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PROPERTIES EXPRESSIBLE IN SMALL FRAGMENTS OF THE THEORY OF THE HYPERFINITE II_1 FACTOR

ISAAC GOLDBRING AND BRADD HART

Abstract. We show that any II_1 factor that has the same 4-quantifier theory as the hyperfinite II_1 factor $\mathcal{R}$ satisfies the conclusion of the Popa Factorial Commutant Embedding Problem (FCEP) and has the Brown property. These results improve recent results proving the same conclusions under the stronger assumption that the factor is actually elementarily equivalent to $\mathcal{R}$. In the same spirit, we improve a recent result of the first-named author, who showed that if (1) the amalgamated free product of embeddable factors over a property (T) base is once again embeddable, and (2) $\mathcal{R}$ is an infinitely generic embeddable factor, then the FCEP is true of all property (T) factors. In this paper, it is shown that item (2) can be weakened to assume that $\mathcal{R}$ has the same 3-quantifier theory as an infinitely generic embeddable factor.

Introduction

The following problem of Popa is the main motivation for the work in this paper:

Problem (Popa's Factorial Commutant Embedding Problem (FCEP)). — *Suppose that M is a separable embeddable factor. Does there exist an embedding $i : M \hookrightarrow \mathcal{R}^{\mathcal{U}}$ with factorial commutant, that is, such that $i(M)' \cap \mathcal{R}^{\mathcal{U}}$ is a factor?*

Until recently, very little progress on the FCEP had been made. In [1], the following theorem was proven:

Theorem 1. — *If M is elementarily equivalent to $\mathcal{R}$, then M satisfies the FCEP.*

Recall that II_1 factors M and N are elementarily equivalent, denoted $M \equiv N$, if, for any sentence σ in the language of tracial von Neumann algebras, one has $\sigma^M = \sigma^N$. A logic-free definition can be given using the Keisler-Shelah Theorem: M and N are elementarily equivalent if and only if they have isomorphic ultrapowers.[1] By [6, Theorem 4.3], any separable II_1 factor M has continuum many nonisomorphic separable II_1 factors elementarily equivalent to it, whence Theorem 1 gave continuum many new examples of separable II_1 factors satisfying the FCEP.

In this paper, we weaken the assumption of the previous theorem and arrive at the same conclusion. We say that II_1 factors M and N are k-elementarily equivalent, denoted $M \equiv_k N$, if they agree on all formulae of quantifier-complexity at most k. (This will be defined precisely in the last section.) The following is an imprecise version of our first main result:

Theorem A. — *If $M \equiv_4 \mathcal{R}$, then M satisfies the FCEP.*

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[1] If one is willing to assume the continuum hypothesis, this can even be improved by saying that M and N are elementarily equivalent if and only if $M^{\mathcal{U}} \cong N^{\mathcal{U}}$ for any nonprincipal ultrafilter on $\mathbb{N}$.

In another direction, one of the main results of [7] was progress on the FCEP problem for embeddable[2] property (T) factors:

Theorem 2. — *Suppose that the following two statements are true:*

(1) *Whenever M_1 and M_2 are embeddable II_1 factors with a common property (T) subfactor N, then the amalgamated free product $M_1 *_N M_2$ is also embeddable.*
(2) *$\mathcal{R}$ is an* infinitely generic *embeddable factor.*

Then every embeddable property (T) factor satisfies the FCEP.

Infinitely generic factors form a large class of "rich" II_1 factors and more information about them can be found in [5]. In [5], it was claimed that $\mathcal{R}$ is an infinitely generic embeddable factor. However, the proof there is incredibly flawed and settling the question of whether or not $\mathcal{R}$ is actually an infinitely generic embeddable factor remains an important open question.

Ideally, one would like to remove the model-theoretic assumption (2) in the previous theorem, leaving only the operator-algebraic obstacle (1). Item (2) in the previous theorem is equivalent to the statement that $\mathcal{R}$ is elementarily equivalent to an infinitely generic embeddable factor. Consequently, the following theorem, a consequence of a more general result proven in Section 3, is a strengthening of the previous result:

Theorem B. — *Suppose that the following two statements are true:*

(1) *Whenever M_1 and M_2 are embeddable II_1 factors with a common property (T) subfactor N, then the amalgamated free product $M_1 *_N M_2$ is also embeddable.*
(2') *There is an infinitely generic embeddable factor M such that $M \equiv_3 \mathcal{R}$.*

Then every embeddable property (T) factor satisfies the FCEP.

It is worth noting that any infinitely generic embeddable factor M satisfies $M \equiv_2 \mathcal{R}$. In Section 3, we also note that the statement that there is an infinitely generic embeddable factor M such that $M \equiv_3 \mathcal{R}$ is already known to be "halfway true."

