W\mathfrak{g})^{\mathfrak{g}}$) is a solution of (10) (notice that $[e(t), e(t)] = 0$) for $c = -p \in W\mathfrak{g}$ and $q(t) = p(t)$. We thus get a homomorphism $D_p\mathfrak{g} \to A_{(e(0))}$. Notice that $e(0) \in (W\mathfrak{g})^{\mathfrak{g}}$ satisfies $de(0) = -p$.

For $n = 2$, one can choose the element e in the form

$$e(t) = -(p(t + f, \theta) - \frac{1}{6}p(\theta, [\theta, \theta])).$$

Since $e(t) - e(0) = -p(t, \theta)$, $i(t)$ maps to $I(t) - p(t, \theta)$. Note that

$$[I(x) - p(x, \theta), I(y) - p(y, \theta)] = -2p(x, y),$$

as required by the relations of $D_p\mathfrak{g}$. In the case of $n = 2$, higher contractions vanish and one actually obtains a DGLA homomorphism $C_p\mathfrak{g} \to A_{(e(0))}$.

For $n \geq 3$, images of some higher contractions are necessarily non-vanishing. For $n = 3$, we can work it out in more detail. Recall that $(D_p\mathfrak{g})^0 \cong \mathfrak{g}$ with generators $l(x)$, $(D_p\mathfrak{g})^{-1} \cong \mathfrak{g}$ with generators $\mathcal{I}(x)$, $(D_p\mathfrak{g})^{-2} \cong S^2\mathfrak{g}$ spanned by $[\mathcal{I}(x), \mathcal{I}(y)]$, $(D_p\mathfrak{g})^{-3} \cong S^2\mathfrak{g} \oplus \ker(\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g})$ spanned by $\mathcal{I}(xy)$ and $[\mathcal{I}(x), [\mathcal{I}(y), \mathcal{I}(z)]]$, and $(D_p\mathfrak{g})^{-4} = (D\mathfrak{g})^{-4} \oplus \mathbb{R}$, where $\mathbb{R}$ is the central line. For the algebra A, we have $A^{-4} = \mathbb{R}$ spanned by the unit of the Weil algebra, $A^{-3} \cong \mathfrak{g}^*$ with generators θ^a, $A^{-2} \cong \mathfrak{g}^* \oplus \wedge^2\mathfrak{g}^*$ spanned by f^a and $\theta^a\theta^b$, $A^{-1} \cong \mathfrak{g} \oplus (W\mathfrak{g})^3$ with $\mathfrak{g}$ spanned by $I(x)$, and $A^0 \cong \mathfrak{g} \oplus (W\mathfrak{g})^4$, where $\mathfrak{g}$ is spanned by $L(x)$.

The image of the homomorphism $\rho: D_p\mathfrak{g} \to A$ is a DGLA B with non-trivial graded components $B^0 \cong \mathfrak{g}$ with generators $L(x)$, $B^{-1} \cong \mathfrak{g}$ with generators $\tilde{I}(x)$, $B^{-2} = \mathfrak{g}^*$ with generators $\mu(\xi)$, $B^{-3} = \mathfrak{g}^*$ with generators $\theta(\xi)$ (here $\xi \in \mathfrak{g}^*$), and $B^{-4} = \mathbb{R}c$. The differential acts as $d\tilde{I}(x) = L(x), d\theta(\xi) = \mu(\xi)$. For the Lie bracket, $L(x)$ act on other components by the adjoint and coadjoint actions, B^{-2}, B^{-3}, B^{-4} form an abelian Lie subalgebra, $[\tilde{I}(x), \theta(\xi)] = \theta(\mathrm{ad}^*(x)\xi)$, $[\tilde{I}(x), \theta(\xi)] = \xi(x)c$, and

$$[\tilde{I}(x), \tilde{I}(y)] = 2\mu(p(x, y, \cdot)).$$

Here $\rho(\mathcal{I}(xy)) = \theta(p(x, y, \cdot))$, and the last relation follows from $[\mathcal{I}(x), \mathcal{I}(y)] = 2d\mathcal{I}(xy)$.

5. Current Algebras

In this section, we introduce a functor associating to a manifold M and to a DGLA A a Lie algebra $\mathcal{CA}(M, A)$. As an application, we give a new interpretation of the FMS cocycles of higher-dimensional current algebras.

5.1. *Current algebra functor*

Let A be a DGLA. Then, the subspace of closed elements of A of degree 0 $A^0_{\mathrm{closed}} \subset A^0$, is a Lie subalgebra of A. Following [11], notice that A^{-1} equipped with the bracket

$$\{\alpha, \beta\} := [\alpha, d\beta] \tag{11}$$

is a Leibniz algebra. That is, the bracket $\{,\}$ satisfies the Jacobi identity

$$\{\alpha, \{\beta, \gamma\}\} = \{\{\alpha, \beta\}, \gamma\} + \{\beta, \{\alpha, \gamma\}\}.$$

As the symmetric part of this bracket has exact values, the quotient space $A^{-1}/A^{-1}_{\text{exact}}$ is a Lie algebra.

Proposition 9. *There is an exact sequence of Lie algebras,*

$$0 \to H^{-1}(A) \to A^{-1}/A^{-1}_{\text{exact}} \to A^{0}_{\text{closed}} \to H^{0}(A) \to 0, \tag{12}$$

where $H^{-1}(A)$ is abelian, the Lie bracket on $H^0(A)$ is induced by the Lie bracket on A^0, and the map $A^{-1}/A^{-1}_{\text{exact}} \to A^{0}_{\text{closed}}$ is induced by the differential of A.

Proof. By definition of cohomology groups $H^{-1}(A)$ and $H^0(A)$, the sequence (12) is exact. The map $H^{-1}(A) \to A^{-1}/A^{-1}_{\text{exact}}$ is a Lie homomorphism since the derived bracket (11) vanishes if α and β are closed. The map $A^{-1}/A^{-1}_{\text{exact}} \to A^{0}_{\text{closed}}$ is a Lie homomorphism because $d\{\alpha, \beta\} = d[\alpha, d\beta] = [d\alpha, d\beta]$. Finally, the map $A^{0}_{\text{closed}} \to H^0(A)$ is a Lie homomorphism by definition of the Lie bracket on $H^0(A)$. □

Remark. The 2-step complex $A^{-1}/A^{-1}_{\text{exact}} \to A^{0}_{\text{closed}}$ inherits a DGLA structure from A. Equivalently, the pair of Lie algebras $(A^{-1}/A^{-1}_{\text{exact}}, A^{0}_{\text{closed}})$ is a crossed module of Lie algebras.

Proposition 10. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of DGLAs. Then, there is an exact sequence of Lie algebras,*

$$\begin{aligned} \cdots \to H^{-2}(A) \to H^{-2}(B) \to H^{-2}(C) \\ \to A^{-1}/A^{-1}_{\text{exact}} \to B^{-1}/B^{-1}_{\text{exact}} \to C^{-1}/C^{-1}_{\text{exact}} \to 0, \end{aligned} \tag{13}$$

where all cohomology groups are viewed as abelian Lie algebras, the map $H^{-2}(C) \to A^{-1}/A^{-1}_{\text{exact}}$ is the composition of the connecting homomorphism $H^{-2}(C) \to H^{-1}(A)$ and the natural map $H^{-1}(A) \to A^{-1}/A^{-1}_{\text{exact}}$.

Proof. Replace complexes A, B and C by their truncations where all components of non-negative degrees are replaced by zero. Then, (13) is the corresponding long exact sequence. Maps between cohomology groups are Lie homomorphisms since the corresponding Lie brackets vanish. Maps between Lie algebras equipped with derived brackets are Lie homomorphisms since $A \to B \to C$ are homomorphisms of DGLAs. Finally, the map $H^{-2}(C) \to A^{-1}/A^{-1}_{\text{exact}}$ is a Lie homomorphism because it factors through the Lie homomorphism $H^{-1}(A) \to A^{-1}/A^{-1}_{\text{exact}}$. □

Note that if C is acyclic, the long exact sequence (13) degenerates to a short exact sequence,

$$0 \to A^{-1}/A^{-1}_{\text{exact}} \to B^{-1}/B^{-1}_{\text{exact}} \to C^{-1}/C^{-1}_{\text{exact}} \to 0.$$

Proposition 11. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of DGLAs. Then, there is an exact sequence of vector spaces,*

$$0 \to A^{0}_{\text{closed}} \to B^{0}_{\text{closed}} \to C^{0}_{\text{closed}} \to H^{1}(A) \to H^{1}(B) \to H^{1}(C) \to \cdots.$$

If $H^0(C)$ or $H^1(A)$ vanishes, it gives rise to a short exact sequence of Lie algebras,

$$0 \to A^0_{\text{closed}} \to B^0_{\text{closed}} \to C^0_{\text{closed}} \to 0.$$

Proof. Replace the complexes A, B and C by their truncations where all components with negative degrees are replaced by zero. The corresponding long exact sequence is the one displayed in the proposition. The connecting homomorphism C^0_{closed} is a composition of the natural projection $C^0_{\text{closed}} \to H^0(A)$ and the standard connecting homomorphism $H^0(A) \to H^1(C)$. If either $H^0(A)$ or $H^1(C)$ (or both) vanishes, this map vanishes as well giving rise to a short exact sequence. This is a short exact sequence of Lie algebras since $0 \to A \to B \to C \to 0$ is an exact sequence of DGLAs. $\square$

For a manifold M and a DGLA A, we consider the DGLA $(\Omega(M) \otimes A)$, where the Lie bracket is induced by the Lie bracket of A, and the differential comes from the differential of A and the de Rham differential on M. We define the current algebra functor as

$$\mathcal{CA}(M, A) = (\Omega(M) \otimes A)^{-1} / (\Omega(M) \otimes A)^{-1}_{\text{exact}}.$$

It associates a Lie algebra (a current algebra) to a pair of a manifold and a DGLA. It is convenient to introduce a special notation

$$\mathcal{SA}(M, A) = (\Omega(M) \otimes A)^0_{\text{closed}}.$$

As before, we have a natural exact sequence of Lie algebras

$$0 \to H^{-1}(\Omega(M) \otimes A) \to \mathcal{CA}(M, A) \to \mathcal{SA}(M, A) \to H^0(M, A) \to 0$$

and $\mathcal{CA}(M, A) \to \mathcal{SA}(M, A)$ is a crossed module of Lie algebras. Note that if A is acyclic, the exact sequence degenerates to an isomorphism $\mathcal{CA}(M, A) \cong \mathcal{SA}(M, A)$.

