Vincel HOANG NGOC MINH

On the solutions of the universal differential equation with three regular singularities (On solutions of $KZ_3$)

<http://cml.centre-mersenne.org/item?id=CML_2019__11_2_25_0>
ON THE SOLUTIONS OF THE UNIVERSAL DIFFERENTIAL EQUATION WITH THREE REGULAR SINGULARITIES
(ON SOLUTIONS OF $KZ_3$)

VINCEL HOANG NGOC MINH

Abstract. This review concerns the resolution of a special case of Knizhnik-Zamolodchikov equations ($KZ_3$) and our recent results on combinatorial aspects of zeta functions on several variables.

In particular, we describe the action of the differential Galois group of $KZ_3$ on the asymptotic expansions of its solutions leading to a group of associators which contains the unique Drinfel’d associator (or Drinfel’d series). Non trivial expressions of an associator with rational coefficients are also explicitly provided, based on the algebraic structure and the singularity analysis of the multi-indexed polylogarithms and harmonic sums.

Contents

1. Knizhnik-Zamolodchikov equations and Drinfel’d series 26
2. Combinatorial framework 31
  2.1. Shuffle and quasi-shuffle algebras 31
  2.2. Diagonal series on bialgebras 32
  2.3. Exchangeable and noncommutative rational series 34
3. Indexation by words and generating series 36
  3.1. Indexation by words 36
  3.2. Indexation by noncommutative rational series 37
  3.3. Noncommutative generating series 40
4. Global asymptotic behaviors at singularities 41
  4.1. The case of positive multi-indices 41
  4.2. Structure of polyzetas 43
  4.3. The case of negative multi-indices 44
5. A group of associators 47
  5.1. The action of the Galois differential group 47
  5.2. Associate $\Phi_{KZ}$ 49
  5.3. Associators with rational coefficients 50
6. Conclusion 54
Appendix A 55
Appendix B 58
Appendix C 59
Appendix D 60
Acknowledgements 62
References 62

Math. classification: 05E16, 11M32, 16T05, 20F10, 33F10, 44A20.
Keywords: Algebraic Basis, Combinatorial Hopf Algebra, Harmonic Sum, Polylogarithm, Polyzeta.
1. KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND DRINFELD'S SERIES

In this paper, we survey our recent results which pertain to an in-depth combinatorial study of the several complex variables zeta functions defined as follows

\[ \forall r \geq 1, \quad \zeta_r : \mathcal{H}_r \rightarrow \mathbb{R}, \quad (s_1, \ldots, s_r) \mapsto \sum_{n_1 > \ldots > n_k > 0} n_1^{-s_1} \cdots n_k^{-s_r}, \]

where \( \mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\} \) [29, 30]. They appear in the regularization of solutions of the following fuchsian first order differential equation with initial condition, with regular singularities in \( \{0, 1, +\infty\} \) and noncommutative indeterminates in \( X = \{x_0, x_1\} \):

\[ (DE) \quad dG(z) = \left( x_0 \frac{dz}{z} + x_1 \frac{dz}{1-z} \right) G(z). \quad (1.1) \]

Let us denote by \( \mathcal{H}(\Omega) \) the ring of holomorphic functions over the simply connected domain \( \Omega := \mathbb{C} \setminus [0, \infty) \cup [1, +\infty] \), with \( 1_\Omega : \Omega \rightarrow \mathcal{H}(\Omega) \) as the neutral element (\( z \mapsto 1 \)). Let us also introduce the following differential forms

\[ \omega_0(z) := \frac{dz}{z} \quad \text{and} \quad \omega_1(z) := \frac{dz}{1-z}. \]

This equation can be considered as the universal fuchsian first order differential equation with three regular singularities. Here, the notation has become essentially classical since Drinfel’d’s papers [24, 25] which emphasized the importance of (1.1). After some elementary transformations [24, 25] one also finds that (1.1) is (equivalent to) the first non trivial Knizhnik-Zamolodchikov KZ. This is connected to the fact that the colored braid group on three strands \( P_3 \) is the direct product of its cyclic center with a copy of the free group on two generators. Although this interpretation of (1.1) does not play an explicit role below, it should be kept in mind with a view towards applications.

We may now return to (1.1) for which a solution can be obtained, as already pointed out by Poincaré, and done for the systems of ordinary linear differential equations with regular singularities in [18, 26, 37, 50], via Picard’s iterative approximation. The differential Galois group of (1.1) is nothing else than the Hausdorff regularization of Lie series in \( \mathcal{L}(\mathbb{C}, X) \) (see Section 5). In this way, on the completion of \( \mathcal{H}(\Omega) \langle X \rangle \), one obtains the so-called Chen series, over \( \omega_0 \) and \( \omega_1 \) along the path \( z_0 \leadsto z \) on \( \Omega \), defined by [9, 33]:

\[ C_{z_0 \leadsto z} := \sum_{w \in X^*} \alpha_{z_0}^z(w)w \in \mathcal{H}(\Omega) \langle X \rangle, \quad (1.2) \]

where \( X^* \) is the free monoid, generated by \( X \) [1, 58] (\( 1_{X^*} \) is the neutral element), \( \alpha_{z_0}^z(1_{X^*}) \) equals \( 1_\Omega \) and, for subdivisions \( (z_0, z_1, \ldots, z_k, z) \) of \( z_0 \leadsto z \) and for \( w = x_{i_1} \cdots x_{i_k} \in X^*X \), the coefficient \( \alpha_{z_0}^z(x_{i_1} \cdots x_{i_k}) \) is defined by

\[ \alpha_{z_0}^z(x_{i_1} \cdots x_{i_k}) := \int_{z_0}^{z} \omega_{i_1}(z_1) \cdots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k) \in \mathcal{H}(\Omega) \quad (1.3) \]

and satisfies the shuffle relation \( \alpha_{z_0}^z(u \omega v) = \alpha_{z_0}^z(u) \alpha_{z_0}^z(v) \), for \( u, v \in X^* \) [11].

By termwise differentiation, the power series \( C_{z_0 \leadsto z_0} \) satisfies (1.1), with initial condition \( C_{z_0 \leadsto z_0} = 1_{X^*} \). From a theorem due to Ree [56], there exists a primitive
series \( L_{z_0 \to z} \in \mathcal{H}(\Omega)(X) \) such that \( e^{L_{z_0 \to z}} = C_{z_0 \to z} \), meaning that \( C_{z_0 \to z} \) is group-like. The challenge is then to determine explicitly \( L_{z_0 \to z} \), via the Magnus’ Lie-integral-functional expansion [54] and to regularize, effectively, \( C_{0 \to 1} \) and \( L_{0 \to 1} \) (although a lot of iterated integrals be divergent). On the other hand, essentially interested in the solutions of (1.1) over the interval \([0, 1]\) and using the involution \( z \mapsto 1 - z \), Drinfel’d stated that (1.1) admits a unique solution \( G_0 \) (resp. \( G_1 \)) satisfying the following asymptotic behaviors [24, 25]:

\[
G_0(z) \sim_0 z^{x_0} \quad \text{and} \quad G_1(z) \sim_1 (1 - z)^{-x_1}.
\]

(1.4)

In particular, since \( G_0 \) and \( G_1 \) are group-like, there is a unique group-like series \( \Phi_{KZ} \in \mathbb{R}\langle\langle X\rangle\rangle \), called the Drinfel’d associator [55] (or Drinfel’d series [34]), such that [24, 25]

\[
G_0 = G_1 \Phi_{KZ}.
\]

(1.5)

Drinfel’d proved also the existence of group-like series in \( \mathbb{Q}\langle\langle X\rangle\rangle \) satisfying similar properties of \( \Phi_{KZ} \), but he neither constructed such an expression nor made explicit \( G_0 \) and \( G_1 \) (similarly for \( \log(G_0), \log(G_1) \) and \( \log(\Phi_{KZ}) \)).

After that, Lê and Murakami expressed, in particular, the divergent coefficients of \( \Phi_{KZ} \) as linear combinations of \( \{ \zeta_r(s_1, \ldots, s_r) \}_{(s_1, \ldots, s_r) \in \mathbb{N}_{\geq 1}, s_1 \geq 2} \) via a regularization based on representation of the chord diagram algebras [52].

One has two ways of considering, for any \( (s_1, \ldots, s_r) \in \mathcal{H}_r \), the quantities \( \zeta_r(s_1, \ldots, s_r) \) as limits fulfilling identities (see Section 3) [13, 16, 46, 47]. Firstly, they are limits at \( z = 1 \) of polylogarithms, and secondly, as truncated sums, they are limits of harmonic sums when the upper bound tends to \( +\infty \):

\[
\text{Li}_{s_1, \ldots, s_k}(z) := \sum_{n_1 > \ldots > n_k > 0} n_1^{-s_1} \ldots n_k^{-s_k} z^{n_1}, \quad \text{for} \quad z \in \mathbb{C}, |z| < 1,
\]

(1.6)

\[
H_{s_1, \ldots, s_k}(n) := \sum_{n_1 > \ldots > n_k > 0} n_1^{-s_1} \ldots n_k^{-s_k}, \quad \text{for} \quad n \in \mathbb{N}_+.
\]

(1.7)

More precisely, if \( (s_1, \ldots, s_r) \in \mathcal{H}_r \), then\(^1\), after a theorem by Abel, one has

\[
\lim_{n \to \infty} \text{Li}_{s_1, \ldots, s_k}(z) = \lim_{n \to \infty} H_{s_1, \ldots, s_k}(n) = \zeta_r(s_1, \ldots, s_k).
\]

(1.8)

This does not hold for \( (s_1, \ldots, s_r) \notin \mathcal{H}_r \), while (1.6) is well defined over \( \{ z \in \mathbb{C}, |z| < 1 \} \) and so are (1.7) as Taylor coefficients of the following function

\[
P_{s_1, \ldots, s_k}(z) := \frac{\text{Li}_{s_1, \ldots, s_k}(z)}{1 - z} = \sum_{n \geq 1} H_{s_1, \ldots, s_k}(n) z^n, \quad \text{for} \quad z \in \mathbb{C}, |z| < 1.
\]

(1.9)

The coefficients in (1.3) are single valued over \( \Omega \); alternatively they can be analytically continued and appear as multivalued functions over \( B := \mathbb{C} - \{0, 1\} \). In fact, we have mappings from the universal cover of \( B \), denoted by \( \tilde{B} \), i.e. we choose a universal covering \( (B, \tilde{B}, p) \), where \( p : \tilde{B} \to B \) is the covering map [9].