A crucial ingredient to the proof of Theorem 1 above is the following result of Nate Brown [3, Theorem 6.9]:

Fact. — *If N is a separable subfactor of $\mathcal{R}^{\mathcal{U}}$, then there is a separable subfactor P of $\mathcal{R}^{\mathcal{U}}$ with $N \subseteq P$ such that $P' \cap \mathcal{R}^{\mathcal{U}}$ is a II_1 factor.*

In [1], we said the II_1 factor M had the **Brown property** if, for all separable subfactors N of $M^{\mathcal{U}}$, then there is a separable subfactor P of $M^{\mathcal{U}}$ with $N \subseteq P$ such that $P' \cap M^{\mathcal{U}}$ is a II_1 factor. It was shown in [1] that any $M \equiv \mathcal{R}$ has the Brown property. In the last section of this paper, we prove a strengthening of this result:

Theorem C. — *If $M \equiv_4 \mathcal{R}$, then M has the Brown property.*

An interesting question arises: are these results actually improvements of their predecessors? Indeed, perhaps it is the case that there is $k \in \mathbb{N}$ such that if $M \equiv_k \mathcal{R}$, then $M \equiv \mathcal{R}$. If this were to happen, then one would say that $\mathrm{Th}(\mathcal{R})$ has *quantifier simplification*. Given recent results showing that the $\mathrm{Th}(\mathcal{R})$ is very complicated

[2] In this paper, we use the term **embeddable** as an abbreviation for $\mathcal{R}^{\mathcal{U}}$-embeddable.

from the model-theoretic perspective (see, e.g., [5] and [9]), we strongly believe in the following:

Conjecture. — $\mathrm{Th}(\mathcal{R})$ *does not admit quantifier simplification.*

For the rest of this paper, we work under the assumption that the previous Conjecture has a positive solution. In this case, Theorem A yields continuum many examples of factors satisfying the FCEP not covered by Theorem 1. Similarly, Theorem C yields continuum many new examples of factors with the Brown property.

Infinitely generic embeddable factors form a subclass of the more general class of **existentially closed embeddable factors**. An embeddable factor M is existentially closed (e.c.) if: whenever N is an embeddable factor with $M \subseteq N$, there is an embedding $N \hookrightarrow M^{\mathcal{U}}$ that restricts to the diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$. It was noted in [5] that $\mathcal{R}$ is an e.c. embeddable factor. Existentially closed embeddable factors have proven very important in applications of model-theoretic ideas to the study of II_1 factors. It is a major open question whether or not there are two non-elementarily equivalent e.c. embeddable factors. If $\mathcal{R}$ is not infinitely generic, then we would have an example of such a pair of e.c. embeddable factors. However, it could still be the case that all e.c. factors have the same 3-quantifier theory, in which case (2') in Theorem B is actually satisfied.

In order to keep this note relatively self-contained, we do not include much model-theoretic or operator-algebraic background. A rather lengthy introduction to model-theoretic ideas as they pertain to problems around factorial commutants can be found in [1].

In Section 1, we prove the main model-theoretic tools needed in the proof of Theorem A. In Section 2 we prove Theorem A, in Section 3 we prove Theorem B, and in Section 4 we prove Theorem C.

1. Weak heirs and weak embeddings

In this section, we fix a continuous language L. We say that a formula φ is in **prenex normal form** if it is of the form

$$Q_1 x_1 \cdots Q_m x_m \psi(x_1, \ldots, x_m, \vec{y}),$$

with each $Q_i \in \{\sup, \inf\}$ and with ψ quantifier-free. If the Q_i's alternate type, then we say that φ is $\forall_m$ (respectively $\exists_m$) if $Q_1 = \sup$ (resp. $Q_1 = \inf$).[3] If a formula is equivalent to a $\forall_m$ or $\exists_m$ formula, we often abuse terminology and refer to the formula itself as $\forall_m$ or $\exists_m$.

By a **fragment** of L-formulae, we mean a set Δ consisting of all $\forall_m$-formulae or of all $\exists_m$-formulae for some m.

Definition 1.1. — Fix an L-structure M, parameter sets $A \subseteq B \subseteq M$, and fragments Δ and Δ'.

(1) For $c \in M$, we set $\mathrm{tp}^M_\Delta(c/A)$ to be the set of all conditions $\varphi(x) = r$, where $\varphi \in \Delta$ has parameters from A and $\varphi(c)^M = r$.
(2) $S^M_\Delta(A)$ denotes the set of all $\mathrm{tp}^M_\Delta(c/A)$ for $c \in M$.
(3) For $p \in S^M_\Delta(A)$ and $\varphi(x)$ a formula from Δ with parameters from A, we set $\varphi(x)^p$ to be the unique r so that $\varphi(x) = r$ belongs to p.

[3] Technically we really should be speaking of $m-1$ alternations of *blocks* of quantifiers of the same length, but we blur this distinction here.

(4) For $c \in M$, let $\mathrm{tp}^M_{\Delta,\Delta'}(c/A,B)$ be the union of $\mathrm{tp}^M_\Delta(c/A)$ and $\mathrm{tp}^M_{\Delta'}(c/B)$.
(5) We let $S_{\Delta,\Delta'}(A,B)$ denote the set of all $\mathrm{tp}^M_{\Delta,\Delta'}(c/A,B)$ for $c \in M$. We extend the notation $\varphi(x)^p$ to $S_{\Delta,\Delta'}(A,B)$ in the obvious way.
(6) If $p \in S_\Delta(A)$, $q \in S_{\Delta,\Delta'}(A,B)$, and $\Delta' \subseteq \Delta$, we say that q is an **heir** of p if, for every $b \in B$, every $\varphi(x,y) \in \Delta'$, and every $\epsilon > 0$, there is $a \in A$ such that $|\varphi(x,a)^p - \varphi(x,b)^q| < \epsilon$.