Remark. Notice that $\mathcal{SA}(M, A) = \mathcal{SA}(M, A_{\text{trunc}})$, where

$$A^i_{\text{trunc}} = \begin{cases} A^i & i < 0, \\ A^0_{\text{closed}} & i = 0, \\ 0 & i > 0. \end{cases}$$

The functor $\mathcal{SA}(\cdot, A)$ is well defined for supermanifolds. The DGLA A_{trunc} can be restored from $\mathcal{SA}(\cdot, A)$ as $\mathcal{SA}(\mathbb{R}^{0|1}, A)$; the grading comes via functoriality from the vector field $\epsilon\partial_\epsilon$ and the differential from ∂_ϵ, where ϵ is the coordinate on $\mathbb{R}^{0|1}$. In general, if F is a contravariant functor from the category of supermanifolds to the category of Lie algebras (functorial with respect to internal Hom's) then $A_F = F(\mathbb{R}^{0|1})$ is a DGLA. We get a natural transformation $F \to \mathcal{SA}(\cdot, A_F)$, and A_F is universal among DGLAs A equipped with a natural transformation $F \to \mathcal{SA}(\cdot, A)$.

Proposition 12. *Let A be a DGLA, and suppose that, as a complex, it is isomorphic to a cone over a graded vector space V. That is, $A \cong CV = V[\varepsilon]$, where $\varepsilon^2 = 0$,*

$\deg \varepsilon = -1$ and $d = d/d\varepsilon$. Then, the current algebra $\mathcal{CA}(M, A)$ is isomorphic to $(\Omega(M) \otimes V)^0$ as a vector space.

Proof. Since $\mathcal{CA}(M, A) \cong \mathcal{SA}(M, A) = (\Omega(M) \otimes A)^0_{\text{closed}}$, we consider an element $\alpha \otimes x + \beta \otimes y\varepsilon \in (\Omega(M) \otimes A)^0$. The closedness condition reads

$$d(\alpha \otimes x + \beta \otimes y\varepsilon) = d\alpha \otimes x - \beta \otimes y + d\beta \otimes y\varepsilon = 0.$$

Hence, $x = y$ and $\beta = d\alpha$, and the projection $(\Omega(M) \otimes A)^0_{\text{closed}} \to (\Omega(M) \otimes V)^0$ mapping $\alpha \otimes x + d\alpha \otimes x\varepsilon \to \alpha \otimes x$ is an isomorphism. □

Remark. Note that $\alpha \otimes x + d\alpha \otimes x\varepsilon = d(\alpha \otimes x\varepsilon)$, where $\alpha \otimes x\varepsilon \in (\Omega(M) \otimes A)^{-1}$ defines an element of $\mathcal{CA}(M, A)$.

The current algebra functor is contravariant with respect to M and covariant with respect to A. If $0 \to A \to B \to C \to 0$ is a short exact sequence of DGLAs, we obtain from Proposition 10 an exact sequence of Lie algebras,

$$\begin{aligned} 0 &\to \mathrm{im}(H^{-2}(\Omega(M) \otimes C) \to H^{-1}(\Omega(M) \otimes A)) \\ &\to \mathcal{CA}(M, A) \to \mathcal{CA}(M, B) \to \mathcal{CA}(M, C) \to 0. \end{aligned} \tag{14}$$

Again, if C is acyclic, it degenerates to a short exact sequence of current algebras,

$$0 \to \mathcal{CA}(M, A) \to \mathcal{CA}(M, B) \to \mathcal{CA}(M, C) \to 0.$$

In many examples, C is equal to $C\mathfrak{g}$ or $D\mathfrak{g}$. These DGLAs are acyclic, and we obtain short exact sequences of current algebras.

If C is acyclic, we have $H^0(\Omega(M \otimes C)) = 0$, and we obtain a short exact sequence of Lie algebras

$$0 \to \mathcal{SA}(M, A) \to \mathcal{SA}(M, B) \to \mathcal{SA}(M, C) \to 0.$$

5.2. *Examples*

In this section, we apply the functor $\mathcal{CA}$ to obtain several examples of current algebras on manifolds.

5.2.1. $A = C\mathfrak{g}$

Let $\mathfrak{g}$ be a Lie algebra. The cone $C\mathfrak{g}$ is an acyclic DGLA. Hence, $\mathcal{CA}(M, C\mathfrak{g}) \cong \Omega^0(M) \otimes \mathfrak{g} = C^\infty(M, \mathfrak{g})$. It is easy to see that the Lie bracket of $\mathcal{CA}(M, C\mathfrak{g})$ coincides with the pointwise Lie bracket on $C^\infty(M, \mathfrak{g})$. Indeed, the derived bracket of two elements $f \otimes I(x), g \otimes I(y) \in (\Omega(M) \otimes C\mathfrak{g})^{-1}$ is given by

$$\begin{aligned} \{f \otimes I(x), g \otimes I(y)\} &= [f \otimes I(x), d(g \otimes I(y))] \\ &= [f \otimes I(x), dg \otimes I(y) + g \otimes L(y)] \\ &= fg \otimes [I(x), L(y)] = fg \otimes I([x, y]), \end{aligned}$$

as required.

5.2.2. $A = D\mathfrak{g}$

In the case of $A = D\mathfrak{g}$, it is difficult to give a compact description of $\mathcal{CA}(M, D\mathfrak{g})$. By Proposition 1, $D\mathfrak{g}$ is acyclic. It is a cone CV over a graded vector space V with $V^0 = \mathfrak{g}, V^{-1} = 0, V^{-2} = S^2\mathfrak{g}, V^3 = \ker(s : \mathfrak{g} \otimes S^2\mathfrak{g} \to S\mathfrak{g}^3)$ etc. Here the map $s : \mathfrak{g} \otimes S^2\mathfrak{g} \to S\mathfrak{g}^3$ is the symmetrization. Let $\pi : D\mathfrak{g} \to C\mathfrak{g}$ be the natural projection. Then, the short exact sequence $0 \to \ker(\pi) \to D\mathfrak{g} \to C\mathfrak{g} \to 0$ gives rise to a short exact sequence of current algebras

$$0 \to \mathcal{CA}(M, \ker(\pi)) \to \mathcal{CA}(M, D\mathfrak{g}) \to C^\infty(M, \mathfrak{g}) \to 0.$$

In particular, $\mathcal{CA}(M, \ker(\pi))$ contains a subspace isomorphic to $\Omega^2(M) \otimes S^2\mathfrak{g}$. Repeating the computation of Sec. 5.2.1, we obtain

$$\begin{aligned}\{f \otimes \mathcal{I}(x), g \otimes \mathcal{I}(y)\} &= [f \otimes \mathcal{I}(x), d(g \otimes \mathcal{I}(y))] \\ &= [f \otimes \mathcal{I}(x), dg \otimes \mathcal{I}(y) + g \otimes l(y)] \\ &= f dg \otimes [\mathcal{I}(x), \mathcal{I}(y)] + fg \otimes [\mathcal{I}(x), l(y)] \\ &= f dg \otimes 2d\mathcal{I}(xy) + fg \otimes \mathcal{I}([x, y]) \\ &= -2df \wedge dg \otimes \mathcal{I}(xy) + fg \otimes \mathcal{I}([x, y]).\end{aligned}$$

Here the element $df \wedge dg \otimes \mathcal{I}(xy) \in (\Omega(M) \otimes D\mathfrak{g})^{-1}/\text{exact}$ is the image of $df \wedge dg \otimes xy \in \Omega^2(M) \otimes S^2\mathfrak{g}$.

5.2.3. $A = D_p\mathfrak{g}$

Recall that for $p \in (S^n\mathfrak{g}^*)^\mathfrak{g}$ we have a short exact sequence of DGLAs $0 \to \mathbb{R}[2n - 2] \to D_p\mathfrak{g} \to D\mathfrak{g} \to 0$ which induces a short exact sequence of current algebras

$$0 \to \mathcal{CA}(M, \mathbb{R}[2n - 2]) \to \mathcal{CA}(M, D_p\mathfrak{g}) \to \mathcal{CA}(M, D\mathfrak{g}) \to 0,$$

where $\mathcal{CA}(M, \mathbb{R}[2n - 2]) = \Omega^{2n-3}(M)/\Omega^{2n-3}_{\text{exact}}(M)$. If M is a compact connected orientable manifold of dimension $2n - 3$, $\mathcal{CA}(M, \mathbb{R}[2n - 2]) \cong \mathbb{R}$ and we obtain a central extension of $\mathcal{CA}(M, D\mathfrak{g})$ by a line.

For $n = 2$, one can choose $M = S^1$. In this case, $\mathcal{CA}(M, D\mathfrak{g}) = \mathcal{CA}(M, C\mathfrak{g})$ for dimensional reasons. Redoing again the calculation of the previous two sections, we obtain

$$\begin{aligned}\{f \otimes I(x), g \otimes I(y)\} &= [f \otimes I(x), d(g \otimes I(y))] \\ &= [f \otimes I(x), dg \otimes I(y) + g \otimes L(y)] \\ &= -2f dg \otimes p(x, y)c + fg \otimes I([x, y]).\end{aligned}$$

The isomorphism $\Omega^1(S^1)/\Omega^1_{\text{exact}}(S^1) \cong \mathbb{R}$ is given by the integral of a 1-form over the circle. Hence, the cocycle term in the Lie bracket reads $-2p(x, y) \int f dg$ which coincides (up to normalization) with the standard Kac–Moody central extension of the loop algebra.