This second point of view will be adopted in the sequel. In this respect, let \( \mathcal{H}(B) \) (resp. \( \mathcal{H}(\tilde{B}) \)) denote the ring of holomorphic functions over \( B \) (resp. \( \tilde{B} \)), with \( 1_B : B \to \mathbb{C} \) (resp. \( \tilde{1}_B : \tilde{B} \to \mathbb{C} \)) as the neutral element (\( z \mapsto 1 \)).

\(^1\zeta_1(s_1) \) is nothing else than the Riemann zeta function. It is convenient to set \( \zeta_0 = 1_R \).
Let \( s : \Omega \to \tilde{B} \) be a lifting of the canonical embedding \( j : \Omega \to B \)

\[
\begin{array}{c}
\tilde{B} \\
\downarrow s \\
\Omega \\
\downarrow j \\
B
\end{array}
\]

In particular, for any \( g : B \to B \) and \( x, y \in \tilde{B} \) such that \( g(p(x)) = p(y) \) there exists a unique lifting \( \tilde{g} \) (depending on \( (x, y) \)) such that \( \tilde{g}(x) = y \) and the following commutes [9]

\[
\begin{array}{c}
\tilde{B} \\
\downarrow \tilde{g} \\
\tilde{B} \\
\downarrow p \\
B \\
\downarrow g \\
B
\end{array}
\]

The work presented in this survey will concern our recent results about polylogarithms, harmonic sums and zeta values, involved in the coefficients of \( C_{z_0 \to z} \) and \( L_{z_0 \to z} \) belonging to \( \mathcal{H}(B) \langle \langle X \rangle \rangle \).

We will base our work essentially on

(1) The isomorphisms of the Cauchy and Hadamard algebras of polylogarithmic functions, as defined in (1.6) and (1.9), respectively, with the shuffle \( (\mathcal{C}(X), \omega, 1_X \cdot) \) and the quasi-shuffle algebras \( (\mathcal{C}(Y), \omega, 1_Y \cdot) \) admitting Lyndon words as pure transcendence bases (recalled in Section 2),

(2) The isomorphisms of the bialgebras \( (A(X), \cdot, 1_X \cdot, \Delta_\omega, e) \) and \( (A(Y), \cdot, 1_Y \cdot, \Delta_\omega, e) \) with, respectively, the enveloping algebras of their primitive elements, leading to the constructions of the pairs of bases in duality to factorize the diagonal series thanks to the Cartier-Quillen-Milnor-Moore (CQMM, in short) and Poincaré-Birkhoff-Witt (PBW, in short) theorems (recalled in Section 2),

(3) The use of commutative and noncommutative generating series to establish combinatorial algebraic and analytical aspects of the polylogarithms \( \{\text{Li}_{s_1, \ldots, s_r}\}_{r \geq 1}^{s_1, \ldots, s_r \in \mathbb{N}} \), the harmonic sums \( \{H_{s_1, \ldots, s_r}\}_{r \geq 1}^{s_1, \ldots, s_r \in \mathbb{N}} \), and the zeta functions \( \{\zeta_r(s_1, \ldots, s_r)\}_{r \geq 1}^{s_1, \ldots, s_r \in \mathbb{N}} \) (recalled in Sections 3–5).

In the sequel, for simplification, we will adopt the notation \( \zeta \) for \( \zeta_r, r \in \mathbb{N} \).

We will examine the following problems:

P1. The renormalization which consists of finding counter terms to eliminate the divergence of the polylogarithms \( \{\text{Li}_{s_1, \ldots, s_r}\}_{r \geq 1}^{s_1, \ldots, s_r \in \mathbb{Z}} \) at \( z = 1 \), and of the harmonic sums \( \{H_{s_1, \ldots, s_r}\}_{r \geq 1}^{s_1, \ldots, s_r \in \mathbb{Z}} \) for \( n \to +\infty \) (see Theorems 4.1 and 4.9 below).

For this, a theorem due to Abel is extended to treat, simultaneously, all convergent cases as well as all divergent cases via their generating series.

P2. The regularization which consists of evaluating \textit{analytically} the finite parts (involved in the coefficients of \( C_{0 \to 1} \) and \( L_{0 \to 1} \)) of the singular expansions of the polylogarithms \( \{\text{Li}_{s_1, \ldots, s_r}\}_{r \geq 1}^{s_1, \ldots, s_r \in \mathbb{N}} \) at \( z = 1 \) with respect to the comparison scale \( \{(1-z)^{-a} \log^b(1-z)\}_{a,b \in \mathbb{N}} \), and the asymptotic expansions
of the harmonic sums \( \{H_{s_1, \ldots, s_r}\}_{(s_1, \ldots, s_r) \in \mathbb{N}^r_{\geq 1}} \) for \( n \to +\infty \) in the scales \( \{n^{-a} \log^b(n)\}_{a, b \in \mathbb{N}} \) and \( \{n^{-a} H^b_1(n)\}_{a, b \in \mathbb{N}} \), via *combinatorial* aspects of their noncommutative generating series (see Proposition 5.9 below).

For this, the definition of the regularization characters over the *algebraic bases* of noncommutative polynomial algebras have to be reduced to match with their analytical meanings.

P3. For any multiindex \((-s_1, \ldots, -s_k)\) in \( \mathbb{N}^r_- \), since the polylogarithms (resp. harmonic sums) are polynomial in \( e^{-\log(1-z)} \) for \( |z| < 1 \) (resp. in \( n \in \mathbb{N} \)) with coefficients in \( \mathbb{Z} \) (resp. \( \mathbb{Q} \)) (see Propositions 4.7 and 4.11 below):

\[
\text{Li}_{-s_1, \ldots, -s_k}(z) = \sum_{k=0}^{r+s_1+\ldots+s_k} p_k e^{-k \log(1-z)} = p(e^{-\log(1-z)}),
\]

(1.10)

\[
H_{-s_1, \ldots, -s_k}(n) = \sum_{k=0}^{r+s_1+\ldots+s_k} \frac{p_k}{k!} (n+k)_n = \hat{p}(n).
\]

(1.11)

Hence, \( \text{Li}_{-s_1, \ldots, -s_k}(1) \) (resp. \( H_{-s_1, \ldots, -s_k}(+\infty) \)), as divergent sums, can be regularized (see Lemma 5.12 below) by the value \( p(1) \in \mathbb{Z} \) (resp. \( \hat{p}(1) \in \mathbb{Q} \)) admitting generating series as *rational* associators (see Theorem 5.15 below).

This way, the previous regularizations are extended algebraically (i.e. by transcendent extension over a subalgebra of noncommutative rational series, see Proposition 5.11 below) and analytically (i.e. by evaluation of their finite parts within the comparison scales \( \{(1-z^{-a} \log^b(1-z))\}_{a, b \in \mathbb{N}} \) and \( \{n^{-a} \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}} \), see Lemma 5.12 below), allowing to regularize, in particular, the iterated integrals and their Taylor coefficients associated with the rational series in \( (C[x_1^*], \omega, 1_{X^*}) \) and \( (C[y_1^*], \underline{\omega}, 1_{Y^*}) \), i.e. the following sums with divergent coefficients (see Theorem 5.15 below)

\[
\sum_{n \geq 0} \text{Li}_{1, \ldots, 1, 1}(1) t^n \quad \text{and} \quad \sum_{n \geq 0} H_{1, \ldots, 1, 1}(+\infty) t^n.
\]

P4. For any multiindex \((s_1, \ldots, s_r)\) in \( \mathbb{N}^r_{\geq 1} \), by expanding \((1-z)^{-1}\) the polylogarithms as in (1.6) can be obtained as iterated integrals over the differential forms \( \omega_0 \) and \( \omega_1 \) along the path \( 0 \sim z \) associated with the words \( x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \) over \( X^* x_1 \), as in (1.3). They induce *shuffle* relations while the Taylor coefficients as in (1.7) induce quasi-shuffle relations among convergent zeta values, as obtained in (1.8) (see Theorem 3.1 below).

In fact, the polynomial relations (*homogenous* in weight) over a commutative \( \mathbb{Q}\)-extension, denoted by \( A \), among convergent zeta values, are *relations obtained at singularities* among elements of a transcendence basis of the algebra of polylogarithms (or harmonic sums, see Proposition 4.3 below). These relations are not due to but imply the *double-shuffle relations* and do not need any regularization. Moreover, if Euler’s constant \( \gamma \notin A \), then they are algebraically independent of \( \gamma \) (see Corollary 5.7 below).

\[\text{Here, } Y = \{y_k\}_{k \geq 1} \text{ and } \underline{\omega} \text{ is the quasi-shuffle (or stuffle, for sticky shuffle) product.}\]
The organization of this paper is as follows:

- **In Section 2**, the algebraic combinatorial framework is introduced. In particular, we will give an explicit isomorphism $\varphi_{\pi_1}$ from the shuffle bialgebra to the quasi-shuffle bialgebra (Theorem 2.1).

  Working with $\varphi_{\pi_1}$, the construction by Mélançon-Reutenauer-Schützenberger (MRS, in short), initially elaborated in the shuffle bialgebra and useful to factorize the group-like series and then rational power series (Theorem 2.3), will be extended in the quasi-shuffle bialgebra for the similar factorizations via the constructions of pairs of bases in duality (see (2.6)–(2.7)).