Definition 1.2. — Suppose that $i : N \hookrightarrow M$ is an embedding between L-structures and Δ is a fragment. We say that i is:

(1) **downward** Δ if, for any nonnegative formula $\varphi(x) \in \Delta$ and any $a \in N$, if $\varphi(i(a))^M = 0$, then $\varphi(a)^N = 0$;
(2) **upward** Δ if, for any nonnegative formula $\varphi(x) \in \Delta$ and any $a \in N$, if $\varphi(a)^N = 0$, then $\varphi(i(a))^M = 0$.

We note one obvious fact:

Lemma 1.3. — *Given an embedding $i : N \hookrightarrow M$, we have that i is downwards $\exists_m$ if and only if i is upwards $\forall_m$.*

Proof. — Suppose that i is not upwards $\forall_m$, so there is a nonnegative $\forall_m$ formula $\varphi(x)$ and $a \in N$ such that $\varphi(a)^N = 0$ but $\varphi(i(a))^M = \epsilon > 0$. Then $(\epsilon \dot- \varphi(i(a)))^M = 0$ and since this formula is equivalent to a $\exists_m$ formula, we have that $(\epsilon \dot- \varphi(a))^N = 0$, a contradiction. The other direction is similar. □

The following is our main technical result concerning the existence of weak heirs. In the remainder of this paper, $\mathcal{U}$ denotes a countably incomplete ultrafilter on some index set (unless otherwise specified).

Theorem 1.4. — *Suppose that M is a separable L-structure. Fix a separable substructure N of $M^\mathcal{U}$ such that the inclusion $N \subseteq M^\mathcal{U}$ is downward $\exists_{m+2}$. Fix also $p \in S_{\forall_m}(N)$. Then for any separable parameter set A with $N \subseteq A \subseteq M^\mathcal{U}$ and any $n < m$, there is $q \in S_{\forall_m,\forall_n}(N,A)$ that is an heir of p.*

Proof. — We seek $a \in M^\mathcal{U}$ satisfying the following two kinds of conditions:

(1) $\psi(a) = \psi(x)^p$ for any $\forall_m$-formula $\psi(x)$ with parameters from N;
(2) $\varphi(a,c)^{M^\mathcal{U}} \geqslant \frac{\epsilon}{2}$ for any $\forall_{n+1}$-formula $\varphi(x,y)$ with parameters from A and any $\epsilon > 0$ such that $\varphi(x,b)^p \geqslant \epsilon$ for all $b \in N$.

Indeed, if a is as above, we claim that $q := \mathrm{tp}^{M^\mathcal{U}}_{\forall_m,\forall_n}(a/A)$ is an heir of p. By (1), q is an extension of p. To see that q is an heir, fix a $\forall_n$-formula $\varphi(x,c)$ with parameters from A and set $s := \varphi(x,c)^q = \varphi(a,c)^{M^\mathcal{U}}$. Suppose, towards a contradiction, that there is $\epsilon > 0$ such that $|\varphi(x,b)^p - s| \geqslant \epsilon$ for all $b \in N$. It follows that $|\varphi(x,b) - s|^p \geqslant \epsilon$ for all $b \in N$. Since $|\varphi(x,b) - s|$ is logically equivalent to a $\forall_{n+1}$, whence, by (2), $|\varphi(a,c)^{M^\mathcal{U}} - s| \geqslant \frac{\epsilon}{2}$, leading to a contradiction.

Suppose now, towards a contradiction, that no such $a \in M^\mathcal{U}$ exists. By countable saturation, it follows that there are:

- a $\forall_m$-formula $\psi(x)$ with parameters from N such that $\psi(x)^p = 0$,
- a $\delta > 0$, and
- formulae $\varphi_1(x,c_1), \ldots, \varphi_k(x,c_k)$ with parameters from A as in (2)

such that, for any $a \in M^\mathcal{U}$, if $\psi(a) < \delta$, then $\varphi_i(a,c_i) < \frac{\epsilon}{2}$ for some $i = 1, \ldots, k$.