For $n = 3$, we choose M to be a compact orientable 3-manifold. In this case,

$$\mathcal{CA}(M, D\mathfrak{g}) = C^\infty(M, \mathfrak{g}) \oplus (\Omega^2(M) \otimes S^2\mathfrak{g}) \oplus (\Omega^3(M) \otimes \ker(\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g})).$$

Here the map $\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g}$ is the symmetrization. The computation of the Lie bracket elements of $C^\infty(M, \mathfrak{g})$ is exactly the same as in Sec. 5.2.2. However, there is a new feature in the following Lie bracket:

$$\begin{aligned}\{\alpha \otimes \tilde{\mathcal{I}}(xy), f \otimes \tilde{\mathcal{I}}(z)\} &= [\alpha \otimes \tilde{\mathcal{I}}(xy), d(f \otimes \tilde{\mathcal{I}}(z))] \\ &= [\alpha \otimes \tilde{\mathcal{I}}(xy), df \otimes \tilde{\mathcal{I}}(z) + f \otimes l(z)] \\ &= -f\alpha \otimes \tilde{\mathcal{I}}(\mathrm{ad}_z(xy)) + \alpha \wedge df \otimes [\tilde{\mathcal{I}}(xy), \tilde{\mathcal{I}}(z)].\end{aligned}$$

The last Lie bracket is of the form

$$\begin{aligned}[\tilde{\mathcal{I}}(xy), \tilde{\mathcal{I}}(z)] &= \frac{1}{3}(2[\tilde{\mathcal{I}}(xy), \tilde{\mathcal{I}}(z)] - [\tilde{\mathcal{I}}(xz), \tilde{\mathcal{I}}(y)] - [\tilde{\mathcal{I}}(yz), \tilde{\mathcal{I}}(x)]) \\ &\quad + \frac{1}{3}([\tilde{\mathcal{I}}(xy), \tilde{\mathcal{I}}(z)] + [\tilde{\mathcal{I}}(xz), \tilde{\mathcal{I}}(y)] + [\tilde{\mathcal{I}}(yz), \tilde{\mathcal{I}}(x)]),\end{aligned}$$

where the first term is an element of $\ker(\mathfrak{g} \otimes S^2\mathfrak{g} \to S^3\mathfrak{g})$, and the second term can be represented as

$$\frac{1}{3}([\tilde{\mathcal{I}}(xy), \tilde{\mathcal{I}}(z)] + [\tilde{\mathcal{I}}(xz), \tilde{\mathcal{I}}(y)] + [\tilde{\mathcal{I}}(yz), \tilde{\mathcal{I}}(x)]) = \tilde{\mathcal{I}}(xyz) - p(xyz)c.$$

Again, the isomorphism $\Omega^3(M)/\Omega^{3,\mathrm{exact}}(M) \cong \mathbb{R}$ is given by the integral over M, and the new cocycle term reads $-p(xyz) \int_M \alpha \wedge df$.

5.2.4. *FMS current algebra*

Recall Sec. 4.4.2: let $p \in (S^n\mathfrak{g}^*)^\mathfrak{g}$, $e \in (S\mathfrak{g}^* \otimes W\mathfrak{g})^g$ such that $d_\mathfrak{g} e = p \otimes 1 - 1 \otimes p$, and let $A_{\mathrm{FMS}} = (C\mathfrak{g} \ltimes W\mathfrak{g}[2n-2])_{(e(0))}$ be the semi-direct product of $C\mathfrak{g}$ and $W\mathfrak{g}[2n-2]$ with differential $d' = d - [e(0), \cdot]$. In fact, the only part of the differential which is changed is $d'I(x) = L(x) - I_{W\mathfrak{g}}(x)e(0)$.

The differential on $W\mathfrak{g}[2n-2]$ is induced by the Weil differential. Hence, the embedding $\mathbb{R}[2n-2] \to W\mathfrak{g}[2n-2]$ is a chain map (and a homomorphism of abelian DGLAs). As a consequence, we obtain a short exact sequence of DGLAs $0 \to \mathbb{R}[2n-2] \to A_{\mathrm{FMS}} \to A' \to 0$, where $A' = (C\mathfrak{g} \ltimes W^+\mathfrak{g}[2n-2])_{(e(0))}$ is an acyclic DGLA. Then, we obtain a short exact sequence of current algebras

$$0 \to \Omega^{2n-3}(M)/\Omega^{2n-3}_{\mathrm{exact}}(M) \to \mathcal{CA}(M, A_{\mathrm{FMS}}) \to \mathcal{CA}(M, A') \to 0.$$

If M is a compact connected orientable manifold of dimension $2n-3$, we have $\Omega^{2n-3}(M)/\Omega^{2n-3}_{\mathrm{exact}}(M) \cong \mathbb{R}$ with an isomorphism defined by integration.

One can view the abelian current algebra $\mathcal{CA}(M, W^+\mathfrak{g}[2n-2])$ as a space of local functional of $\mathfrak{g}$-connections on M. Recall that a $\mathfrak{g}$-connection on M defines a homomorphism of graded commutative algebras $W\mathfrak{g} \to \Omega(M)$. Taking into account that $(W^+\mathfrak{g}[2n-2] \otimes \Omega(M))^{-1} \cong (W^+\mathfrak{g} \otimes \Omega(M))^{2n-3}$, we obtain (for each $\mathfrak{g}$-connection) a map

$$\mathcal{CA}(M, W^+\mathfrak{g}[2n-2]) \to \Omega(M)^{2n-3}/\Omega(M)^{2n-3}_{\text{exact}} \cong \mathbb{R}.$$

The Lie algebra $\mathcal{CA}(M, A')$ is therefore an abelian extension of $\mathcal{CA}(M, C\mathfrak{g}) = C^\infty(M, \mathfrak{g})$ by functionals on the space of $\mathfrak{g}$-connections.

Computing the Lie bracket of the elements $f \otimes x, g \otimes y \in C^\infty(M, \mathfrak{g})$ yields

$$\begin{aligned}\{f \otimes I(x), g \otimes I(y)\} &= [f \otimes I(x), d(g \otimes I(y))]\\ &= [f \otimes I(x), dg \otimes I(y) + g \otimes L(y) - I_{W\mathfrak{g}}(y)e(0)]\\ &= fg \otimes (I([x,y]) - I_{W\mathfrak{g}}(x)I_{W\mathfrak{g}}(y)e(0)).\end{aligned}$$

The cocycle term reads $-fg \otimes I_{W\mathfrak{g}}(x)I_{W\mathfrak{g}}(y)e(0)$. For $n = 2$, it reads $-fg \otimes p([x,y], \cdot)$. For higher n, this formula defines the FMS cocycle [6, 7, 13] on the Lie algebra of maps from M to $\mathfrak{g}$ with values in local functionals of $\mathfrak{g}$-connections.

5.2.5. *Truncated FMS current algebra*

The construction of Sec. 4.4.2 defines a DGLA homomorphism

$$\rho : D_p\mathfrak{g} \to (C\mathfrak{g} \ltimes W\mathfrak{g}[2n-2])_{(e(0))} = A. \tag{15}$$

Hence, we obtain an induced homomorphism of current algebras $\mathcal{CA}(M, D_p\mathfrak{g}) \to \mathcal{CA}(M, A_{\text{FMS}})$. Note that the map ρ restricts to the identity mapping the central line $\mathbb{R}[2n-2] \subset D_p\mathfrak{g}$ to the line $\mathbb{R}[2n-2] \subset W\mathfrak{g}[2n-2]$. As a consequence, we obtain an induced homomorphism $\rho' : D\mathfrak{g} \to A' = A_{\text{FMS}}/\mathbb{R}[2n-2]$ and a homomorphism of the corresponding current algebras $\mathcal{CA}(M, D\mathfrak{g}) \to \mathcal{CA}(M, A')$.

The image of the map ρ is a DGLA $B_{\text{FMS}} \subset A_{\text{FMS}}$. We refer to $\mathcal{CA}(M, B_{\text{FMS}})$ as to truncated FMS current algebra. We work out in detail the example of $n = 3$. In this case, we have an exact sequence of DGLAs $0 \to \mathbb{R}[4] \to B_{\text{FMS}} \to B' \to 0$, where $B' = B_{\text{FMS}}/\mathbb{R}[4]$ is acyclic. Hence, we obtain an exact sequence of current algebras

$$0 \to \Omega^3(M)/\Omega^3_{\text{exact}}(M) \to \mathcal{CA}(M, B_{\text{FMS}}) \to \mathcal{CA}(M, B') \to 0,$$

where $\mathcal{CA}(M, B') = C^\infty(M, \mathfrak{g}) \oplus (\Omega^2(M) \otimes \mathfrak{g}^*)$. For the Lie bracket between elements $f \otimes x, g \otimes y \in C^\infty(M, \mathfrak{g})$, we have

$$\begin{aligned}\{f \otimes \tilde{I}(x), g \otimes \tilde{I}(y)\} &= [f \otimes \tilde{I}(x), d(g \otimes \tilde{I}(y))]\\ &= [f \otimes \tilde{I}(x), dg \otimes \tilde{I}(y) + g \otimes L(y)]\\ &= fdg \otimes [\tilde{I}(x), \tilde{I}(y)] + fg \otimes [\tilde{I}(x), L(y)]\end{aligned}$$

$$= 2f dg \otimes \mu(p(x, y, \cdot) + fg \otimes \tilde{I}([x, y])$$
$$= 2df \wedge dg \otimes \theta(p(x, y, \cdot)) + fg \otimes \tilde{I}([x, y]).$$

Here the term $2df \wedge dg \otimes \theta(p(x, y, \cdot))$ is another representative of the FMS cocycle.

In a somewhat different language, the Lie algebra $\mathcal{CA}(M, B_{\mathrm{FMS}})$ was introduced in [4].

6. Groups Integrating Current Algebras $\mathcal{SA}(M, A)$

In this section, we construct sheaves of groups integrating sheaves of Lie algebras $\mathcal{SA}(M, A)$, and in particular we apply this technique to $\mathcal{SA}(M, D\mathfrak{g})$ and $\mathcal{SA}(M, D_p\mathfrak{g})$.