- **In Section 3**, to study their structure via generating series, polylogarithms and harmonic sums at integral multiindices will be encoded by words over various alphabets (Theorems 3.1, 3.2 and Lemmas 2.4–3.5). In particular, the bi-integro differential algebra of polylogarithms will be examined (Proposition 3.6) and their noncommutative generating series will be put in the MRS form (their logarithms will be also provided, Proposition 3.8).

  Concerning the polylogarithms at positive indices, we will insist on the fact that their noncommutative generating series is the actual solution of (1.1), and the noncommutative generating series of the finite parts of their singular expansions corresponds to the associator $\Phi_{KZ}$ which will be also put in MRS form without divergent zeta values as local coordinates.

- **In Section 4**, with noncommutative generating series, the global renormalizations of polylogarithms and harmonic sums will provide *associators* (Theorems 4.1 and 4.9). In particular, using the bridge equations connecting shuffle structures (Propositions 4.2 and 4.3), the enumerable families of *irreducible* zetas values will be implemented (see (4.8)–(4.9)) and Euler’s $\gamma$ constant will be generalized as finite parts of harmonic sums (Corollary 4.6). This will be achieved by identifying the local coordinates in *infinite* dimension and by obtaining algebraic relations among zeta values.

  With commutative generating series, many functions (algebraic functions with singularities in \{0, 1, +\infty\}, see Example 3.3) forgotten in the straight algebra of polylogarithms, at positive indices, will be recovered.

- **In Section 5**, the elements of the differential Galois group $\text{Gal}_C(DE)$ containing the groups of monodromy and of associators will be considered as *regularized solutions* of (1.1). The actions of $\text{Gal}_C(DE)$ on the singular expansions of the solutions of (1.1) will be then discussed (Theorem 5.2): on the one hand, since the group of associators contains itself $\Phi_{KZ}$ and the local coordinates of each associator are homogenous in weight polynomials on zeta values over $A$, the independence of the convergent zeta values with respect to $\gamma$ will be discussed according to $A$ (Corollary 5.7), and $\log(\Phi_{KZ})$ will be also expressed (Proposition 5.9); on the other hand, since the polylogarithms at negative indices are polynomial in $(1 - z)^{-1}$ with coefficients in $\mathbb{Z}$ (Propositions 4.7–4.11), the generating series of the finite parts of their singular expansions will specify the *regularization characters* (Propositions 5.6–5.11) and give examples of *rational* associators (Theorem 5.15).
2. Combinatorial framework

2.1. Shuffle and quasi-shuffle algebras. Let $A$ be a commutative and associative $\mathbb{Q}$-algebra with unit.

Let $X = \{x_0, x_1\}$ (resp. $Y_0 = \{y_s\}_{s \geq 0}$) be an alphabet equipped with the total order $x_0 < x_1$ (resp. $y_0 > y_1 > y_2 > \ldots$) and let $Y = Y_0 - \{y_0\}$. The free monoid generated by $X$ (resp. $Y$, or $Y_0$) is denoted by $X^*$ (resp. $Y^*$, or $Y_0^*$) and admits the empty word, $1_{X^*}$ (resp. $1_{Y^*}$ and $1_{Y_0^*}$) as unit [1].

The sets of polynomials and formal power series over $X^*$ (resp. $Y^*$ or $Y_0^*$) with coefficients in $A$ are denoted respectively by $A\langle X \rangle$ (resp. $A\langle Y \rangle$ or $A\langle Y_0 \rangle$) and $A\langle X \rangle$ (resp. $A\langle Y \rangle$ or $A\langle Y_0 \rangle$) [1]. The sets of polynomials are $A$-modules admitting $\{w\}_{w \in X^*}$ (resp. $\{w\}_{w \in Y^*}$ and $\{w\}_{w \in Y_0^*}$) as linear bases, i.e.

$$A\langle X \rangle \cong A[X^*], \quad A\langle Y \rangle \cong A[Y^*], \quad A\langle Y_0 \rangle \cong A[Y_0^*].$$

Therefore, their full duals are

$$A\langle X \rangle^* = A^{X^*}, \quad A\langle Y \rangle^* = A^{Y^*}, \quad A\langle Y_0 \rangle^* = A^{Y_0^*}$$

and the natural pairing is given by the scalar product

$$\langle S \mid P \rangle = \sum_{u \in Z^*} S(u)P(u) \quad \text{with} \quad Z \in \{X, Y, Y_0\},$$

where, $S(u)$ and $P(u)$ are the coefficients of $u$ in the series $S$ and the polynomial $P$, respectively.

As algebras (see (2.1)) the $A$-modules $A\langle X \rangle$ (resp. $A\langle Y \rangle$ and $A\langle Y_0 \rangle$) come equipped with the associative concatenation product and

1. In $A\langle X \rangle$, the associative commutative shuffle product [11, 27, 56] is defined, for any $u, v, w \in X^*$ and $x, y \in X$, as follows [33]

$$w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad xu \shuffle yv = x(u \shuffle yv) + y(xu \shuffle v),$$

2. In $A\langle Y \rangle$ and $A\langle Y_0 \rangle$, the associative commutative quasi-shuffle product [49] is defined for all $y_i, y_j \in Y_0$ and $u, v, w \in Y_0^*$ as follows [48]

$$w \shuffle 1_{Y_0^*} = 1_{Y_0^*} \shuffle w = w,$n$$y_iu \shuffle y_jv = y_i(u \shuffle y_jv) + y_j(y_iu \shuffle v) + y_{i+j}(u \shuffle v).$$

Their associated coproducts, $\Delta_{\shuffle}$ and $\Delta_{\shuffle'}$, are defined for $u_1, v_1, w_1 \in X^*$ and $u_2, v_2, w_2 \in Y_0^*$ as follows

$$\langle u_1 \shuffle v_1 \mid w_1 \rangle = \langle u_1 \otimes v_1 \mid \Delta_{\shuffle}(w_1) \rangle,$n$$\langle u_2 \shuffle' v_2 \mid w_2 \rangle = \langle u_2 \otimes v_2 \mid \Delta_{\shuffle'}(w_2) \rangle.$$n

These operators are morphisms for the concatenation defined on the letters $x \in X$ and $y_k \in Y_0$ by

$$\Delta_{\shuffle}(x) = 1 \otimes x + x \otimes 1,$n$$\Delta_{\shuffle'}(y_k) = 1 \otimes y_k + y_k \otimes 1 + \sum_{i+j=k} y_i \otimes y_j.$$

The algebras $(A\langle X \rangle, \shuffle, 1_{X^*})$ and $(A\langle Y \rangle, \shuffle', 1_{Y_0^*})$ admit the sets of Lyndon words denoted, respectively, by $\text{Lyn}X$ and $\text{Lyn}Y$, as pure transcendence bases [57].

\[3\text{This coefficient is then } \langle S \mid u \rangle \text{ and } \langle P \mid u \rangle.\]
(resp. [46, 47]). A pair of Lyndon words \((l_1, l_2)\) is called the standard factorization of \(l\) if \(l = l_1 l_2\) and \(l_2\) is the smallest nontrivial proper right factor of \(l\) (for the lexicographic order) or, equivalently, its (Lyndon) longest such [53].

2.2. Diagonal series on bialgebras. Let \(\mathrm{Lie}_A\langle X \rangle\) and \(\mathrm{Lie}_A\langle X \rangle\) denote the sets of, respectively, Lie polynomials and Lie series over \(X\) with coefficients in \(A\) [53, 57]. The CQMM theorem [7] guarantees that the connected \(\mathbb{N}\)-graded, co-commutative Hopf algebra\(^4\) is the enveloping algebra of its primitive elements (hence, \(A(X) = U(\mathrm{Lie}_A\langle X \rangle)\)). Classically, the pair of dual bases, \(\{P_w\}_{w \in X^*}\) expanded over the basis \(\{P_l\}_{l \in \mathrm{Lyn}_X}\) of \(\mathrm{Lie}_A\langle X \rangle\) and \(\{S_w\}_{w \in X^*}\) containing the pure transcendence basis of the shuffle algebra denoted by \(\{S_l\}_{l \in \mathrm{Lyn}_X}\), permits an expression of the diagonal series as follows [57]

\[
\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathrm{Lyn}_X} e^{S_l \otimes P_l}.
\]  

(2.2)

We also get two other connected \(\mathbb{N}\)-graded, co-commutative Hopf algebras isomorphic to the enveloping algebras of their Lie algebras of their primitive elements:

\[
\mathcal{H}_{\langle \rangle} := (A(Y), \left<, Y^*, \Delta_{\langle \rangle}, e \right>) \cong U(\mathrm{Lie}_A(Y)),
\]

\[
\mathcal{H}_{\langle \rangle} := (A(Y), \left<, Y^*, \Delta_{\langle \rangle}, e \right>) \cong U(\mathrm{Prim}(\mathcal{H}_{\langle \rangle})),
\]

where \(\mathrm{Prim}(\mathcal{H}_{\langle \rangle}) = \mathrm{Im}(\pi_1) = \mathrm{span}_A\{\pi_1(w) \mid w \in Y^*\}\) and \(\pi_1\) is the extended eulerian projector defined, for any \(w \in Y^*\), by [46, 47]

\[
\pi_1(w) = w + \sum_{k=2}^{(w)} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} (w \mid u_1 \shuffle \ldots \shuffle u_k) u_1 \ldots u_k.
\]  

(2.3)

Denoting by \((l_1, l_2)\) the standard factorization of \(l \in \mathrm{Lyn}_Y - Y\), let us consider

1. The PBW basis \(\{p_w\}_{w \in Y^*}\) of \(U(\mathrm{Lie}_A(Y))\) constructed recursively as follows [57]

\[
\begin{cases}
p_{y_n} = y_n, & \text{for } y_n \in Y, \\
p_l = [p_{l_1}, p_{l_2}], & \text{for } l \in \mathrm{Lyn}_Y - Y, \; \text{st}(l) = (l_1, l_2), \\
p_w = p_{l_1}^{i_1} \ldots p_{l_k}^{i_k}, & \text{for } w = l_1^{i_1} \ldots l_k^{i_k} \text{ with } l_1, \ldots, l_k \in \mathrm{Lyn}_Y, \; l_1 > \ldots > l_k.
\end{cases}
\]  

(2.4)

2. and, by duality\(^5\), the basis \(\{s_w\}_{w \in Y^*}\) of \((A(Y), \shuffle, 1_{Y^*})\), i.e.

\[
\langle p_u \mid s_v \rangle = \delta_{u,v} \quad \text{for all } u, v \in Y^*.
\]

This linear basis can be computed recursively as follows [57].

\[
\begin{cases}
s_{y_n} = y_s, & \text{for } y_n \in Y, \\
s_l = y_n s_{u}, & \text{for } l = y_n u \in \mathrm{Lyn}_Y, \\
s_w = \frac{s_{\shuffle}^{i_1} \ldots s_{\shuffle}^{i_k} \shuffle^{i_k}}{i_1! \ldots i_k!}, & \text{for } w = l_1^{i_1} \ldots l_k^{i_k} \text{ with } l_1, \ldots, l_k \in \mathrm{Lyn}_Y, \; l_1 > \ldots > l_k.
\end{cases}
\]  

(2.5)

\(^4\)Here, \(e\) denotes the counit defined by \(e(P) = \langle P \mid 1_{X^*} \rangle\) (for any \(P \in \mathrm{A}(Y)\)).