In other words,

$$\left(\sup_x \min\left(\delta \dot{-} \psi(x), \min_{1\leqslant i\leqslant k}\left(\varphi_i(x,c_i) \dot{-} \frac{\epsilon}{2}\right)\right)\right)^{M^{\mathcal{U}}} = 0.$$

Consequently,

$$\left(\inf_{y_1}\cdots\inf_{y_k}\sup_x \min\left(\delta \dot{-} \psi(x), \min_{1\leqslant i\leqslant k}\left(\varphi_i(x,y_i) \dot{-} \frac{\epsilon}{2}\right)\right)\right)^{M^{\mathcal{U}}} = 0,$$

and thus, since the inclusion $N \subseteq M^{\mathcal{U}}$ is downward $\exists_{m+2}$, we have

$$\left(\inf_{y_1}\cdots\inf_{y_m}\sup_x \min\left(\delta \dot{-} \psi(x), \min_{1\leqslant i\leqslant m}\left(\varphi_i(x,y_i) \dot{-} \frac{\epsilon}{2}\right)\right)\right)^{N} = 0.$$

Set $\eta := \min(\delta, \frac{\epsilon}{2})$ and take $d_1, \dots, d_k \in N$ such that

$$\left(\sup_x \min\left(\delta \dot{-} \psi(x), \min_{1\leqslant i\leqslant k}\left(\varphi_i(x,d_i) \dot{-} \frac{\epsilon}{2}\right)\right)\right)^{N} < \eta;$$

since the inclusion $N \subseteq M^{\mathcal{U}}$ is upward $\forall_{m+1}$, we have

$$\left(\sup_x \min\left(\delta \dot{-} \psi(x), \min_{1\leqslant i\leqslant k}\left(\varphi_i(x,d_i) \dot{-} \frac{\epsilon}{2}\right)\right)\right)^{M^{\mathcal{U}}} < \eta.$$

Take $a \in M^{\mathcal{U}}$ realizing p. Then $\psi(a)^{M^{\mathcal{U}}} = \psi(x)^p = 0$, whence, since $\eta \leqslant \delta$, we have $\min_{1\leqslant i\leqslant k}(\varphi_i(x,d_i) \dot{-} \frac{\epsilon}{2})^{M^{\mathcal{U}}} < \eta \leqslant \frac{\epsilon}{2}$. Choosing i such that $(\varphi_i(a,d_i) \dot{-} \frac{\epsilon}{2})^{M^{\mathcal{U}}} < \eta$, we get that $\varphi_i(x,d_i)^p = \varphi_i(a,d_i)^{M^{\mathcal{U}}} < \epsilon$, a contradiction. ☐

We will be interested in the following special case of Theorem 1.4:

Corollary 1.5. — *Suppose that M is a separable L-structure. Fix a separable substructure N of $M^{\mathcal{U}}$ such that the inclusion $N \subseteq M^{\mathcal{U}}$ is downward $\exists_3$. Fix also $p \in S_{\forall_1}(N)$. Then for any separable parameter set A with $N \subseteq A \subseteq M^{\mathcal{U}}$, there is $q \in S_{\forall_1,\forall_0}(N,A)$ that is an heir of p.*

Definition 1.6. — Given a fragment Δ and an L-structure M, we set

$$\mathrm{Th}_\Delta(M) := \{\sigma \ : \ \sigma \text{ is a nonnegative } L\text{-sentence from } \Delta \text{ and } \sigma^M = 0\}.$$

If N is another L-structure, we write $N \models \mathrm{Th}_\Delta(M)$ if $\sigma^N = 0$ for all $\sigma \in \mathrm{Th}_\Delta(M)$.

We now prove a result connecting small quantifier-fragments of theories of structures with the existence of embeddings as in the previous theorem.

Proposition 1.7. — *Suppose that M and N are separable L-structures and $m \in \mathbb{N}$. Then there is an embedding $i : N \hookrightarrow M^{\mathcal{U}}$ that is downwards $\exists_{m+2}$ if and only if $M \models \mathrm{Th}_{\exists_{m+3}}(N)$.*

Proof. — First suppose that a downwards $\exists_{m+2}$-embedding $i : N \hookrightarrow M^{\mathcal{U}}$ exists and σ is a nonnegative $\exists_{m+3}$-sentence such that $\sigma^N = 0$. Write $\sigma = \inf_x \varphi(x)$ with φ a $\forall_{m+2}$-formula. Fix $\epsilon > 0$ and take $a \in N$ such that $\varphi(a) < \epsilon$. Then $(\varphi(a) \dot{-} \epsilon)^N = 0$, and since this formula is equivalent to a $\forall_{m+2}$-formula and i is upwards $\forall_{m+2}$, we have that $(\varphi(i(a)) \dot{-} \epsilon)^{M^{\mathcal{U}}} = 0$. Consequently, $(\inf_x(\varphi(x) \dot{-} \epsilon))^M = 0$; since M is arbitrary, we have that $\sigma^M = 0$, as desired.