6.1. *The group $\mathcal{SG}(M, A, G)$ integrating the current algebra $\mathcal{SA}(M, A)$*

To simplify the task, we first consider a simpler Lie algebra $\overline{\mathcal{SA}}(M, A) = (\Omega(M) \otimes A)^0$. Denote $A^0_{\mathrm{closed}} = \mathfrak{g}$, let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $A' = \bigoplus_{n<0} A^n$ be the sum of graded components of A of negative degrees. Suppose that the adjoint action of $\mathfrak{g}$ on A' lifts to an action of G. Denote by

$$G(M) = C^\infty(M, G)$$

the group of smooth maps from M to G (with pointwise multiplication). Let $\mathcal{U}(A')$ be the degree completion of the universal enveloping algebra of A', and let

$$\mathcal{H}(M) = \Omega(M) \otimes \mathcal{U}(A')$$

be the Hopf algebra with coproduct induced by the one of $\mathcal{U}(A')$. Group-like elements of degree 0 in $\mathcal{H}(M)$ form a group

$$H(M) = \exp((\Omega(M) \otimes A')^0) \subset \mathcal{H}(M).$$

The group $G(M)$ acts on $\mathcal{H}(M)$ by Hopf algebra automorphisms. This action induces an action of $G(M)$ on $H(M)$ by group automorphisms. We define

$$\overline{\mathcal{SG}}(M, A, G) = G(M) \ltimes H(M)$$

as the semi-direct product of $G(M)$ and $H(M)$. Note that the group $\overline{\mathcal{SG}}(M, A, G)$ depends of the choice of the connected Lie group G integrating the Lie algebra $\mathfrak{g} = A^0_{\mathrm{closed}}$.

It is easy to see that $\overline{\mathcal{SG}}(M, A, G)$ is an integration of $\overline{\mathcal{SA}}(M, A)$. Indeed, let (h_t, g_t) be a one-parameter subgroup of $\overline{\mathcal{SG}}(M, A)$, where $h_t \in H(M), g_t \in G(M), t \in \mathbb{R}$. Then, its derivative at $t = 0$ is a pair (u, x), where $x \in C^\infty(M, \mathfrak{g})$ and $u \in (\Omega(M) \otimes A')^0$. The pair (u, x) defines an element of $(\Omega(M) \otimes A)^0$. In the other direction, every such an element can be exponentiated to a one-parameter subgroup of $\overline{\mathcal{SG}}(M, A, G)$.

Remark. Let $A = C\mathfrak{g}$. Note that $C\mathfrak{g}^0_{\text{closed}} = C\mathfrak{g}^0 = \mathfrak{g}$. Let G be a connected Lie group integrating $\mathfrak{g}$. We have $A' = \mathfrak{g}\varepsilon \subset C\mathfrak{g}$, and $H(M) = \{\exp(u);\ u \in \Omega^1(M) \otimes \mathfrak{g}\varepsilon\}$. Elements of $\overline{\mathcal{SG}}(M, C\mathfrak{g}, G)$ are of the form hg, where $h \in H(M)$ and $g \in G(M)$.

The Lie subalgebra $\mathcal{SA}(M, A) \subset \overline{\mathcal{SA}}(M, A)$ is singled out by the closedness condition. That is, a pair $(u, x) \in \overline{\mathcal{SA}}(M, A)$ belongs to $\mathcal{SA}(M, A)$ if $du + dx = 0$. The analogue of this condition at the group level is as follows:

$$\mathcal{SG}(M, A, G) = \{(h, g) \in \overline{\mathcal{SG}}(M, A);\ h^{-1}dh + dg\, g^{-1} = 0 \in \Omega(M) \otimes A\}. \tag{16}$$

Note that

$$h^{-1}dh + dg\, g^{-1} = h^{-1}d(hg)\, g^{-1},$$

and Eq. (16) expresses the fact that $hg \in \mathcal{SG}(M, A, G)$ is closed. For the product $hg = (h_1g_1)(h_2g_2)$, we have

$$h^{-1}d(hg)g^{-1} = (g_1h_2g_1^{-1})\left(h_1^{-1}d(h_1g_1)g_1^{-1}\right)(g_1h_2g_1^{-1})^{-1} + g_1\left(h_2^{-1}d(h_2g_2)g_2^{-1}\right)g_1^{-1}.$$

Hence, $\mathcal{SG}(M, A, G)$ is indeed a subgroup of $\overline{\mathcal{SG}}(M, A, G)$.

Again, it is easy to see that $\mathcal{SG}(M, A, G)$ is an integration of $\mathcal{SA}(M, A)$. Indeed, let $(h_t, g_t) \in \mathcal{SG}(M, A, G)$ be a one-parameter subgroup, and let $(u, x) \in \overline{\mathcal{SA}}(M, A)$ be its derivative at $t = 0$. Then, Eq. (16) implies $du + dx = 0$. In the other direction, for $\exp(t(u + x)) = h_tg_t$ we have

$$h_t^{-1}d(h_tg_t)g_t^{-1} = \mathrm{Ad}_{g_t^{-1}} \frac{1 - \exp(-t\,\mathrm{ad}_{u+x})}{t\,\mathrm{ad}_{u+x}}\, d(u + x) = 0$$

if $du + dx = 0$. Here we have used the standard expression for the derivative of the exponential map.

Note that for every manifold M the functor $\mathcal{SA}(\cdot, A)$ produces a sheaf of Lie algebras, where the Lie algebra of sections over $U \subset M$ is defined as $\mathcal{SA}(U, A)$. Similarly, $\mathcal{SG}(\cdot, A, G)$ defines a sheaf of groups.

Remark. For $A = C\mathfrak{g}$, we consider Eq. (16), where $h = \exp(u)$ and $u \in \Omega^1(M) \otimes \mathfrak{g}\varepsilon$. Note that

$$h^{-1}dh = e^{-u}de^u = \frac{1 - \exp(-\mathrm{ad}_u)}{\mathrm{ad}_u}\, du = du - \frac{1}{2}[u, du].$$

Here we have used that $\mathrm{ad}_u^k = 0$ for $k \geq 2$. Let e_a be a basis of $\mathfrak{g}$. Then, $u = u^a \otimes e_a\varepsilon$, $du = du^a \otimes e_a\varepsilon - u^a \otimes e_a$, and

$$du - \frac{1}{2}[u, du] = -u^a \otimes e_a + \left(du^a \otimes e_a - \frac{1}{2}[u^a \otimes e_a, u^b \otimes e_b]\right)\varepsilon.$$

Denote $\hat{u} = u^a \otimes e_a \in \Omega^1(M) \otimes \mathfrak{g}$, and compute

$$h^{-1}dh + dgg^{-1} = (dgg^{-1} - \hat{u}) + \left(d\hat{u} - \frac{1}{2}[\hat{u}, \hat{u}]\right)\varepsilon.$$

Hence, a pair $(\exp(u), g)$ defines an element of $\mathcal{SG}(M, C\mathfrak{g}, G)$ if and only if $\hat{u} = dgg^{-1}$ and $\hat{u}$ is a Maurer–Cartan element. The second condition follows from the first one since $d(dgg^{-1}) = (dgg^{-1})^2 = [dgg^{-1}, dgg^{-1}]/2$.

In conclusion, h is uniquely determined by g, and the forgetful map $(h, g) \mapsto g$ defines a group isomorphism $\mathcal{SA}(M, C\mathfrak{g}, G) \cong G(M)$. The inverse map $G(M) \to \mathcal{SA}(M, C\mathfrak{g}, G)$ is given by $g \mapsto (\exp(I(dgg^{-1})), g)$.

Remark. Let (A, G) be a pair, where A is a DGLA and G is a connected Lie group integrating $\mathfrak{g} = A^0_{\text{closed}}$ such that the adjoint action of $\mathfrak{g}$ on A lifts to an action of G on A. We define a morphism of such pairs $(A, G) \to (B, H)$ as pairs of a DGLA homomorphism $A \to B$ and a group homomorphism $G \to H$ integrating the Lie algebra homomorphism $A^0_{\text{closed}} \to B^0_{\text{closed}}$. Then, a morphism of pairs $(A, G) \to (B, H)$ induces a group homomorphism $\mathcal{SG}(M, A, G) \to \mathcal{SG}(M, B, H)$. In particular, if $A^0_{\text{closed}} = B^0_{\text{closed}}$ and $G = H$, we obtain a canonical group homomorphism.

6.2. *The group $C_pG(M) = \mathcal{SG}(M, C_p\mathfrak{g}, G)$*

Let $p \in (S^2\mathfrak{g}^*)^{\mathfrak{g}}$. The group $C_pG(M) := \mathcal{SG}(M, C_p\mathfrak{g}, G)$ is contained in the preimage of $\mathcal{SG}(M, C\mathfrak{g}, G)$ under the projection map $\overline{\mathcal{SG}}(M, C_p\mathfrak{g}, G) \to \overline{\mathcal{SG}}(M, C\mathfrak{g}, G)$. Therefore, it consist of elements of the form

$$(h = \exp(\omega \otimes c + \tilde{I}(dgg^{-1})),\ g),$$

where $g : M \to G$, $\omega \in \Omega^2(M)$, and $h^{-1}dh + dgg^{-1} = 0$. A straightforward calculation (see, e.g. [1, Proposition 5.7]) shows that

$$h^{-1}dh + dg\, g^{-1} = (d\omega + g^*\eta_p) \otimes c,$$

where $\eta_p \in \Omega^3(G)$ is the Cartan 3-form (that is, a bi-invariant differential form on G defined by the map $(x, y, z) \mapsto p(x, [y, z])$ at the group unit). The group $C_pG(M)$ can therefore be identified with the set of pairs

$$(g : M \to G,\ \omega \in \Omega^2(M)) \quad \text{such that } d\omega + g^*\eta_p = 0.$$

Since $\exp(\tilde{I}(u))\exp(\tilde{I}(v)) = \exp(\tilde{I}(u + v) + p(u, v)c/2)$, the group law is expressed in terms of these pairs,

$$(g_1, \omega_1)(g_2, \omega_2) = \left(g_1g_2, \omega_1 + \omega_2 + \frac{1}{2}(g_1 \times g_2)^*\rho_p\right),$$

where $\rho_p \in \Omega^2(G \times G)$ is defined by $\rho_p = p(\pi_1^*\theta^L, \pi_2^*\theta_R)$ with θ^L and θ^R left-invariant and right-invariant Maurer–Cartan forms of G and $\pi_{1,2} : G \times G \to G$ projections on the first and second factor, respectively.