\(^5\)The dual family, i.e. the set of coordinates forms, is linearly free (but not a basis in general) in the algebraic dual which is the space of noncommutative series, but as the enveloping algebra under consideration is graded in finite dimension by multidegree. In fact it consists of multi-homogeneous polynomials.
As in (2.2), let $D_{\mathbb{I}_\mathbb{I}}$ be the diagonal series on $\mathcal{H}_{\mathbb{I}_\mathbb{I}}$. Then [57]

$$D_{\mathbb{I}_\mathbb{I}} := \sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} s_w \otimes p_w = \prod_{l \in \mathcal{L}Y} e^{s_l \otimes p_l}.$$ 

**Theorem 2.1 ([47]).** — Let $\varphi_{\pi_1} : (A(Y), 1_Y) \to (A(Y), 1_Y)$ be the endomorphism of algebras mapping $y_k$ to $\pi_1(y_k)$. Then $\varphi_{\pi_1}$ is an automorphism of $A(Y)$ and it realizes an isomorphism from the bialgebra $\mathcal{H}_{\mathbb{I}_\mathbb{I}}$ to the bialgebra $\mathcal{H}_{\mathbb{U}_\mathbb{U}}$. In particular, the following diagram is commutative

$$\begin{array}{ccc}
Q\langle \bar{Y} \rangle & \xrightarrow{\Delta_{\mathbb{I}_\mathbb{I}}} & Q\langle \bar{Y} \rangle \otimes Q\langle \bar{Y} \rangle \\
\varphi_{\pi_1} & & \varphi_{\pi_1} \otimes \varphi_{\pi_1} \\
Q\langle Y \rangle & \xrightarrow{\Delta_{\mathbb{U}_\mathbb{U}}} & Q\langle Y \rangle \otimes Q\langle Y \rangle
\end{array}$$

and

$$\mathcal{H}_{\mathbb{U}_\mathbb{U}} \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\mathbb{U}_\mathbb{U}})) \quad \text{and} \quad \mathcal{H}_{\mathbb{U}_\mathbb{U}}^\vee \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\mathbb{U}_\mathbb{U}}))^\vee.$$ 

Moreover, the bases $\{\Pi_w\}_{w \in Y^*}$ and $\{\Sigma_w\}_{w \in Y^*}$ of, respectively, $\mathcal{U}(\text{Prim}(\mathcal{H}_{\mathbb{U}_\mathbb{U}}))$ and $\mathcal{U}(\text{Prim}(\mathcal{H}_{\mathbb{U}_\mathbb{U}}))^\vee$, are images by $\varphi_{\pi_1}$ and $\check{\varphi}_{\pi_1}$ of $\{p_w\}_{w \in Y^*}$ and $\{s_w\}_{w \in Y^*}$.

Algorithmically, the families $\{\Pi_w\}_{w \in Y^*}$ and $\{\Sigma_w\}_{w \in Y^*}$ of polynomials homogeneous for the weight can be constructed directly and recursively as follows [3, 46, 47]

(1) The PBW basis $\{\Pi_w\}_{w \in Y^*}$ of $\mathcal{U}(\text{Prim}(\mathcal{H}_{\mathbb{U}_\mathbb{U}}))$:

$$
\begin{cases}
\Pi_{y_s} = \pi_1(y_s), & \text{for } y_s \in Y, \\
\Pi_l = [\Pi_{l_1}, \Pi_{l_2}], & \text{for } l \in \mathcal{L}Y - Y, \text{ st}(l) = (l_1, l_2), \\
\Pi_w = \Pi_{l_1} \cdots \Pi_{l_k}, & \text{for } w = l_1 \cdots l_k \text{ with } l_1, \ldots, l_k \in \mathcal{L}Y, l_1 > \ldots > l_k.
\end{cases}
$$

(2) and, by duality, i.e.

$$\langle \Pi_u | \Sigma_v \rangle = \delta_{u,v} \quad \text{for all } u, v \in Y^*,$$

the basis $\{\Sigma_w\}_{w \in Y^*}$ of $(Q(Y), \mathbb{U}, 1_Y)$:

$$
\begin{cases}
\Sigma_{y_s} = y_s, & \text{for } y_s \in Y, \\
\Sigma_l = \sum_{(s)} \frac{1}{l!} y_{s_{k_1} + \ldots + s_{k_i}} \Pi_{l_1} \ldots \Pi_{l_n}, & \text{for } l = y_{s_1} \ldots y_{s_k} \in \mathcal{L}Y, \\
\Sigma_w = \sum_{(s)} \frac{\Pi_{l_1} \ldots \Pi_{l_k}}{\Pi_{l_1} \ldots \Pi_{l_k}}, & \text{for } w = l_1 \cdots l_k \text{ with } l_1, \ldots, l_k \in \mathcal{L}Y, l_1 > \ldots > l_k.
\end{cases}
$$

(2.7)

In $(s)$, the sum is taken over all $\{k_1, \ldots, k_i\} \subset \{1, \ldots, k\}$ and $l_1 \geq \ldots \geq l_n$ such that $(y_{s_1}, \ldots, y_{s_k}) \searrow (y_{s_{k_1}}, \ldots, y_{s_{k_i}}, l_1, \ldots, l_n)$, where $\searrow$ denotes the transitive closure of the relation on standard sequences, denoted by $\Leftarrow$ [3].

---

In [4], other pairs of bases in duality for $\mathcal{H}_{\mathbb{U}_\mathbb{U}}$ are also proposed.
We have

$$\sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{i \in \text{Lyn}Y} e^{\Sigma_i \otimes \Pi_i}. \quad (2.8)$$

More generally, under suitable conditions these factorizations still hold for the $\varphi$-deformed shuffle product, thanks to an extension of Theorem 2.1 [6, 7, 31].

Now, let us consider the following morphism

$$\pi^0_Y : (A X \ast \oplus A \langle X \rangle x_1, \ldots) \to (A \langle Y \rangle, \ldots),$$

$$x_0^{s_1} \ldots x_0^{s_r} x_1 \mapsto y_s \ldots y_s, \quad \text{for } r \geq 1,$

and $\pi^0_Y(a) = a$ for any $a \in A$. The extension of $\pi^0_Y$ over $A(X)$ is the map $\pi_Y : (A(X), \ldots) \to (A(Y), \ldots)$ satisfying $\pi_Y(p) = 0$ for any $p \in A(X)x_0$. Hence, $\ker \pi_Y = A(X)x_0$ and $\text{Im} \pi_Y = A(Y)$. Let $\pi_X$ be the inverse of $\pi^0_Y$:

$$\pi_X : (A \langle Y \rangle, \ldots) \to (A \oplus A \langle X \rangle x_1, \ldots),$$

$$y_s \ldots y_s \mapsto x_0^{s_1} \ldots x_0^{s_r} x_1, \quad \text{for } r \geq 1.$$

For the scalar products, the projectors $\pi_X$ and $\pi^0_Y$ are then mutually adjoints:

$$\forall p \in A \oplus A \langle X \rangle x_1, \quad \forall q \in A \langle Y \rangle, \quad \langle \pi^0_Y(p) \mid q \rangle = \langle p \mid \pi_X(q) \rangle.$$

We have $\pi_Y \circ \pi_X = \text{Id}_X$. But $\pi_X \circ \pi_Y \neq \text{Id}_Y$. It is an orthogonal projector of $A(X)$ on $A \oplus A \langle X \rangle x_1$ parallel to $A \oplus A \langle X \rangle x_0$. Indeed $\ker(\pi_X \circ \pi_Y) = A(X)x_0$ and $\text{Im}(\pi_X \circ \pi_Y) = A(Y)$.

The map $\pi_X$ is a morphism of associative algebras with unity (AAU) and the map $\pi_Y$ is multiplicative on $A.1_{X_\ast} \oplus A \langle X \rangle x_1$ but not on $A(X)$. For example,

$$0 = \pi_Y(x_0) \pi_Y(x_1) = \pi_Y(x_0 x_1) = \pi^0_Y(x_0 x_1) = y_2.$$

These can be extended by linearity and continuity over $A \langle X \rangle$ and $A \langle Y \rangle$, respectively.