Conversely, suppose that $M \models \mathrm{Th}_{\exists_{m+3}}(N)$. Let L_N be the language obtained by adding constants c_a for $a \in N$. Set Γ to be the following collection of L_N sentences:

(1) $\theta(c_{a_1}, \ldots, c_{a_n})$, where θ is a nonnegative quantifier-free formula such that $\theta(a_1, \ldots, a_n)^N = 0$;

(2) $\epsilon \dot{-} \varphi(c_{a_1}, \ldots, c_{a_n})$, where φ is a $\exists_{m+2}$-formula with $\varphi(a_1, \ldots, a_n)^N \geqslant \epsilon$

If Γ can be shown to be approximately finitely satisfiable in an expansion of M, then by countable saturation there is an expansion of $M^{\mathcal{U}}$ which is a model of Γ, and this yields the desired embedding. So suppose $\theta_1, \ldots, \theta_k$ are as in (1) and $\epsilon_j \dot{-} \varphi_j$, $j = 1, \ldots, l$, are as in (2). Then

$$\inf_x \left(\max \left(\max_{i=1,\ldots,k} \theta_i(x), \max_{j=1,\ldots,l} (\epsilon_j \dot{-} \varphi_j(x) \right) \right)$$

is equivalent to an $\exists_{m+3}$-sentence that evaluates to 0 in N, whence, by assumption, also evaluates to 0 in M. This completes the proof. □

Combining Theorem 1.4 and Proposition 1.7, we arrive at:

Corollary 1.8. — *Suppose that M is a separable L-structure. Fix a separable substructure N of $M^{\mathcal{U}}$ such that $M \models \mathrm{Th}_{\exists_{m+3}}(N)$. Fix also $p \in S^{M^{\mathcal{U}}}_{\forall_m}(N)$. Then for any separable parameter set A with $N \subseteq A \subseteq M^{\mathcal{U}}$ and any $n < m$, there is $q \in S^{M^{\mathcal{U}}}_{\forall_m,\forall_n}(N, A)$ that is an heir of p. In particular, if $M \models \mathrm{Th}_{\exists_4}(N)$, then for any $p \in S^{M^{\mathcal{U}}}_{\forall_1}(N)$ and any separable parameter set A with $N \subseteq A \subseteq M^{\mathcal{U}}$, there is $q \in S^{M^{\mathcal{U}}}_{\forall_1,\forall_0}(N, A)$ that is an heir of p.*

2. Proof of Theorem A

In this section, we apply the abstract results from the previous section to the setting of II_1 factors. Throughout this section, L is the language of tracial von Neumann algebras and T is the universal theory of embeddable tracial von Neumann algebras. All structures considered in this section will be models of T.

Lemma 2.1. — *Suppose that M and N are separable with $N \subseteq M^{\mathcal{U}}$. Suppose also that $a, b \in M^{\mathcal{U}}$ are such that $a \in Z(N' \cap M^{\mathcal{U}})$ and $\mathrm{tp}^{M^{\mathcal{U}}}_{\forall_1}(a/N) = \mathrm{tp}^{M^{\mathcal{U}}}_{\forall_1}(b/N)$. Then $b \in Z(N' \cap M^{\mathcal{U}})$.*

Proof. — Since $\mathrm{tp}^{M^{\mathcal{U}}}_{\forall_0}(a/N) = \mathrm{tp}^{M^{\mathcal{U}}}_{\forall_0}(b/N)$, we have $b \in N' \cap M^{\mathcal{U}}$. Now fix $\epsilon > 0$. By countable saturation, there are $e_1, \ldots, e_n \in N$ and $\delta > 0$ such that, for all $c \in M^{\mathcal{U}}$, if $\|[c, e_i]\|_2 < \delta$ for all $i = 1, \ldots, n$, then $\|[c, a]\|_2 < \epsilon$. Consequently,

$$\sup_x \min\left(\delta \dot{-} min_i \|[x, e_i]\|_2, \|[x, y]\|_2 \dot{-} \epsilon\right)$$

belongs to $\mathrm{tp}^{M^{\mathcal{U}}}_{\forall_1}(a/N)$, whence it also belongs to $\mathrm{tp}^{M^{\mathcal{U}}}_{\forall_1}(b/N)$. It follows that $b \in Z(N' \cap M^{\mathcal{U}})$. So, if $c \in N' \cap M^{\mathcal{U}}$, then $\|[b, c\|_2 \leqslant \epsilon$. Since ϵ was arbitrary, it follows that $[b, c] = 0$, and thus $b \in Z(N' \cap M^{\mathcal{U}})$, as desired. □

Corollary 2.2. — *Suppose that $N \subseteq P \subseteq M^{\mathcal{U}}$, $P' \cap M^{\mathcal{U}}$ is a factor, and every element of $S^{M^{\mathcal{U}}}_{\forall_1}(N)$ admits an heir to $S^{M^{\mathcal{U}}}_{\forall_1,\forall_0}(N, P)$. Then $N' \cap M^{\mathcal{U}}$ is a factor.*

Proof. — Take $a \in Z(N' \cap M^{\mathcal{U}})$ and let $p := \mathrm{tp}_{\forall_1}(a/N)$. Let $q \in S_{\forall_1,\forall_0}(N, P)$ be an heir of p. Let $b \in M^{\mathcal{U}}$ satisfy q. By the heir property, $b \in P' \cap M^{\mathcal{U}}$. If $c \in P' \cap M^{\mathcal{U}}$, then $c \in N' \cap M^{\mathcal{U}}$, whence, by the previous lemma, $[b, c] = 0$. It follows that $b \in Z(P' \cap M^{\mathcal{U}}) = \mathbb{C}$. So $b = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$, so $d(x, \lambda \cdot 1) = 0$ belongs to q, whence it also belongs to p, and thus $a = \lambda \cdot 1$, as desired. □