6.3. *The group* $DG(M) = \mathcal{SG}(M, D\mathfrak{g}, G)$

In this section, we consider the group $\mathcal{SG}(M, D\mathfrak{g}, G)$. We will use a shorthand notation $DG(M) := \mathcal{SG}(M, D\mathfrak{g}, G)$.

Observe that the exact sequence of DGLAs

$$0 \to \ker(D\mathfrak{g} \to C\mathfrak{g}) \to D\mathfrak{g} \to C\mathfrak{g} \to 0$$

gives rise (since $C\mathfrak{g}$ is acyclic) to an exact sequence of current algebras

$$0 \to \mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g})) \to \mathcal{SA}(M, D\mathfrak{g}) \to \mathcal{SA}(M, C\mathfrak{g}) \to 0,$$

which in turn lifts to an exact sequence of groups

$$1 \to \mathcal{SG}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}), 1) \to DG(M) \to G(M).$$

Here 1 stands for the trivial group ($\ker(D\mathfrak{g} \to C\mathfrak{g})^0 = 0$), we have used that $D\mathfrak{g}^0 \cong C\mathfrak{g}^0 = \mathfrak{g}$, and in both cases we have chosen the same connected Lie group G integrating $\mathfrak{g}$.

6.3.1. *The group* $DG(M)$ *and* $\mathfrak{g}$*-connections*

We begin by observing an interesting relation between the group $DG(M)$ and the space $\mathcal{G}(M) = \Omega^1(M) \otimes \mathfrak{g}$ of $\mathfrak{g}$-connections on M. Let $\widehat{\mathcal{G}}(M)$ denote the action groupoid of $G(M)$ on $\mathcal{G}(M)$, where the action is by gauge transformations,

$$g : A \mapsto A^g = \mathrm{Ad}_{g^{-1}} A + g^{-1} dg.$$

Again, constructions of $\mathcal{G}(M)$ and of the gauge action are local, and we obtain a sheaf of groupoids over M.

There is a natural morphism of sheaves of groupoids $\widehat{\mathcal{G}}(M) \to G(M)$ by forgetting a connection (here we view the group $G(M)$ as a groupoid with an object set consisting of one point). Recall that a $\mathfrak{g}$-connection $A \in \mathcal{G}(M)$ gives rise to a homomorphism of $\mathfrak{g}$-differential algebras $W\mathfrak{g} \to \Omega(M)$, defined by $\theta \mapsto A$, $t \mapsto F_A = dA + [A, A]/2$. Under this map, the image of an element $\alpha(\theta, t)$ is $\alpha(A, F_A)$. Let $g \in G(M)$, $A \in \mathcal{G}(M)$, and $m_{g,A} : A \to A^g$ be the corresponding morphism in $\widehat{\mathcal{G}}(M)$. We define a map $\mu : \widehat{\mathcal{G}}(M) \to \overline{\mathcal{SG}}(M, D\mathfrak{g}, G)$ given by the following formula,

$$\mu(m_{g,A}) = \Phi(A, F_A)\, g\, \Phi(A^g, F_{A^g})^{-1},$$

where Φ is defined in Theorem 3.

Theorem 6. *The map* $\mu : \widehat{\mathcal{G}}(M) \to \overline{\mathcal{SG}}(M, D\mathfrak{g}, G)$ *is a morphism of groupoids. It takes values in* $DG(M) = \mathcal{SG}(M, D\mathfrak{g}, G)$*, and its composition with the natural projection* $DG(M) \to G(M)$ *coincides with the forgetful map* $m_{g,A} \mapsto g$*.*

Proof. To simplify notation, we denote $\Phi(A) = \Phi(A, F_A)$. For the composition of morphisms, we have

$$\begin{aligned}\mu(m_{g,A})\mu(m_{h,A^g}) &= (\Phi(A)g\Phi(A^g)^{-1})(\Phi(A^g)h\Phi((A^g)^h)) \\ &= \Phi(A)(gh)\Phi(A^{gh}) \\ &= \mu(m_{gh,A}) \\ &= \mu(m_{g,A} \circ m_{h,A^g}).\end{aligned}$$

Hence, μ is a morphism of groupoids.

Next, we verify that $\mu(m_{g,A})$ is indeed an element of $DG(M)$. We compute,

$$\begin{aligned}d(\mu(m_{g,A})) &= \Phi(A)((-\iota(F_A) + l(A))g + dg - g(-\iota(F_{A^g}) + l(A^g)))\Phi(A^g)^{-1} \\ &= \Phi(A)g(\iota(F_{A^g}) - \iota(\mathrm{Ad}_{g^{-1}}F_A) + l(\mathrm{Ad}_{g^{-1}}A + g^{-1}dg - A^g))\Phi(A^g)^{-1} \\ &= 0,\end{aligned}$$

where we have used Eq. (6).

Finally, by compositing μ with the projection map $\pi : DG(M) \to G(M)$ we obtain

$$\begin{aligned}(\pi \circ \mu)(m_{g,A}) &= \pi(\Phi(A)g\Phi(A^g)^{-1}) \\ &= \pi(\Phi(A)\mathrm{Ad}_g(\Phi(A^g)^{-1})g) \\ &= g,\end{aligned}$$

as required. □

6.3.2. *Structure of $DG(M)$*

Using the results of the previous section, we can now prove the following proposition.

Proposition 13. *There is an exact sequence of groups*

$$1 \to \mathcal{SG}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}), 1) \to DG(M) \to G(M) \to 1,$$

and the map $g \mapsto \mu(m_{g,0})$ is a section of the natural projection $DG(M) \to G(M)$.

Proof. By Theorem 6, the map $G(M) \to DG(M) \to G(M)$ defined by composing the map $g \mapsto \mu(m_{g,0})$ and the natural projection $DG(M) \to G(M)$ is the identity map. Hence, the projection $DG(M) \to G(M)$ is surjective, and the sequence of groups in the proposition is exact. □

Note that the DGLA $\ker(D\mathfrak{g} \to C\mathfrak{g})$ is acyclic and negatively graded. As a complex, it can be represented as a cone $CU = U[\varepsilon]$ for some negatively graded vector space U. The corresponding current algebra $\mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}))$ is isomorphic (as a vector space) to

$$\mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g})) \cong \bigoplus_{i>0} \Omega^i(M) \otimes U^{-i}.$$

Using the grading induced by the degree of differential forms, we infer that this Lie algebra is nilpotent. Hence, by composing with the exponential map we obtain a bijection

$$\nu\colon \bigoplus_{i>0} \Omega^i(M) \otimes U^{-i} \to \mathcal{SG}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}), 1).$$

Proposition 14. *The map* $(g, u) \mapsto \mu(m_{g,0})\,\nu(u)$ *defines a bijection*

$$G(M) \times \left(\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i} \right) \to DG(M). \tag{17}$$

Proof. This follows from the facts that the map $g \mapsto \mu(m_{g,0})$ defines a section of the projection $DG(M) \to G(M)$, and that the map ν is bijective. □

6.4. *The group $D_pG(M) = \mathcal{SG}(M, D_p\mathfrak{g}, G)$*

In this section, we study the group $\mathcal{SG}(M, D_p\mathfrak{g}, G)$, the shorthand notation is $D_pG(M)$.

As before, the short exact sequence of DGLAs

$$0 \to \ker(D_p\mathfrak{g} \to C\mathfrak{g}) \to D_p\mathfrak{g} \to C\mathfrak{g} \to 0$$

gives rise to a short exact sequence of current algebras

$$0 \to \mathcal{SA}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g})) \to \mathcal{SA}(M, D_p\mathfrak{g}) \to \mathcal{SA}(M, C\mathfrak{g}) \to 0,$$

which lifts to an exact sequence of groups

$$1 \to \mathcal{SG}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}), 1) \to D_pG(M) \to G(M).$$

In this case, the natural projection $D_pG(M) \to G(M)$ may no longer be surjective. Again, an important tool in studying this question is the gauge groupoid.

6.4.1. *Central extensions of the groupoid $\widehat{\mathcal{G}}(M)$*

Let $p \in (S^n\mathfrak{g}^*)^\mathfrak{g}$ be an invariant polynomial of degree $n \geq 2$, and let $e_p \in (W\mathfrak{g})^\mathfrak{g}$ be such that $d\,e_p = p$. The ambiguity in the choice of e_p is by a closed $\mathfrak{g}$-invariant element of $W\mathfrak{g}$ of degree $2n-2$. Since $W\mathfrak{g}$ is acyclic, for any other primitive e'_p we have $e'_p - e_p = df$.

Let us describe a central extension $\widehat{\mathcal{G}}_p(M)$ of the groupoid $\widehat{\mathcal{G}}(M)$ by an abelian group $\Omega^{2n-2}_{\mathrm{closed}}(M)$. The set of objects is again $\mathcal{G}(M)$, and the set of morphisms is labeled by triples (g, A, α), where $g \in G(M)$, $A \in \mathcal{G}(M)$ and $\alpha \in \Omega^{2n-2}(M)$ such that $d\alpha = e_p(A, F) - e_p(A^g, F_{A^g})$. The composition of morphisms is given by

$$m_{g,A,\alpha} m_{h,A^g,\beta} = m_{gh,A,\alpha+\beta},$$

where

$$\begin{aligned} d(\alpha+\beta) &= e_p(A, F_A) - e_p(A^g, F_{A^g}) + e_p(A^g, F_{A^g}) - e_p(A^{gh}, F_{A^{gh}}) \\ &= e_p(A, F_A) - e_p(A^{gh}, F_{A^{gh}}). \end{aligned}$$

Note that for the groupoid of global sections of $\widehat{\mathcal{G}}_p(M)$, the natural projection to $G(M)$ may no longer be surjective. Indeed, the cohomology class of $e_p(A, F_A) - e_p(A^g, F_{A^g})$ in $H^{2n-1}(M)$ coincides with $[g^*\eta_p] \in H^{2n-1}(M)$, where $\eta_p = e_p(\theta, 0) \in (\wedge\mathfrak{g}^*)^{\mathfrak{g}} \subset \Omega(G)$. If $g^*\eta_p \neq 0$, $\widehat{\mathcal{G}}_p(M)$ does not contain elements which project to g.