**Lemma 2.2** ([53, 58]). — $l \in \text{Lyn}X - \{x_0 \}$ if and only if $\pi_Y(l) \in \text{Lyn}Y$.

### 2.3. Exchangeable and noncommutative rational series.

Recall that a formal power series $R$ is **exchangeable** if and only if two words have the same coefficient in $R \in A \langle X \rangle$ whenever they have the same commutative image, i.e. for any $u, v \in X^*$, if $|u|_x = |v|_x$ for any $x \in X$ then $\langle R \mid u \rangle = \langle R \mid v \rangle$ [33]. It follows that an exchangeable series $R$ takes the following form [33]

$$R = \sum_{i_0, i_1 \geq 0} r_{i_0, i_1} x_0^{i_0} \omega x_1^{i_1} = \sum_{i_0, i_1 \geq 0} r_{i_0, i_1} x_0^{i_0} \frac{\omega}{i_0!} x_1^{i_1} \frac{\omega}{i_1!}. \quad (2.9)$$

The set of exchangeable series is denoted by $A_{\text{exc}} \langle X \rangle$.

Let $A_\text{rat} \langle X \rangle$ denote the closure of $A \langle X \rangle$ in $A \langle X \rangle$ under $\{+, \cdot, ^*\}$. It is closed under shuffle [1]. A power series $R \in A_\text{rat} \langle X \rangle$ is said to be **rational**.

---

7The set-theoretical object is the same, but the different indexing here expresses the fact that they will be considered as living in different algebras.

8In fact associative commutative dualizable and moderate, see [6, 7, 31].

9Let $R \in A \langle X \rangle$ be such that $\langle R \mid 1_{X_\ast} \rangle = 0$. Then $R^* = 1_{X_\ast} + R + R^2 + \cdots$. 
Let $R \in A^{\text{rat}}\llangle X \rrangle$. By the Kleene-Schützenberger theorem [1] there exists a linear representation $(\beta, \mu, \eta)$ of dimension $n \geq 1$, where
\[
\beta \in \mathcal{M}_{n,1}(A), \quad \mu : X^* \to \mathcal{M}_{n,n}(A), \quad \eta \in \mathcal{M}_{1,n}(A)
\] such that
\[
R = \sum_{w \in X^*} (\beta \mu(w)\eta) w = \beta \left( \sum_{x \in X} \mu(x)x \right)^* \eta.
\]
Hence, letting $M(x) := \mu(x)x$ for $x \in X$, one has $M(X) = M(x_0) + M(x_1)$ as morphism of monoids, and, using Lazard’s elimination [53, 58], one gets
\[
M(X^*) = (M(x_1^*)M(x_0))M(x_1^*) = (M(x_0^*)M(x_1))M(x_0^*).
\]

Via the diagonal series $D_X$ given in (2.2), the Kleene-Schützenberger theorem [1] can also be extended as follows

**Theorem 2.3 ([36, 37, 43]).** — A series $R \in A\llangle X \rrangle$ is rational if and only if there exists a linear representation $(\beta, \mu, \eta)$ of dimension $n \geq 1$, where
\[
\beta \in \mathcal{M}_{n,1}(A), \quad \mu : X^* \to \mathcal{M}_{n,n}(A), \quad \eta \in \mathcal{M}_{1,n}(A)
\] such that
\[
R = \beta((\text{Id} \otimes \mu)D_X)\eta = \beta \left( \prod_{t \in \mathcal{L} \cap \mathcal{M}_n X} e^{\Sigma_i \mu(F_i)} \right)\eta.
\]

Now, let $(\beta, \mu, \eta)$ be a minimal linear representation of $R \in A^{\text{rat}}\llangle X \rrangle$ [1], and let $\mathcal{L}(\mu)$ be the Lie algebra generated by $\{\mu(x)\}_{x \in X}$. Moreover, if the matrices $\{\mu(x)\}_{x \in X}$ are triangular, then there are diagonal and nilpotent matrices, $\{D(x)\}_{x \in X}$ and $\{N(x)\}_{x \in X}$ in $\mathcal{M}_{n,n}(AX)$ such that $M(X) = D(X) + N(X)$. Hence, again by Lazard’s elimination, one also gets
\[
M(X^*) = ((D(X^*)T(X))^*D(X^*)).
\]

The set of exchangeable rational series, i.e. $A^{\text{rat}}\llangle X \rrangle \cap A_{\text{exc}}\llangle X \rrangle$, is denoted by $A^{\text{rat}}_{\text{exc}}\llangle X \rrangle$. As examples, one can consider the following forms $(F_0), (F_1)$ and $(F_2)$ of rational series in $C^{\text{rat}}\llangle X \rrangle$ [35, 36, 37] :

$(F_0)$ \quad $E_1x_i \ldots E_jx_i E_{j+1}$, where $x_i, \ldots, x_i \in X$ and $E_1, \ldots, E_j \in C^{\text{rat}}\llangle x_0 \rrangle$,
$(F_1)$ \quad $E_1x_i \ldots E_jx_i E_{j+1}$, where $x_i, \ldots, x_i \in X$ and $E_1, \ldots, E_j \in C^{\text{rat}}\llangle x_1 \rrangle$,
$(F_2)$ \quad $E_1x_i \ldots E_jx_i E_{j+1}$, where $x_i, \ldots, x_i \in X$ and $E_1, \ldots, E_j \in C^{\text{exc}}\llangle x \rrangle$.

One has

**Lemma 2.4.** — (1) Let $k \in \mathbb{N}_+, t_0, t_1 \in C$. Then $(x_i^*)^{\omega i k} = (kx_i^*)$,
\[
(t_0x_0 + t_1x_1)^* = (t_0x_0)^* \omega (t_1x_1)^* \quad \text{and} \quad (t_1x_1)^* = (t_1x_1)^* \omega (1 - t_1x_1)^{k-1}.
\]

(2) The series of form $(F_0), (F_1)$ and $(F_2)$ generate sub-bialgebras of $(C^{\text{rat}}\llangle X \rrangle, \otimes, 1_X, \Delta_{\text{conc}}, \varepsilon)$.

(3) Let $(\beta, \mu, \eta)$ be a minimal linear representation of $R \in C^{\text{rat}}\llangle X \rrangle$ and $\mathcal{L}(\mu)$ be the Lie algebra generated by $\{\mu(x)\}_{x \in X}$. Since $R = \beta M(X^*)\eta$,
(a) $R$ is a linear combination of expressions of the form $(F_0)$ (resp. $(F_1)$) if and only if $M(x_1^0)M(x_0)$ (resp. $M(x_0^0)M(x_1)$) is nilpotent$^{11}$. Hence, if $R \in \mathbb{C}^{\text{rat}}\langle x_0 \rangle \bowtie \mathbb{C}(X)$ (resp. $\mathbb{C}^{\text{rat}}\langle x_1 \rangle \bowtie \mathbb{C}(X)$) then $M(x_1)M(x_0)$ (resp. $M(x_0^0)M(x_1)$) is nilpotent.

(b) $R$ is a linear combination of expressions of the form $(F_2)$ if and only if $\mathcal{L}(\mu)$ is solvable$^{12}$. Hence, if $R \in \mathbb{C}^{\text{rat}}\langle X \rangle \bowtie \mathbb{C}(X)$ then $\mathcal{L}(\mu)$ is solvable.

(c) $R \in \mathbb{C}(X)$ if and only if for any $P \in \mathcal{L}(\text{ie} \mathbb{C}(X))$ the matrix $\mu(P)$, belonging to $\mathcal{L}(\mu)$, is nilpotent.

(d) $R \in \mathbb{C}^{\text{rat}}\langle X \rangle \Leftrightarrow [\mu(x_0), \mu(x_1)] = 0 \Leftrightarrow R \in \mathbb{C}^{\text{rat}}\langle x_0 \rangle \bowtie \mathbb{C}^{\text{rat}}\langle x_1 \rangle$.

To end this section, let us note that for any $R \in \mathbb{C}^{\text{rat}}\langle X \rangle$ of minimal linear representation $(\beta, \mu, \eta)$ of dimension $n$ and, for any $x, y \in X$ one has

$$\langle S \mid xy \rangle = \beta \mu(x)\mu(y)\eta = \sum_{i=1}^{n} (\beta \mu(x)e_i)(e_i^T \mu(y)\eta) = \sum_{i=1}^{n} \langle S_i^{(1)} \mid x \rangle \langle S_i^{(2)} \mid y \rangle,$$

where $e_i$ is the vector such that $e_i^T = (0 \ldots 0 1 0 \ldots 0)$. Hence $S_i^{(1)}$ (resp. $S_i^{(2)}$) admits $(\beta, \mu, e_i)$ (resp. $(e_i^T, \mu, \eta)$) as a linear representation, and

$$\langle \mathbb{C}^{\text{rat}}\langle X \rangle, \omega, 1_{X^*}, \Delta_{\text{conc}}, e \rangle$$

is nothing but the Sweedler dual of the bialgebra $(\mathbb{C}(X), \text{conc}, 1_{X^*}, \Delta_{\text{conc}}, e)$$^{[57]}$.

3. INDEXATION BY WORDS AND GENERATING SERIES

3.1. Indexation by words. For any $r \in \mathbb{N}$, any multiindex $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ can be associated with the words $x_1^{s_1-1} \ldots x_1^{s_r-1}x_1 \in X^*x_1 \sqcup \{1_{X^*}\}$. Similarly, any$^{13} (s_1, \ldots, s_r) \in \mathbb{N}^r$ can be associated with the word $y_{s_1} \ldots y_{s_r} \in Y_0^*$. Put $\text{Li}_{x_0^r}(z) := (\log(z))^r/r!$.

(1) Let $\text{Li}_{x_1^{s_1}, \ldots, s_k}$ and $H_{s_1, \ldots, s_k}$ be indexed by words $[38, 39]$:

$$\text{Li}_{x_1^{s_1}, \ldots, s_k} := \text{Li}_{s_1, \ldots, s_k} \quad \text{and} \quad H_{y_{s_1}, \ldots, y_{s_r}} := H_{s_1, \ldots, s_r}.$$

(2) Let $\text{Li}_{y_{s_1}, \ldots, s_k}$ and $H_{s_1, \ldots, s_k}$ be indexed by words $[21, 22]$:

$$\text{Li}_{y_{s_1}, \ldots, s_k} := \text{Li}_{s_1, \ldots, s_k} \quad \text{and} \quad H_{y_{s_1}, \ldots, s_k} := H_{s_1, \ldots, s_k}.$$

In particular, $\text{Li}_{y_{s_1}, \ldots, s_k}(z) := (z/(1 - z))^r$ and $H_{y_{s_1}, \ldots, s_k}(n) := (n)^r = (n)_r/r!$, where $(n)_r = (n + r) \ldots (n)$.