Recall the following fact of Nate Brown mentioned in the introduction:

FACT 2.3. — *For every separable $N \subseteq \mathcal{R}^{\mathcal{U}}$, there is a separable $P \subseteq \mathcal{R}^{\mathcal{U}}$ with $N \subseteq P$ such that $P' \cap \mathcal{R}^{\mathcal{U}}$ is a factor.*

We are now able to prove the following more precise version of Theorem A:

THEOREM 2.4. — *Suppose that N is an embeddable factor such that $\mathcal{R} \models \mathrm{Th}_{\exists_4}(N)$. Then N satisfies the FCEP.*

Proof. — Fix P as in the previous fact, so $N \subseteq P \subseteq \mathcal{R}^{\mathcal{U}}$ with $P' \cap \mathcal{R}^{\mathcal{U}}$ a factor. The proof then follows from Corollary 1.8 and Corollary 2.2. ☐

3. Proof of Theorem B

Let (*) denote the statement: the amalgamated free product of embeddable factors over a property (T) base is once again embeddable.

LEMMA 3.1. — *Suppose that (*) holds. Then whenever N is a w-spectral gap subfactor of the e.c. embeddable factor M, then $(N' \cap M)' \cap M = N$.*

Proof. — In [8], this was proven without a restriction to embeddable factors. The proof goes through in the embeddable case if one assumes (*) holds. ☐

Recall that if N is a property (T) factor, then N has a **Kazhdan set**, which is a finite subset F of N that satisfies the following property: there is a $K > 0$ such that for any II_1 factor M containing N as a subfactor, any $b \in M_1$, and any sufficiently small $\eta > 0$, if $\|[a,b]\|_2 < \eta$ for all $a \in F$, then there is $c \in N' \cap M$ such that $\|b - c\|_2 < K\eta$. Since $\|b - E_{N'\cap M}(b)\|_2 \leqslant \|b - c\|_2 < K\eta$ and $E_{N'\cap M}$ is operator norm-contractive, it follows that we may assume that $c \in M_1$ as well. (See [4, Proposition 1] for a proof.)

THEOREM 3.2. — *Suppose that (*) holds. Suppose further that N is an embeddable property (T) II_1 factor, M is an e.c. embeddable factor containing N, and $j : M \hookrightarrow \mathcal{R}^{\mathcal{U}}$ is downward Σ_2. Then $j(N)' \cap \mathcal{R}^{\mathcal{U}}$ is a factor.*

Proof. — Suppose $a \in Z(j(N)' \cap \mathcal{R}^{\mathcal{U}})$ but $d(a, \mathrm{tr}(a) \cdot 1) = \epsilon > 0$; we aim for a contradiction. Without loss of generality, suppose a is in the unit ball. Let $\{z_1, \ldots, z_n\}$ be a Kazhdan set for N with Kazhdan constant K. Note that

$$\mathcal{R}^{\mathcal{U}} \models \forall w \left(\max_{1 \leqslant i \leqslant n} \|[w, j(z_i)]\|_2 = 0 \rightarrow \|[w, a]\|_2 = 0 \right),$$

whence, by [2, Proposition 7.14], there is a continuous, nondecreasing function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that

$$\mathcal{R}^{\mathcal{U}} \models \sup_w \left(\|[a, w]\|_2 \dot{-} \alpha \left(\max_{1 \leqslant i \leqslant n} \|[w, j(z_i)]\|_2 \right) \right) = 0.$$

Set $\psi(x, \vec{t}) := \sup_w (\|[x, w]\|_2 \dot{-} \alpha(\max_{1 \leqslant i \leqslant n} \|[w, t_i]\|_2))$, a universal formula such that $\mathcal{R}^{\mathcal{U}} \models \psi(a, j(\vec{z})) = 0$ whence

$$\mathcal{R}^{\mathcal{U}} \models \inf_x \max \left(\max_{1 \leqslant i \leqslant n} \|[x, j(z_i)]\|_2, \psi(x, j(\vec{z})), \epsilon \dot{-} d(x, tr(x) \cdot 1) \right) = 0.$$

Since the latter displayed formula is equivalent to a $\exists_2$-formula, by assumption we have

$$M \models \inf_x \max\left(\max_{1\leqslant i\leqslant n} \|[x,z_i]\|_2, \psi(x,\vec{z}), \epsilon \dot{-} d(x, tr(x)\cdot 1)\right) = 0.$$

Fix $\eta > 0$ sufficiently small and take $b \in M_1$ such that

$$M \models \max\left(\max_{1\leqslant i\leqslant n} \|[b,z_i]\|_2, \psi(b,\vec{z}), \epsilon \dot{-} d(b, tr(b)\cdot 1)\right) < \eta.$$