A different choice of e_p gives rise to an isomorphic sheaf of groupoids with an isomorphism given by $\alpha \mapsto \alpha' = \alpha + f(A, F) - f(A^g, F_{A^g})$. In the physics literature, α is called the Wess–Zumino action.

Recall that the DGLA $D_p\mathfrak{g}$ is a central extension of $D\mathfrak{g}$ by the line $\mathbb{R}[2n-2]$. Note that $\mathcal{SA}(\mathbb{R}[2n-2], M) \cong \Omega(M)^{2n-2}_{\text{closed}}$. Since $D\mathfrak{g}$ is acyclic, we obtain an exact sequence of sheaves of Lie algebras

$$0 \to \Omega(M)^{2n-2}_{\text{closed}} \to \mathcal{SA}(M, D_p\mathfrak{g}) \to \mathcal{SA}(M, D\mathfrak{g}) \to 0.$$

This exact sequence integrates to an exact sequence of sheaves of groups,

$$1 \to \Omega(M)^{2n-2}_{\text{closed}} \to D_pG(M) \to DG(M).$$

Consider a map $\mu_p : \widehat{\mathcal{G}}_p(M) \to \overline{\mathcal{SG}}(M, D_p\mathfrak{g}, G)$ defined by formula

$$\mu_p(m_{g,A,\alpha}) = \Phi_p(A, F_A) g\, \Phi_p(A^g, F_{A^g})^{-1} e^{\alpha\otimes c}, \tag{18}$$

where c is the generator of the central line of $D_p\mathfrak{g}$, and Φ_p is defined in Sec. 5.

Proposition 15. *The map μ_p is a morphism of groupoids, it takes values in $D_pG(M) \subset \overline{\mathcal{SG}}(M, D_p\mathfrak{g}, G)$, and it restricts to identity on $\Omega^{2n-2}_{\text{closed}}$.*

Proof. The proof is similar to Theorem 6. We have

$$d\mu_p(m_{g,A,\alpha}) = \mu_p(m_{g,A,\alpha})(e(A^g, F_{A^g}) - e(A, F_A) + d\alpha) \otimes c = 0.$$

Hence, μ_p takes values in $D_pG(M)$. For the morphism of groupoids, one follows the proof of Proposition 6 and uses the fact that $m_{g,A,\alpha} \circ m_{f,A^g,\beta} = m_{g,A,\alpha+\beta}$. Finally, for $g = 1$ we obtain $\mu_p(m_{1,A,\alpha}) = \exp(\alpha \otimes c)$ which coincides with the image of α under injection $\Omega^{2n-2}_{\text{closed}} \to D_pG(M)$. ∎

6.4.2. *Structure of the group $D_pG(M)$*

The DGLA $\ker(D_p\mathfrak{g} \to C\mathfrak{g})$ fits into a short exact sequence

$$0 \to \mathbb{R}[2n-2] \to \ker(D_p\mathfrak{g} \to C\mathfrak{g}) \to \ker(D\mathfrak{g} \to C\mathfrak{g}) \to 0$$

giving rise to a short exact sequence of current algebras

$$0 \to \Omega^{2n-2}_{\text{closed}}(M) \to \mathcal{SA}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g})) \to \mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g})) \to 0.$$

All of these Lie algebras being nilpotent, the exact sequence lifts to an exact sequence of groups

$$1 \to \Omega^{2n-2}_{\text{closed}} \to \mathcal{SG}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}), 1) \to \mathcal{SG}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}), 1) \to 1. \quad (19)$$

Furthermore, by choosing a section of the projection $\mathcal{SA}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g})) \to \mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}))$, and by composing with the exponential map we obtain a section

$$\nu_p : \left(\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i}\right) \cong \mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g})) \to \mathcal{SG}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}), 1).$$

Recall that $\eta_p = e_p(\theta, 0) \in \Omega^{2n-1}(G)$.

Proposition 16. *The image of the natural projection $D_pG(M) \to DG(M)$ is the set of elements of $DG(M)$ which project to maps $g: M \to G$ with vanishing $[g^*\eta_p] \in H^{2n-1}(M)$.*

Proof. Let f be an element of $DG(M)$, and g be the projection of f to $G(M)$. Then, $f_0 = \mu(m_{g,0})^{-1} f$ projects to the group unit of $G(M)$. That is, $f_0 \in \mathcal{SG}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}), 1)$. The exact sequence (19) implies that f_0 admits a lift to $\mathcal{SG}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}), 1) \subset D_pG(M)$. Hence, f admits a lift to D_pG if and only if so does $\mu(m_{g,0})$.

Recall that, as a graded Lie algebra, $D_p\mathfrak{g}$ is a direct sum $D_p\mathfrak{g} = D\mathfrak{g} \oplus \mathbb{R}c$ of $D\mathfrak{g}$ and the central line $\mathbb{R}c$ with c the generator of degree $2-2n$. Hence, we have $\overline{\mathcal{SG}}(M, D_p\mathfrak{g}, G) = \overline{\mathcal{SG}}(M, D\mathfrak{g}, G) \times \exp(\Omega^{2n-2}(M) \otimes c)$. Let us consider the subgroup

$$Q = DG(M) \times \exp(\Omega^{2n-2}(M) \otimes c) \subset \overline{\mathcal{SG}}(M, D_p\mathfrak{g}, G)$$

containing $D_pG(M)$. Lifts of $\mu(m_{g,0})$ to Q are of the form $\hat{g} = \mu(m_{g,0}) \exp(\alpha \otimes c)$, where $\alpha \in \Omega^{2n-2}(M)$. Since $\mu(m_{g,0})^{-1} d_p \mu(m_{g,0}) = g^*\eta_p \otimes c$, we have

$$\hat{g}^{-1} d_p \hat{g} = (g^*\eta_p + d\alpha) \otimes c.$$

If $[g^*\eta_p] \neq 0$, then $\hat{g}^{-1} d_p \hat{g} \neq 0$, and $\mu(m_{g,0})$ does not lift to $D_pG(M)$. If $[g^*\eta_p] = 0$, we can achieve $\hat{g} \in D_pG(M)$ for a suitable choice of α. ∎

Let us denote by ${}_pG(M)$ the set of pairs $(g : M \to G, \alpha \in \Omega^{2n-2}(M))$ such that $g^*\eta_p + d\alpha = 0$. In other words, ${}_pG(M)$ is the set of morphisms in the groupoid $\hat{\mathcal{G}}_p(M)$ starting at the object $A = 0$.

Proposition 17. *The map* $(g, \alpha, u) \to \mu_p(m_{g,0,\alpha})\nu_p(u)$ *defines a bijection*

$$ {}_pG(M) \times \left(\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i} \right) \to D_pG(M). \tag{20}$$

The maps (17) *and* (20) *and the natural projections* ${}_pG(M) \to G(M)$ *and* $D_pG(M) \to DG(M)$ *form a commutative diagram.*

Proof. For the first statement, let $f \in D_pG(M)$, and denote its image in $G(M)$ by g. Then, there is a form $\alpha \in \Omega^{2n-2}(M)$ such that $g^*\eta_p + d\alpha = 0$, and $\hat{g} := \mu_p(m_{g,0,\alpha}) \in D_pG(M)$ with the same projection $g \in G(M)$. Hence, $\hat{g}^{-1}f \in \mathcal{SG}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}))$, and it is of the form $\exp(u')$ for some $u' \in \mathcal{SA}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}))$. The element u' projects to $u \in \left(\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i}\right) \cong \mathcal{SA}(M, \ker(D\mathfrak{g} \to C\mathfrak{g}))$. Then, $\hat{g}^{-1}f\nu_p(u)^{-1}$ projects to the group unit in $\mathcal{SG}(M, \ker(D_p\mathfrak{g} \to C\mathfrak{g}), 1)$. Hence, it is of the form $\exp(\beta \otimes c)$ for some $\beta \in \Omega^{2n-2}_{\text{closed}}(M)$. For the element f we obtain $f = \mu_p(m_{g,0,\alpha}) \exp(\beta \otimes c)\nu_p(u) = \mu_p(m_{g,0,\alpha+\beta})\nu_p(u)$, as required.

For the second statement, the natural projections ${}_pG(M) \to G(M)$ and $D_pG(M) \to DG(M)$ are forgetful maps with respect to differential forms $\alpha \in \Omega^{2n-2}(M)$. This implies commutativity of the diagram announced in the proposition. □

6.5. *Torsors and obstructions*

A torsor over the sheaf of groupoids $\widehat{\mathcal{G}}(M)$ is a principal G-bundle over M with a choice of a principal $\mathfrak{g}$-connection. More explicitly, if U_i is an open cover of M, a (descent data for a) $\widehat{\mathcal{G}}(M)$-torsor on M is defined by a choice of local $\mathfrak{g}$-connections $A_i \in \Omega^1(U_i) \otimes \mathfrak{g}$ and of the gluing maps $g_{ij} : U_{ij} \to G$ such that g_{ij} is a cocycle, and $A_j = A_i^{g_{ij}}$.

Theorem 7. *Let* $\mathcal{T}$ *be a* $\widehat{\mathcal{G}}(M)$*-torsor over* M *with underlying principal* G*-bundle* P*. It lifts to a* $\widehat{\mathcal{G}}_p(M)$*-torsor if and only if the Chern–Weil class* $\mathrm{cw}(p) = [p(F)]$ *of* P *vanishes.*

Proof. A lift of a $\widehat{\mathcal{G}}(M)$-torsor to a $\widehat{\mathcal{G}}_p(M)$-torsor amounts to a choice of a Cech cocycle $\alpha_{ij} \in \Omega^{2n-2}(U_{ij})$ such that $d\alpha_{ij} = e_p(A_i, F_{A_i}) - e_p(A_j, F_{A_j})$. Assume that such a cocycle exists. Since the Cech cohomology $H^1(\Omega^{2n-2}(M))$ vanishes, there exist (for a sufficiently fine cover) local forms $\beta_i \in \Omega^{2n-2}(U_i)$ such that $\alpha_{ij} = \beta_i - \beta_j$. Then, the local forms $e_p(A_i, F_{A_i}) - d\beta_i = e_p(A_j, F_{A_j}) - d\beta_j$ define a global section of $\Omega^{2n-1}(M)$. The de Rham differential of this globally defined differential form is $d(e_p(A_i, F_{A_i}) - d\beta_i) = de_p(A_i, F_{A_i}) = p(F_{A_i})$. Hence, $\mathrm{cw}(p) = [p(F_{A_i})] = 0$ in $H^{2n}(M)$.