All of $\{\text{Li}_{w}^\gamma \}_{w \in Y_0^*}$ and $\{H_w^\gamma \}_{w \in Y_0^*}$ are divergent at their singularities.

Theorem 3.1 ([41, 38, 42]). — (1) The following morphisms of algebras are injective (and surjective by definition)

$$\text{H}_w : (Q(Y), \omega, 1_{Y^*}) \to (Q\{H_w^\gamma \}_{w \in Y^*}, \times, 1), \quad w \mapsto \text{H}_w,$$

$$\text{Li}_w : (Q(X), \omega, 1_{X^*}) \to (Q\{Li_w^\gamma \}_{w \in X^*}, \times, 1_{\mathcal{B}}), \quad w \mapsto \text{Li}_w.$$

(2) The families $\{\text{H}_w \}_{w \in Y^*}$ and $\{\text{Li}_w \}_{w \in X^*}$ are $Q$-linearly independent.

$^{11}$Using (2.10), one gets the expected expression for $R$.

$^{12}$By Lie's theorem [15], using (2.11), one gets the expected expression for $R$.

$^{13}$The weight of $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ (resp. $\mathbb{N}^r$) is defined as the integer $s_1 + \ldots + s_r$ which corresponds to the weight, denoted $(w)$, of its associated word $w \in Y^*$ (resp. $Y_0^*$) and, if $w \in Y^*$, it corresponds also to the length, denoted by $|u|$, of its associated word $u \in X^*$. 
The families \( \{H_i\}_{i \in \mathcal{L}Y} \) and \( \{Li_i\}_{i \in \mathcal{L}X} \) are \( \mathbb{Q} \)-algebraically independent.

But at the singularities \( \{1, +\infty\} \), for any \( u \in x_0 X^* x_1 \) (resp. \( u \in Y^* - y_1 Y^* \)) \( Li_u \) (resp. \( H_u \)) receives the value \( \zeta(v) := Li_v(1) \) (resp. \( \zeta(u) := H_u(+\infty) \)) and are no more linearly independent (and then the values \( \{H_i(+\infty)\}_{i \in \mathcal{L}Y} - \{y_1\} \) (resp. \( \{Li_i(1)\}_{i \in \mathcal{L}X} \)) are no longer algebraically independent) [38, 40, 59].

There also exists a law of algebra, denoted by \( \top \), in \( \mathbb{Q}\langle Y_0 \rangle \) (which is not dualizable) [6, 31] such that

\[ \text{ker } H^{-\top} = \text{ker } Li^{-\top} = \mathbb{Q}\langle \{w - w^\top 1Y_0^* | w \in Y_0^*\} \rangle \] and the families \( \{H_{y_k}\}_{k \geq 0} \) and \( \{Li_{y_k}\}_{k \geq 0} \) are \( \mathbb{Q} \)-linearly independent.

Moreover, let \( \top' : \mathbb{Q}\langle Y_0 \rangle \times \mathbb{Q}\langle Y_0 \rangle \to \mathbb{Q}\langle Y_0 \rangle \) be a law such that \( Li^{-\top'} \) is a morphism for \( \top' \) and \( (1Y_0^* \top Q\langle Y_0 \rangle) \cap \ker(Li^{-\top'}) = \{0\} \). Then \( \top' = g \circ \top \), where \( g \in GL(\mathbb{Q}\langle Y_0 \rangle) \)

is such that \( Li^{-\top'} \circ g = Li^{-\top} \).

Now, for any \( i \in \mathbb{N} \) let \( t_i \in \mathbb{C} \) be such that \( |t_i| < 1 \) and \( z \in \mathbb{C} \) satisfying \( |z| < 1 \). Then [35] (to be compared with (1.4) and (1.5))

\[
\sum_{n \geq 0} Li_{x_0^n \top} (z) t_0^n = z^{t_0} \quad \text{and} \quad \sum_{n \geq 0} Li_{x_1^n \top} (z) t_1^n = (1 - z)^{-t_1}.
\]

(3.1)

What precedes suggests to extend the domain of \( Li \), which is, up to now and through linear extension, restricted to \( \mathbb{C}\langle X \rangle \), to some rational series as follows.

### 3.2. Indexation by noncommutative rational series

Let us call \( \text{Dom}(Li_\bullet) \) the set of series of \( \mathbb{C}\langle X \rangle \)

\[
S = \sum_{n \geq 0} S_n \quad \text{with} \quad S_n := \sum_{|w| = n} \langle S \mid w \rangle w
\]

such that the following sum converges uniformly on all compacts of \( \hat{B} \)

\[
\sum_{n \geq 0} Li_{S_n}.
\]

(3.2)

One can check easily that [22] :

- The set \( \text{Dom}(Li_\bullet) \) is closed under shuffle products.
- For any \( S, T \in \text{Dom}(Li_\bullet) \) one has \( Li_{S \shuffle T} = Li_S Li_T \).
- One has \( \mathbb{C}\langle X \rangle \shuffle \mathbb{C}_{\text{rat}}\langle x_0 \rangle \shuffle \mathbb{C}_{\text{rat}}\langle x_1 \rangle \subset \text{Dom}(Li_\bullet) \).

This extension is compatible with identities between rational series such as Lazard’s elimination [53, 58], for instance (see Appendix C) :

\[
Li_S(z) = \sum_{n \geq 0} \langle S \mid x_0^n \rangle \frac{\log^n(z)}{n!} + \sum_{k \geq 1} \sum_{w \in x_0^\bullet \shuffle x_1^k} \langle S \mid w \rangle Li_w(z),
\]
and explains that, for \( R \) as given in (2.9), \( \text{Li}_R \) is expressible as analytic composition of \( \log(z) \) and \( \log(1 - z) \):

\[
\text{Li}_R(z) = \sum_{i_0, i_1 \geq 0} \frac{r_{i_0, i_1}}{i_0! i_1!} \log^{i_0}(z) (-\log(1 - z))^{i_1}.
\]

**Example 3.3.** — Consider the extension of \( \text{Li}_* \) defined in (3.2). Then [35, 36, 37]

1. By (3.1), \( \text{Li}_{(t_0 x_0)^*} (z) = z^{i_0} \) and \( \text{Li}_{(t_1 x_1)^*} (z) = (1 - z)^{-t_1} \). More generally, for any \( i, j \in \mathbb{N}_+ \), one has by Lemma 2.4

\[
\text{Li}_{((t_0 x_0)^* \cup i \cup (t_1 x_1)^* \cup j)} (z) = z^{i_0} (1 - z)^{-j_1},
\]

\[
\text{Li}_{(t_0 x_0 + t_1 x_1)^* \cup x_0 \cup x_1^*} (z) = z^{i_0} \frac{\log^i(z) \log^j((1 - z)^{-1})}{i! j!}.
\]

2. For \( a \in \mathbb{C} \) and \( i \in \mathbb{N}_+ \), one has by Lemma 2.4

\[
\text{Li}_{(a x_0)^*} (z) = z^a \sum_{k=0}^{i-1} \left( \frac{(a \log(z))^k}{k!} \right), \quad (3.3)
\]

\[
\text{Li}_{(a x_1)^*} (z) = \frac{1}{(1 - z)^{a}} \sum_{k=0}^{i-1} \left( \frac{(-a \log(1 - z))^k}{k!} \right). \quad (3.4)
\]

3. From the previous points, one has (see Lemma 2.4)

\[
\{ \text{Li}_S \}_{S \in \mathbb{C}[x_0^*] \cup \mathbb{C}(-x_0^*) \cup \mathbb{C}[x_1^*]} = \text{span}_C \{ z^a (1 - z)^{-b} \}_{a \in \mathbb{Z}, b \in \mathbb{N}},
\]

\[
\{ \text{Li}_S \}_{S \in \mathbb{C}^{\text{rat}}(x_0) \cup \mathbb{C}^{\text{rat}}(x_1)} = \text{span}_C \{ z^a (1 - z)^{b} \}_{a, b \in \mathbb{C}},
\]

\[
\{ \text{Li}_S \}_{S \in \mathbb{C}(x) \cup \mathbb{C}[x_0^*] \cup \mathbb{C}(-x_0^*) \cup \mathbb{C}[x_1^*]} = \text{span}_C \left\{ \frac{z^a}{(1 - z)^b} \text{Li}_w(z) \right\}_{a \in \mathbb{Z}, b \in \mathbb{N}}
\]

\[
\subset \text{span}_C \{ \text{Li}_{i_1, \ldots, i_r} \} \}_{i_1, \ldots, i_r \in \mathbb{Z}^r},
\]

\[
\oplus \text{span}_C \{ z^a | a \in \mathbb{Z} \},
\]

\[
\{ \text{Li}_S \}_{S \in \mathbb{C}(x) \cup \mathbb{C}^{\text{rat}}(x_0) \cup \mathbb{C}^{\text{rat}}(x_1)} = \text{span}_C \left\{ \frac{z^a}{(1 - z)^b} \text{Li}_w(z) \right\}_{a, b \in \mathbb{C}}
\]

\[
\subset \text{span}_C \{ \text{Li}_{i_1, \ldots, i_r} \} \}_{i_1, \ldots, i_r \in \mathbb{C}^r},
\]

\[
\oplus \text{span}_C \{ z^a | a \in \mathbb{C} \},
\]

4. For any \( (s_1, \ldots, s_r) \in \mathbb{N}_+^r \) and \( |t_i| < 1 \), let

\[
W = (t_1 x_0)^{s_1} x_0^{s_1 - 1} x_1 \ldots (t_r x_0)^{s_r} x_0^{s_r - 1} x_1
\]

(which is of the form\(^{14}\) (\( F_0 \)) of Lemma 2.4). Then\(^ {15}\)

\[
\text{Li}_W(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{(n_1 - t_1)^{s_1} \ldots (n_r - t_r)^{s_r}}.
\]

\(^{14}\)For the form (\( F_0 \)) one can apply Theorems 2.3 and 2.4 of [35].