If η is sufficiently small, there is $b' \in N' \cap M$ such that $d(b,b') < K\eta$. For simplicity, set $\beta := K\eta$. Now suppose that $c \in N' \cap M$ is in the unit ball. Then $\|[b,c]\|_2 < \eta$, whence $\|[b',c]\|_2 < \eta + 2\beta$. Since $c \in N' \cap M$ was arbitrary, we have $d(b', (N' \cap M)' \cap M) \leqslant \eta + 2\beta$.[4] By Lemma 3.1, since M is e.c. and N has w-spectral gap in M, we have that $(N' \cap M)' \cap M = N$, so $d(b', N) \leqslant \eta + 2\beta$, that is, $d(b', E_N(b')) \leqslant \eta + 2\beta$. However, $b' \in N' \cap M$ implies $E_N(b') \in Z(N) = \mathbb{C}$. It follows that $d(b', \mathrm{tr}(b')\cdot 1) = d(b, \mathbb{C}) \leqslant d(b, E_N(b')) \leqslant \eta + 2\beta$. Since $\epsilon \dot{-} d(b, \mathrm{tr}(b)\cdot 1) < \eta$, we have that $\epsilon \dot{-} d(b', \mathrm{tr}(b')\cdot 1) < \eta + 2d(b,b') < \eta + 2\beta$, which is a contradiction as long as $2\eta + 4\beta < \epsilon$. Recalling that $\beta = K\eta$, we have that $2\eta + 4\beta = (2+4K)\eta$, whence choosing $\eta < \frac{\epsilon}{2+4K}$, we arrive at the desired contradiction. □

The following is a more precise version of Theorem D; it follows immediately from Proposition 1.7 and Theorem 3.2.

Corollary 3.3. — *Suppose that (*) holds and every embeddable factor N embeds into an e.c. embeddable factor M such that $M \models \mathrm{Th}_{\forall_3}(\mathcal{R})$. Then every embeddable property (T) factor satisfies the FCEP.*

The assumption in the previous corollary should be compared to:

Lemma 3.4. — *If M is an e.c. embeddable factor, then $M \models \mathrm{Th}_{\exists_3}(\mathcal{R})$.*

Proof. — Since M is a II_1 factor, we may assume that $\mathcal{R} \subseteq M$. Fix an $\exists_3$-sentence $\sigma = \inf_x \sup_y \inf_z \varphi(x,y,z)$ such that $\sigma^{\mathcal{R}} = 0$. Fix $\epsilon > 0$ and $a \in \mathcal{R}$ such that $(\inf_y \sup_z \varphi(a,y,z))^{\mathcal{R}} < \epsilon$. Fix $b \in M$ and an embedding $i : M \hookrightarrow \mathcal{R}^{\mathcal{U}}$. Then $(\inf_z \varphi(i(a), i(b), z)^{\mathcal{R}^{\mathcal{U}}} < \epsilon$, whence there is $c \in \mathcal{R}^{\mathcal{U}}$ such that $(\varphi(i(a), i(b), c)^{\mathcal{R}^{\mathcal{U}}} < \epsilon$. Since M is e.c. there is $b' \in M$ such that $\varphi(a,b,c') < 2\epsilon$. Since ϵ is arbitrary, we have that $\sigma^M = 0$. □

Thus, the assumption of Corollary 3.3 comes tantalizingly close to removing any model-theoretic assumption at all, leaving only the operator-algebraic assumption (*).

4. Proof of Theorem C

We begin by explaining exactly what we mean for two structures to be k-elementarily equivalent.

Definition 4.1. — If φ is a formula and k is a nonnegative integer, we recall what it means for φ to have **quantifier depth at most** k, written $\mathrm{depth}(\varphi) \leqslant k$, by induction on the complexity of φ:

- If φ is atomic, then $\mathrm{depth}(\varphi) \leqslant 0$.

[4] This follows from the general fact that, for a subfactor P of a II_1 factor Q and $a \in Q_1$, one has $d(a, P' \cap Q) \leqslant \sup_{b \in P_1} \|[a,b]\|_2$.

- If $\varphi_1, \ldots, \varphi_n$ are formulae, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varphi = f(\varphi_1, \ldots, \varphi_n)$, then $\operatorname{depth}(\varphi) \leqslant \max_{1 \leqslant i \leqslant n} \operatorname{depth}(\varphi_i)$.
- If $\varphi = \sup_{\vec{x}} \psi$ or $\varphi = \inf_{\vec{x}} \psi$, then $\operatorname{depth}(\varphi) \leqslant \operatorname{depth}(\psi) + 1$.

Definition 4.2. — If M and N are L-structures, we write $M \equiv_k N$ if $\sigma^M = \sigma^N$ whenever $\operatorname{depth}(\sigma) \leqslant k$.

Remark 4.3. — If σ is an $\forall_m$-sentence or a $\exists_m$-sentence, then clearly $\operatorname{depth}(\sigma) = m$. Consequently, if $M \equiv_m N$, then $M \models \mathrm{Th}_{\forall_m}(N)$ and $N \models \mathrm{Th}_{\forall_m}(M)$.

We recall the following Ehrenfeucht-Fraisse game for continuous logic.