In the other direction: if $[p(F_{A_i})] = 0$, there is a differential form $\gamma \in \Omega^{2n-1}(M)$ such that $d\gamma = p(F_{A_i})$ on U_i. Let $\omega_i = e_p(A_i, F_{A_i}) - \gamma \in \Omega^{2n-1}(U_i)$. We have $d\omega_i = 0$, and if the cover is sufficiently fine we find $\beta_i \in \Omega^{2n-2}(U_i)$ such that $d\beta_i = \omega_i$. Put $\alpha_{ij} = \beta_i - \beta_j$. We have

$$\begin{aligned} d\alpha_{ij} &= \omega_i - \omega_j = (e_p(A_i, F_{A_i}) - \gamma) - (e_p(A_j, F_{A_j}) - \gamma) \\ &= e_p(A_i, F_{A_i}) - e_p(A_j, F_{A_j}), \end{aligned}$$

as required. □

The groupoid morphism $\mu : \widehat{\mathcal{G}}(M) \to DG(M)$ can be used to map $\widehat{\mathcal{G}}(M)$-torsors to $DG(M)$-torsors. Since its lift μ_p is equal to identity on $\Omega^{2n-2}_{\text{closed}}$ (see Proposition 15), the obstruction to lifting the corresponding $DG(M)$-torsor to a $D_pG(M)$-torsor is again $\mathrm{cw}(p)$.

Finally, let us describe general $DG(M)$-torsors. It is enough to notice that $DG(M)$ is an extension of $G(M)$ by a sheaf of nilpotent groups and that this sheaf is acyclic (this follows from the acyclicity of the kernel of $D\mathfrak{g} \to C\mathfrak{g}$). Therefore, the classification of $DG(M)$-torsors is the same as the classification of principal G-bundles ($G(M)$-torsors). We thus have the following result.

Theorem 8. *The classification of $DG(M)$-torsors is the same as the classification of principal G-bundles; the correspondence is given by the morphism $DG(M) \to G(M)$. The obstruction to lifting a $DG(M)$-torsor to a $D_pG(M)$-torsor is exactly the Chern–Weil class* $\mathrm{cw}(p)$.

7. Groups Integrating Current Algebras $\mathcal{CA}(M, A)$

The purpose of this section is to construct groups integrating current algebras $\mathcal{CA}(M, A)$. The construction is similar to the integration methods in [12, 15].

7.1. *Integration of $\mathcal{CA}(M, A)$*

We will need the following notation: for an embedding of manifolds $f : Y \to X$, we denote $\Omega(X, Y) := \ker(f^* : \Omega(X) \to \Omega(Y))$ (this complex is quasi-isomorphic to the standard relative de Rham complex). Note that $\mathcal{SA}(X, Y, A) = (\Omega(X, Y) \otimes A)^0_{\text{closed}}$ is a Lie subalgebra of $\mathcal{SA}(X, A)$, and $(\Omega(X, Y) \otimes A)^0_{\text{exact}} \subset \mathcal{SA}(X, A)$ is a Lie ideal.

Proposition 18. *Let $I = [0, 1]$ be the unit interval with coordinate s. The map $\tau : \alpha \mapsto d(s\alpha)$ induces a Lie algebra isomorphism*

$$\mathcal{CA}(M, A) \cong \frac{\mathcal{SA}(M \times I, M \times \{0\}, A)}{(\Omega(M \times I, M \times \{0, 1\}) \otimes A)^0_{\text{exact}}}. \tag{21}$$

Proof. The map τ is well defined since $\tau(d\omega) = d(sd\omega) = -d(ds \wedge \omega)$, and $ds \wedge \omega$ vanishes when restricted to $M \times \{0\}$ and $M \times \{1\}$. To show that τ is an isomorphism of vector spaces, observe that the fiber integral $\lambda \mapsto \int_I \lambda$ is an inverse of τ. Finally, τ is a Lie homomorphism since for $\alpha, \beta \in (\Omega(M) \otimes A)^{-1}$ we have

$$\tau([\alpha, d\beta]) = d(s[\alpha, d\beta]) = [d(s\alpha), d(s\beta)] + d[s\alpha, d((1-s)\beta)] \equiv [d(s\alpha), d(s\beta)].$$

Here we used that $[s\alpha, d((1-s)\beta)]$ vanishes on $M \times \{0\}$ and $M \times \{1\}$. □

We can reformulate the isomorphism (21) as follows. Let a *path* in $\mathcal{SA}(M, A)$ be an element $\gamma \in \mathcal{SA}(M \times I, A)$. The endpoints of γ are the elements $\gamma|_{M\times\{0\}}, \gamma|_{M\times\{1\}} \in \mathcal{SA}(M, A)$. Let $\gamma_0, \gamma_1 \in \mathcal{SA}(M \times I, A)$ be two paths with the same endpoints ϵ_0 and ϵ_1, i.e. such that

$$\gamma_0|_{M\times\{0\}} = \gamma_1|_{M\times\{0\}} = \epsilon_0 \quad \text{and} \quad \gamma_0|_{M\times\{1\}} = \gamma_1|_{M\times\{1\}} = \epsilon_1.$$

A *homotopy* between γ_0 and γ_1 is an element $\chi \in \mathcal{SA}(M \times I \times I)$ such that $\gamma_0 = \chi|_{M\times I\times\{0\}}$, $\gamma_1 = \chi|_{M\times I\times\{1\}}$, and $\chi|_{M\times\{0\}\times I}$ is the pullback of ϵ_0 and $\chi|_{M\times\{1\}\times I}$ is the pullback of ϵ_1 under projection $M \times I \to M$.

Equation (21) says that $\mathcal{CA}(M, A)$ is isomorphic to the Lie algebra of paths in $\mathcal{SA}(M, A)$ starting at $0 \in \mathcal{SA}(M, A)$, modulo homotopy of paths. Indeed, for $\gamma_1 = \gamma_0 + d\mu$ the desired homotopy is $\chi = \gamma_0 + d(t\mu)$ (here t is the parameter on the unit segment). In the other direction, if χ is a homotopy interpolating between γ_0 and γ_1, we have

$$\gamma_1 - \gamma_0 = d \int_t \chi \in (\Omega(M \times I, M \times \{0, 1\}) \otimes A)^0_{\text{exact}}$$

as required.

In the same way, we can introduce paths and their homotopies in $\mathcal{SG}(M, A, G)$. The group $\mathcal{CG}(M, A, G)$ is then defined as the group of paths in $\mathcal{SG}(M, A)$ starting at the group unit, modulo homotopy of paths. Again, it depends on the choice of a connected Lie group G integrating the Lie algebra $\mathfrak{g} = A^0_{\text{closed}}$.

Remark. The groups $\mathcal{CG}(M, A, G)$ and $\mathcal{SG}(M, A, G)$ form a crossed module (i.e. a strict 2-group): The homomorphism $\mathcal{CG}(M, A, G) \to \mathcal{SG}(M, A, G)$ associates to a path in $\mathcal{SG}(M, A)$ its endpoint, and the action of $\mathcal{SG}(M, A)$ on paths, and hence on $\mathcal{CG}(M, A)$, is by conjugation.

Remark. Let $(A, G) \to (B, H)$ be a morphism of pairs consisting of a DGLA homomorphism $A \to B$ and a group homomorphism $G \to H$ integrating the Lie homomorphism $\mathfrak{g} = A^0_{\text{closed}} \to B^0_{\text{closed}} = \mathfrak{h}$. Then, similar to the $\mathcal{SG}$ functor, we obtain a group homomorphism $\mathcal{CG}(M, A, G) \to \mathcal{CG}(M, B, G)$. For example, consider the DGLA homomorphism $D_p\mathfrak{g} \to A_{\text{FMS}}$. Note that $(A_{\text{FMS}})^0_{\text{closed}} \cong \mathfrak{g} \ltimes W\mathfrak{g}^{2n-2}_{\text{closed}}$. If G is a connected Lie group integrating $\mathfrak{g}$, the Lie algebra A^0_{FMS} integrates to the semi-direct product $H = G \ltimes W\mathfrak{g}^{2n-2}_{\text{closed}}$. Obviously, we have a morphism of pairs $(D_p\mathfrak{g}, G) \to (A_{\text{FMS}}, H)$. It induces a group homomorphism

$\mathcal{CG}(M, D_p\mathfrak{g}, G) \to \mathcal{CG}(M, A_{\rm FMS}, H)$ from $\mathcal{CG}(M, D_p\mathfrak{g}, G)$ to the group $CG(M, A_{\rm FMS}, H)$ integrating FMS current algebra (for details about the group integrating the FMS current algebra see the book [14]). The image of this map is the group $\mathcal{CG}(M, B_{\rm FMS}, G)$ integrating the truncated FMS current algebra (here $B_{\rm FMS} = \mathrm{im}(D_p\mathfrak{g} \to A_{\rm FMS})$ and $\mathrm{im}(G \to H) = G$).

7.2. *The case of $A = D_p\mathfrak{g}$ and M a sphere*

In this section we restrict our attention to examples of $A = C\mathfrak{g}, D\mathfrak{g}, D_p\mathfrak{g}$ and $M = S^n$ a sphere. Let us introduce the shorthand notation $\widetilde{G}(M) = \mathcal{CG}(M, C\mathfrak{g}, G)$, $\widetilde{DG}(M) = \mathcal{CG}(M, D\mathfrak{g}, G)$, and $\widetilde{D_pG}(M) = \mathcal{CG}(M, D_p\mathfrak{g}, G)$.