\(^{15}\)This holds for \( t_i \in \mathbb{C} - \mathbb{N}_+, i \in \mathbb{N} \), by analytic continuation [39].
In particular, for \( s_1 = \ldots = s_r = 1 \) one has
\[
\text{Li}(t_1 x_0) \ast x_1 (t_r x_0) \ast x_1 = \sum_{n_1, \ldots, n_r > 0} \text{Li}_{k_0}^{n_0-1} x_{n_0} \ast x_{n_0}^{n_0-1} x_1 \ast x_1^{n_1-1} \ast x_1^{n_1-1} \ast \ldots \ast x_1^{n_1-1} = \sum_{n_1 > \ldots > n_r > 0} (n_1 - t_1) \ldots (n_r - t_r).
\]

Let \( \partial_z := d/dz \) and let us recall that, for any \( k \geq 1 \),
\[
\frac{1}{(1 - z)^k} = \frac{\partial_z^{k-1}}{(k-1)!} \left( \frac{1}{1 - z} \right) \quad \text{and} \quad \frac{1}{z^k} = (-1)^{k-1} \frac{\partial_z^{k-1}}{(k-1)!} \left( \frac{1}{z} \right)
\]
and the Taylor coefficients of \((1 - z)^{-k}\) are expressed as follows for all \( n \geq 1 \)
\[
\langle (1 - z)^{-k} | z^n \rangle = \Gamma^{-1}(k)(n + k - 1)_{k-1}.
\]

Let \( \mathcal{G} \) denote the group of transformations of \( \mathbf{16} \) \( B \) generated by \( \{ z \mapsto 1 - z, z \mapsto 1/z \} \), permuting the singularities in \( \{ 0, 1, +\infty \} \) as a copy of \( \mathfrak{S}_3 \).

Let us also consider the differential rings
\[
\begin{align*}
\mathcal{C}_0 &= \mathbb{C}[z^{-1}], & \mathcal{C}_1 &= \mathbb{C}[1 - z^{-1}], & C_0 &= \mathbb{C}[z, z^{-1}], \\
\mathcal{C}_1 &= \mathbb{C}[z, (1 - z)^{-1}], & C' &= \mathbb{C}[z^{-1}, (1 - z)^{-1}], & C &= \mathbb{C}[z, z^{-1}, (1 - z)^{-1}]
\end{align*}
\]
(considered as subrings of \( \mathcal{H}(B) \)). It follows that

**Lemma 3.4.** —
1. The differential ring \( \mathcal{C} \) is closed under the action of \( \mathcal{G} \), i.e. \( G(g(z)) \in \mathcal{C} \) for all \( G \in \mathcal{C} \) and \( g \in \mathcal{G} \).
2. For any \( G = p_1(z) + p_2(1 - z^{-1}) + p_3((1 - z)^{-1}) \in \mathcal{C} \), with \( p_1, p_2, p_3 \in \mathbb{C}[z] \), \( p_2(0) = p_3(0) = 0 \) and \( p_2, p_3 \neq 0 \). Letting \( G_0(z) := P_2(z^{-1}) \in \mathcal{C}'_0 \) and \( G_1(z) := P_3((1 - z)^{-1}) \in \mathcal{C}'_1 \), one has \( G(z) \sim_0 G_0(z) \) and \( G(z) \sim_1 G_1(z) \).
3. The following morphisms of algebras is surjective
\[
\lambda : (\mathcal{C}[x_0^*, (-x_0)^*, x_1^*], \omega, 1_X) \longrightarrow (\mathcal{C}, x, 1_B), \quad R \mapsto \text{Li}_R.
\]
Moreover, \( \ker(\lambda) \) is the shuffle-ideal generated by \( x_0^* \omega x_1^* - x_1^* + 1 \).
4. The following morphisms of algebras are bijective
\[
\begin{align*}
\lambda' : (\mathcal{C}[x_0^*, x_1^*], \omega, 1_X) &\longrightarrow (\mathcal{C}', x, 1_B), \quad R \mapsto \text{Li}_R, \\
\lambda'_i : (\mathcal{C}[x_i^*], \omega, 1_X) &\longrightarrow (\mathcal{C}'_i, x, 1_B), \quad R \mapsto \text{Li}_R \quad \text{for } i = 0, 1.
\end{align*}
\]

In fact, one has

**Lemma 3.5** ([21]). —
1. The family \( \{ x_0^*, x_1^* \} \) is algebraically independent over \( (\mathcal{C}(X), \omega, 1_X) \) in \( (\mathcal{C}(X), \omega, 1_X) \). In particular, the power series \( x_0^* \) and \( x_1^* \) are transcendent over \( \mathcal{C}(X) \).
2. The module \( (\mathcal{C}(X), \omega, 1_X) \omega x_0^*, x_1^* \) is \( \mathcal{C}(X) \)-free and the family \( \{ (x_0^*)^k \omega (x_1^*)^l \}_{k,l}^{\in \mathbb{Z} \times \mathbb{N}} \) forms a \( \mathcal{C}(X) \)-basis of it.
   Hence, \( \{ w \omega x_0^*, \omega x_1^* \}_{w \in X^*}^{\in \mathcal{C}(X)} \) is a \( \mathcal{C} \)-basis of it.
3. One has \( \mathcal{C}^{\text{rat}} \langle \langle x_i \rangle \rangle = \text{span}_{\mathcal{C}} \{ (t x_i) \omega x_i \mid t \in \mathcal{C} \} \) for any \( x_i \in X \).

\(^{16}\text{Any } g \in \mathcal{G} \text{ maps bijectively } B \text{ to itself, one can apply the Monodromy Principle to lift } \mathcal{G} \text{ as a group of transformations of } B.\)
Now, let us also consider the following differential integration operators acting on $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ [46]:

$$\theta_0 := z\partial_z \quad \text{and} \quad \theta_1 := (1 - z)\partial_z,$$

$$\forall f \in \mathcal{C}, \quad \iota_0(f) = \int_{z_0}^{z} f(s)\omega_0(s) \quad \text{and} \quad \iota_1(f) = \int_{0}^{z} f(s)\omega_1(s).$$

The operator $\iota_0$ is well-defined on $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$. (see Definition 6.5 in Appendix D, where the choice of $z_0$ is recalled). One can check easily

**Proposition 3.6** ([22, 38, 41]). — (1) The operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ satisfy

$$\theta_0 + \theta_1 = [\theta_1, \theta_0] = \partial_z \quad \text{and} \quad \theta_{k\iota_k} = \text{Id} \quad \text{for} \quad k = 0, 1,$$

$$[\theta_0\iota_1, \theta_1\iota_0] = 0 \quad \text{and} \quad (\theta_0\iota_1)(\theta_1\iota_0)(\theta_0\iota_1) = \text{Id}.$$

(2) The subspace $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ is closed under the action of $\{\theta_0, \theta_1\}$ and $\{\iota_0, \iota_1\}$. Thus, for any $w = y_{s_1} \ldots y_{s_r} \in Y^*$ (whence $\pi_X(w) = x_{s_1}^{-1}x_1 \ldots x_{s_r}^{-1}x_1$) and $u = y_{t_1} \ldots y_{t_r} \in Y^*$, the functions $\text{Li}_w$ and $\text{Li}_u$ satisfy

$$\text{Li}_w = (\theta_0^{s_1 - 1} \iota_1 \ldots \theta_0^{s_r - 1} \iota_1)1\Omega \quad \text{and} \quad \text{Li}_u = (\theta_0^{s_1 + 1} \iota_1 \ldots \theta_0^{-1} \iota_1)1\Omega,$$

$$\theta_0 \text{Li}_{x_{s_1} \ldots x_{s_r}}(w) = \text{Li}_{x_{0s_1} \ldots x_{0s_r}}(w) \quad \text{and} \quad \iota_1 \text{Li}_w = \text{Li}_{x_1 x_{s_1} \ldots x_{s_r}}(w),$$

$$\theta_0 \text{Li}_{x_{s_1} \ldots x_{s_r}}(w) = \text{Li}_{x_{s_1} x_{s_2} \ldots x_{s_r}}(w) \quad \text{and} \quad \iota_1 \text{Li}_{x_{1s_1} \ldots x_{1s_r}}(w) = \text{Li}_{x_{s_1} x_{s_2} \ldots x_{s_r}}(w).$$

(3) The bi-integro differential ring $(\mathcal{C}\{\text{Li}_w\}_{w \in X^*}, \theta_0, \iota_0, \theta_1, \iota_1)$ is stable under the action of $\mathcal{G}$, i.e. for all $h \in \mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ and $g \in \mathcal{G}$

$$h(g(z)) \in \mathcal{C}\{\text{Li}_w\}_{w \in X^*}.$$

(4) $\theta_0 \iota_1$ and $\theta_1 \iota_0$ are scalar operators in $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$, respectively with eigenvalues $\lambda := z \rightarrow z(1 - z)$ and $1/\lambda$. I.e. for all $f \in \mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ one has

$$(\theta_0 \iota_1)f = \lambda f \quad \text{and} \quad (\theta_1 \iota_0)f = (1/\lambda)f.$$

### 3.3. Noncommutative generating series

The graphs (typed as series) of the isomorphisms of algebras, $\text{Li}_\bullet$ and $\text{H}_\bullet$, defined in Theorem 3.1, then become

**Definition 3.7** ([13, 38, 40, 42]). — Let us consider the following power series

$$L := \sum_{w \in X^*} \text{Li}_w \quad w \quad \text{and} \quad H := \sum_{w \in Y^*} \text{H}_w \cdot w.$$