Definition 4.4. — Let M and N be L-structures and let $k \in \mathbb{N}$. $\mathfrak{G}(M, N, k)$ denotes the following game played by two players. First, player I plays either a tuple[5] $\vec{x_1} \in M$ or a tuple $\vec{y_1} \in N$. Player II then responds with a tuple $\vec{y_1} \in N$ or $\vec{x_1} \in M$. The play continues in this way for k rounds. We say that *Player II wins* $\mathfrak{G}(M, N, k)$ if there is an isomorphism between the substructures generated by $\{\vec{x_1}, \ldots, \vec{x_k}\}$ and $\{\vec{y_1}, \ldots, \vec{y_k}\}$ that maps $\vec{x_i}$ to $\vec{y_i}$.

Definition 4.5. — If M and N are L-structures, we write $M \equiv_k^{EF} N$ if II has a winning strategy for $\mathfrak{G}(M, N, k)$.

It is a routine induction to show that $M \equiv_k^{EF} N$ implies $M \equiv_k N$. Conversely, one has the following result (see [10, Lemma 2.4]):

Fact 4.6. — *Suppose that M and N are countably saturated L-structures. Then $M \equiv_k N$ if and only if $M \equiv_k^{EF} N$.*

We are now ready to prove Theorem C. Recall from the introduction that a II_1 factor M has the Brown property if: for every separable subfactor N of $M^{\mathcal{U}}$, there is a separable subfactor P of $M^{\mathcal{U}}$ with $N \subseteq P$ such that $P' \cap M^{\mathcal{U}}$ is a II_1 factor.

Theorem 4.7. — *Suppose that $M \equiv_4 \mathcal{R}$. Then M has the Brown property.*

Proof. — Suppose N is a separable subfactor of $M^{\mathcal{U}}$. It suffices to find a separable subfactor P of $M^{\mathcal{U}}$ containing N such that $P' \cap M^{\mathcal{U}}$ is a factor. Indeed, since $M \equiv_2 \mathcal{R}$, M is McDuff, whence $P' \cap M^{\mathcal{U}}$ will contain a copy of $\mathcal{R}^{\mathcal{U}}$ and will thus be a II_1 factor, as desired.

Since $M \equiv_4 \mathcal{R}$ and $M^{\mathcal{U}}$ and $\mathcal{R}^{\mathcal{U}}$ are $\aleph_1$-saturated, we know that player II has a winning strategy in $\mathfrak{G}(M^{\mathcal{U}}, \mathcal{R}^{\mathcal{U}}, 4)$. We assume in the following run of the game that player II plays according to this strategy. Let player I begin with $\vec{a}_1$, which is a countable sequence from the unit ball of N which generates N. Let player II respond with $\vec{b}_1$ and let N^* denote the separable subfactor of $\mathcal{R}^{\mathcal{U}}$ generated by $\vec{b}_1$. Since $\mathcal{R}$ has the Brown property, there is a separable subfactor P^* of $\mathcal{R}^{\mathcal{U}}$ containing N^* such that $(N^*)' \cap \mathcal{R}^{\mathcal{U}}$ is a factor. Let $\vec{b}_2$ be a countable subset of the unit ball of P^* which, together with $\vec{b}_1$, generates P^*. Let player II respond with $\vec{a}_2$ and let P be the separable subfactor of $M^{\mathcal{U}}$ generated by $\vec{a}_1$ and $\vec{a}_2$. We claim that this P is as desired.

To see this, suppose that $a_3 \in Z(P' \cap M^{\mathcal{U}})$. We wish to show that $a_3 \in \mathbb{C}$. To see this, let player II respond with $b_3 \in \mathcal{R}^{\mathcal{U}}$. We claim that $b_3 \in Z((P^*)' \cap \mathcal{R}^{\mathcal{U}})$, whence $b_3 \in \mathbb{C}$. To see this, suppose that $b_4 \in (P^*)' \cap \mathcal{R}^{\mathcal{U}}$. Let player II respond

[5] Here, tuples can be either of finite or countably infinite length.

with $a_4 \in M^{\mathcal{U}}$. Since the map $\vec{a}_1\vec{a}_2a_3a_4 \mapsto \vec{b}_1\vec{b}_2b_3b_4$ extends to an isomorphism between the subalgebras they generate, we see that $a_4 \in P' \cap M^{\mathcal{U}}$. It follows that a_3 and a_4 commute, whence so do b_3 and b_4.

Now that we have established that $b_3 \in \mathbb{C}$, the fact that the strategy is winning also shows that $a_3 \in \mathbb{C}$, as desired. □

Recall that a McDuff II_1 factor is **super McDuff** if $M' \cap M^{\mathcal{U}}$ is a II_1 factor. In [1, Proposition 4.2.4], it was proven that M has the Brown property if and only if all N elementarily equivalent to M are super McDuff. Consequently, we arrive at:

Corollary 4.8. — *If $M \equiv_4 \mathcal{R}$, then M is super McDuff.*

As mentioned in the introduction, if $\mathrm{Th}(\mathcal{R})$ does not admit quantifier simplification, then these results yield continuum many new examples of separable factors that are super McDuff and have the Brown property.

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