Proposition 19. *Let G be a simply connected Lie group. Then, there is an exact sequence of groups,*

$$1 \to \pi_{n+1}(G) \to \widetilde{G}(S^n) \to G(S^n) \to \pi_n(G) \to 1.$$

Proof. The group $\widetilde{G}(S^n)$ consists of paths g_t in the group $G(S^n)$ which start at the group unit ($g_0 = 1$) modulo homotopy. Note that $\pi_0(G(S^n)) \cong \pi_n(G)$ and, if G is simply connected, $\pi_1(G(S^n)) \cong \pi_{n+1}(G)$. This implies the exact sequence in the Proposition. □

Proposition 20. *Let G be a simply connected Lie group. Then, there is an exact sequence of groups,*

$$1 \to \pi_{n+1}(G) \to \widetilde{DG}(S^n) \to DG(S^n) \to \pi_n(G) \to 1.$$

Proof. The group $\widetilde{DG}(S^n)$ can be described using the bijection (17). Let us define paths and their homotopies in $\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i}$ in the same way as above. In this sense, $\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i}$ is 1-connected. We infer from (17) that there is a bijection

$$\widetilde{DG}(M) \to \widetilde{G}(M) \times \left(\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i} \right).$$

Together with the exact sequence of the Proposition 19, it implies the required exact sequence. □

For $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$, let $\eta_p = e_p(\theta, 0) \in \Omega^{2n-1}(G)$ be the bi-invariant differential form on G defined by transgression, and $\Pi : \pi_{2n-1}(G) \to \mathbb{R}$ be the group homomorphism defined by the integration map: $C \mapsto \int_C \eta_p$. The image of Π is a subgroup of $\mathbb{R}$. We will be interested in the quotient $\mathbb{R}/\mathrm{im}(\Pi)$. If $p \in ((S^+\mathfrak{g})^{\mathfrak{g}})^2$, then $\eta_p = 0$ and the quotient is equal to $\mathbb{R}$. If $\mathfrak{g}$ is a Lie algebra of a compact simple Lie group, and p is a generator of $(S\mathfrak{g})^{\mathfrak{g}}$ of degree $m_i + 1$ (here m_i is one of the exponents of $\mathfrak{g}$), and the multiplicity of m_i is equal to one (this is always the case with the exception of one of the exponents of the group $\mathrm{SO}(2n)$), then $\mathbb{R}/\mathrm{im}(\Pi) \cong S^1$.

Theorem 9. *Let $p \in (S^n\mathfrak{g}^*)^{\mathfrak{g}}$. Then, there is an exact sequence of groups*

$$1 \to \mathbb{R}/\mathrm{im}(\Pi) \to \widetilde{D_pG}(S^{2n-3}) \to \widetilde{DG}(S^{2n-3}) \to 1,$$

where $\mathbb{R}/\mathrm{im}(\Pi)$ is a central subgroup.

Proof. The bijection (20) implies the bijection

$$\widetilde{D_pG}(M) \to {}_p\widetilde{G}(M) \times \left(\bigoplus_{i>0} \Omega^i(M) \otimes U^{-i}\right),$$

where ${}_p\widetilde{G}(M)$ stands (as usual) for the set of paths in ${}_pG(M)$ starting at the group unit modulo homotopy.

For $M = S^{2n-3}$, $\Omega^{2n-2}(S^{2n-3}) = 0$ and therefore ${}_pG(S^{2n-3}) = G(S^{2n-3})$. A path in ${}_pG(S^{2n-3})$ is a pair (g_t, α) where $g_t : S^{2n-3} \times I \to G$ and $\alpha \in \Omega^{2n-2}(S^{2n-3} \times I)$. Since α is a top degree form, there are no conditions imposed on it and the group homomorphism $\widetilde{D_pG}(S^{2n-3}) \to \widetilde{DG}(S^{2n-3})$ is surjective.

Let us determine the kernel of the group homomorphism $\widetilde{D_pG}(S^{2n-3}) \to \widetilde{DG}(S^{2n-3})$, or equivalently, the kernel of the map ${}_p\widetilde{G}(S^{2n-3}) \to \widetilde{G}(S^{2n-3})$. It consists of homotopy classes of paths in ${}_pG(S^{2n-3})$ of the form $(1, \alpha)$, where $\alpha \in \Omega^{2n-2}(S^{2n-3} \times I)$ and 1 denotes the constant map to $1 \in G$. A homotopy between two such paths $(1, \alpha_0)$, $(1, \alpha_1)$ is a pair (h, β), where h is a map $h : S^{2n-3} \times I \times I \to G$ such that $h|_{S^{2n-3} \times \partial(I \times I)} = 1$ and $\beta \in \Omega^{2n-2}(S^{2n-3} \times I \times I)$ is such that $d\beta + h^*\eta_p = 0$, and $\beta|_{S^{2n-3} \times \{0,1\} \times I} = 0$, $\beta|_{S^{2n-3} \times I \times \{0\}} = \alpha_0$, $\beta|_{S^{2n-3} \times I \times \{1\}} = \alpha_1$. The Stokes theorem implies

$$\int_{S^{2n-3} \times I} (\alpha_1 - \alpha_0) = -\int_{S^{2n-3} \times I \times I} h^*\eta_p \in \mathrm{im}(\Pi).$$

In the other direction, let $(1, \alpha_0)$ and $(1, \alpha_1)$ be paths in ${}_pG(S^{2n-3})$ such that $\int_{S^{2n-3} \times I} (\alpha_1 - \alpha_0) = \Pi(a)$ for some $a \in \pi_{2n-1}(G)$. Choose a smooth map $h : S^{2n-3} \times I \times I \to G$, $h|_{S^{2n-3} \times \partial(I \times I)} = 1$ representing the class $(-a)$. Then, there exists a differential form $\beta \in \Omega^{2n-2}(S^{2n-3} \times I \times I)$ such that (h, β) is a homotopy between the paths $(1, \alpha_0)$ and $(1, \alpha_1)$.

As a result, two paths $(1, \alpha_0)$ and $(1, \alpha_1)$ are homotopic if and only if

$$\int_{S^{2n-3} \times I} (\alpha_1 - \alpha_0) \in \mathrm{im}(\Pi).$$

The kernel of the map $\widetilde{D_pG}(S^{2n-3}) \to \widetilde{DG}(S^{2n-3})$ is therefore isomorphic to $\mathbb{R}/\mathrm{im}(\Pi)$. □

7.3. *Example: central extensions of loop groups*

Let $\mathfrak{g}$ be a Lie algebra of a simple simply connected compact Lie group G, and let $p \in (S^2\mathfrak{g})^{\mathfrak{g}}$ be a non-vanishing element (note that $(S^2\mathfrak{g})^{\mathfrak{g}} \cong \mathbb{R})$). In this case, $\mathbb{R}/\mathrm{im}(\Pi) \cong S^1$.

Taking $M = S^1$ in the discussion of the previous section, we observe the following. Since $\pi_1(G) = \pi_2(G) = 0$, we have $\tilde{G}(S^1) \cong G(S^1)$. In this very special case, we have $\mathcal{CA}(S^1, D\mathfrak{g}) = \mathcal{CA}(S^1, C\mathfrak{g})$, and $\widetilde{DG}(S^1) \cong G(S^1)$. Finally, for $\widetilde{D_pG}(S^1)$ we obtain an exact sequence of groups

$$1 \to S^1 \to \widetilde{D_pG}(S^1) \to G(S^1) \to 1.$$

Proposition 21. *The group $\widetilde{D_pG}(S^1)$ is the standard central extension of the loop group $LG = G(S^1)$.*

Proof. Let us recall the construction of the standard central extension $\widetilde{LG}$ of the loop group LG. Normalize $p \in (S^2\mathfrak{g}^*)^{\mathfrak{g}}$ (which is unique up to multiple) by the condition $\mathrm{im}(\Pi) = \mathbb{Z}$, and by requiring p to be positive-definite. Denote by D^2 the unit two-dimensional disc. First, one introduces a central extension $\widehat{G}(D^2)$ of the group $G(D^2)$ by $U(1)$. As a set, $\widehat{G}(D^2)$ is the direct product $G(D^2) \times U(1)$, and the group law is given by formula

$$(g_1, u_1)(g_2, u_2) = \left(g_1 g_2, u_1 u_2 \exp\left(\pi i \int_{D^2} p(g_1^{-1} dg_1, dg_2 g_2^{-1})\right)\right).$$

Note that the integrand is (up to a factor of $2\pi i$) the pull back of the 2-form ρ_p under the map $(g_1, g_2) \colon D^2 \to G \times G$.

Next, one introduces an equivalence relation on $\widehat{G}(D^2) \colon (g_1, u_1) \sim (g_2, u_2)$ if there exists a homotopy $h \colon D^2 \times I \to G$ between g_1 and g_2 relative to the boundary S^1 of D^2, such that $u_2 = u_1 \exp \int_{D^2 \times I} h^* \eta_p$. The group $\widetilde{LG}$ is then defined as the quotient of the group $\widehat{G}(D^2)$ by the equivalence relation $\sim$.

Equivalently, $\widetilde{LG}$ is quotient of the group $C_pG(D^2) = \mathcal{SG}(D^2, C_p\mathfrak{g}, G)$ by the following equivalence relation: (g_1, ω_1) and (g_2, ω_2) are equivalent if there exists a homotopy $(h, \chi) \in C_pG(D^2 \times I)$ between (g_1, ω_1) and (g_2, ω_2) relative to ∂D^2. Indeed, the map $C_pG(D^2) \to \widehat{G}(D^2)G$ given by $(g, \alpha) \mapsto (g, \exp 2\pi i \int_{D^2} \alpha)$ is a surjective group homomorphism, and two elements of $C_pG(D^2)$ are equivalent if and only if their images are equivalent.

Let us introduce a modification $\widehat{G}'(D^2)$ of $\widehat{G}(D^2)$: in its definition we replace the group $G(D^2)$ by the subgroup of $G(S^1 \times I)$ of maps $g \colon S^1 \times I \to G$ such that $g|_{S^1 \times \{0\}} = 1$, i.e. by the group of paths in $G(S^1) = LG$ starting at $1 \in G(S^1)$. We also replace the homotopies h by homotopies of paths. Then, it is easy to see that $\widehat{G}'(D^2)$ modulo the equivalence is again equal to $\widetilde{LG}$. Equivalently, it is the group of paths in $C_pG(S^1)$ starting at 1 modulo homotopy, i.e. the group $\mathcal{CG}(S^1, C_p\mathfrak{g}, G)$. Hence, $\widetilde{LG} \cong \mathcal{CG}(S^1, C_p\mathfrak{g}, G)$. □

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