With suitable structures (topological ring [8]), by (2.2) and (2.8), one can write $H = (\text{H}_\bullet \otimes \text{Id}_Y)\mathcal{D}_{\text{Li}_w}$ and $L = (\text{Li}_\bullet \otimes \text{Id}_X)\mathcal{D}_{\text{x}}$. Thus, by Theorem 3.1, one obtains

**Proposition 3.8** ([38, 40, 46, 47]). — One has

$$\Delta_{\text{Li}_w}(H) = H \otimes H \quad \text{and} \quad \langle H \mid 1_{Y^*} \rangle = 1,$$

$$\Delta_{\text{Li}_w}(L) = L \otimes L \quad \text{and} \quad \langle L \mid 1_{X^*} \rangle = 1,$$

$$H = \prod_{l \in \mathcal{L}_{Y^*}} e^{\text{H}_{\Sigma,l} \cdot \Pi_l} \quad \text{and} \quad L = \prod_{l \in \mathcal{L}_{X^*}} e^{\text{Li}_{\Sigma,l} \cdot P_l}.$$
Hence\(^\text{17}\), their logarithms are primitive, for the corresponding co-products, and\(^\text{18}\)

\[
\log(H) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\mathcal{u} \in \gamma_{Y^+}} H_{u_1 \mathcal{u} \cdots u_k} u_1 \cdots u_k,
\]

\[
\log(L) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{\mathcal{u} \in \gamma_{X^+}} Li_{u_1 \cdots u_k} u_1 \cdots u_k.
\]

One can then set the following:

**Definition 3.9.** — Let us consider the following power series

\[
Z_{\mathcal{u}} := \prod_{l \in \mathcal{L} \forall n \neq \{y_1\}} e^{\mathcal{H}_i L (\infty) \Pi l} \quad \text{and} \quad Z_{\mathcal{u}} := \prod_{l \in \mathcal{L} \forall n \neq \{y_1\}} e^{\Pi l}.
\]

By termwise differentiation, the power series \(L\) given in Definition 3.7 satisfies the noncommutative differential equation (1.1) and, via the factorization form given in Proposition 3.8, it also satisfies the boundary condition \([38, 41]\)

\[
L(z) \sim \log(1 - z) \quad \text{and} \quad L(z) \sim_{\mathcal{u}} e^{x_0 \log(z)}.
\]

Equation (1.8) and Theorem 3.1 lead to

**Definition 3.10.** — We define \(\zeta\) to be the following polymorphism (which is surjective by definition):

\[
\zeta : (Q_1 \otimes \cdots \otimes_{\mathcal{u}} Q_n, Y^*, \Sigma^*) \rightarrow (Z, x, 1),
\]

\[
x_0 x_1^{-1} \cdots x_n^{-1} y_{s_1} \cdots y_{s_k} \rightarrow n_1^{-s_1} \cdots n_k^{-s_k},
\]

where \(Z\) is the \(Q\)-algebra (algebraically) generated by \(\zeta(l)\) \(l \in \mathcal{L} \forall n \neq \{y_1\}\) (resp. \(\zeta(S_l)\) \(l \in \mathcal{L} \forall n \neq \{y_1\}\)) or, equivalently, \(\{\zeta(l)\} \in \mathcal{L} \forall n \neq \{y_1\}\) (resp. \(\{\zeta(S_l)\} \in \mathcal{L} \forall n \neq \{y_1\}\)).

### 4. Global Asymptotic Behaviors at Singularities

#### 4.1. The case of positive multi-indices

The analysis of singularities on the coefficients of the noncommutative generating series of \(\{Li_{\mathcal{u}}\}_{w \in \mathcal{X}^*}\), put in the factorized form (see Proposition 3.8) leads to\(^\text{20}\) \([38, 41]\)

\[
\lim_{z \rightarrow 0} L(z) e^{-x_0 \log z} = 1 \quad \text{and} \quad \lim_{z \rightarrow 1} e^{x_1 \log(1-z)} L(z) = Z_{\mathcal{u}}.
\]

Knowing that \(G_0\) and \(G_1\), as interpreted in (1.4), are unique and by (1.5), it turns out that, through the interpretation given, \(Z_{\mathcal{u}}\) coincides with \(\Phi_{KZ}\) \([34, 55]\) and, via an identity of type Newton-Girard \([51]\), we obtain \([14, 16, 45]\)

\[
H(n) \sim_{+\infty} \sum_{k \geq 0} H_{y_1} y_1^k \pi_Y(Z_{\mathcal{u}}) \quad \text{and} \quad \sum_{k \geq 0} H_{y_1} y_1^k = e^\sum_{k \geq 1} H_{y_1}^k (n-y_1)^k/k.
\]

\(^\text{17}\) via Friedrich’s criterion \([57]\) and its extension \([46, 47]\).

\(^\text{18}\) From \(\log(L)\), one can extract the expression of the eulerian projector on \(H_{\mathcal{u}}\) \([57]\) and similarly, from \(\log(H)\), for the extended eulerian projector, as given in (2.3), on \(H_{\mathcal{u}}\) \([46, 47]\).

\(^\text{19}\) We will describe relations among \(\{\zeta(S_l)\}_{l \in \mathcal{L} \forall n \neq \{y_1\}}\) (resp. \(\{\zeta(S_l)\}_{l \in \mathcal{L} \forall n \neq \{y_1\}}\)) by local coordinate identification in Section 4.2.

\(^\text{20}\) i.e. \(L(z) \sim_{0} z^{x_0}\) and \(L(z) \sim_{1} (1-z)^{-x_1} Z_{\mathcal{u}}\).
In other terms, we have the following global renormalization

**Theorem 4.1** (First Abel like theorem, [14, 16, 45]). —

$$\lim_{z \to 1} e^{y_1 \log(1-z)} \pi_Y(L(z)) = \lim_{n \to \infty} e^{\sum_{k \geq 1} H_{w_k}(n)(-y_1)^k / k} H(n) = \pi_Y(Z_{\Delta_{\pi_Y}}).$$

Thus, the coefficients \( \{ (Z_{\Delta_{\pi_Y}}) \}_{u \in Y^*} \) (i.e. \( \{ \zeta_{\Delta_{\pi_Y}}(u) \}_{u \in Y^*} \)) and \( \{ (Z_{\Delta_{\pi_Y}}) \}_{v \in Y^*} \) (i.e. \( \{ \zeta_{\Delta_{\pi_Y}}(v) \}_{v \in Y^*} \)) represent, respectively, the finite part of the singular expansion, in the comparison scale \( \{ (1 - z)^{-a} \log^b(1 - z) \}_{a,b \in \mathbb{N}}, \) of \( \text{Li}_w \) at \( z = 1 \)

\[
\text{f.p.} z \to 1 \text{Li}_w(z) = \zeta_{\Delta_{\pi_Y}}(w), \quad \{ (1 - z)^{-a} \log^b((1 - z)^{-1}) \}_{a \in \mathbb{Z}, b \in \mathbb{N}},
\]

and the asymptotic expansion, in \( \{ n^{-a} \text{H}_1^b(n) \}_{a,b \in \mathbb{N}}, \) of \( H_w \) for \( n \to +\infty : \)

\[
\text{f.p.} n \to +\infty \text{H}_w(n) = \zeta_{\Delta_{\pi_Y}}(w), \quad \{ n^{-a} \text{H}_1^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\]

For commodity, we will denote

\[
\text{F.P.} z \to 1 L(z) = Z_{\Delta_{\pi_Y}}, \quad \{ (1 - z)^{-a} \log^b((1 - z)^{-1}) \}_{a \in \mathbb{Z}, b \in \mathbb{N}},
\]

\[
\text{F.P.} n \to +\infty H(n) = Z_{\Delta_{\pi_Y}}, \quad \{ n^{-a} \text{H}_1^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\]

On the other hand, by a transfer theorem [32], let \( \{ \gamma_w \}_{w \in Y^*} \) be the finite part of an asymptotic expansion, in \( \{ n^{-a} \log^b(n) \}_{a,b \in \mathbb{N}}, \) of \( \{ H_w \}_{w \in Y^*} \) for \( n \to +\infty : \)

\[
\text{f.p.} n \to +\infty H_w(n) = \gamma_w, \quad \{ n^{-a} \log^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\]

Then let \( Z_\gamma \) be the noncommutative generating series of \( \{ \gamma_w \}_{w \in Y^*} \). One has

\[
\text{F.P.} n \to +\infty H(n) = Z_\gamma := \sum_{w \in Y^*} \gamma_w w, \quad \{ n^{-a} \log^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\]

**Proposition 4.2** ([46, 47]). — (1) The following map is a character

\[
\gamma : (Q(Y), \Delta_{\pi_Y}, 1_{Y^*}) \to (Z[\gamma], \times, 1), \quad w \mapsto \gamma_w.
\]

(2) Equivalently, one has \( \Delta_{\pi_Y}(Z_\gamma) = Z_\gamma \otimes Z_\gamma \) and \( \langle Z_\gamma | 1_{Y^*} \rangle = 1. \) Hence,

\[
Z_\gamma = e^{y_1 \sum_{l \in \mathcal{L}Y_{\pi_Y} - \{ y_1 \}} e^{\zeta_l} \Pi_l} = e^{y_1 Z_{\Delta_{\pi_Y}}}
\]

and \( \Delta_{\pi_Y}(\log(Z_\gamma)) = \log(Z_\gamma) \otimes 1_{Y^*} + 1_{Y^*} \otimes \log(Z_\gamma). \) It follows then

\[
\log(Z_\gamma) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^*} \gamma u_1 u_2 \ldots u_k u_1 \ldots u_k.
\]

The asymptotic behaviors on (4.2) and Proposition 4.2 lead to

**Proposition 4.3** (Bridge equation, [14, 16, 45, 46, 47]). — Put \( B(y_1) = \exp \left( \gamma y_1 - \sum_{k \geq 2} \frac{\zeta(k)}{k} (-y_1)^k \right) \quad \text{and} \quad B'(y_1) = \exp \left( -\sum_{k \geq 2} \frac{\zeta(k)}{k} (-y_1)^k \right). \)

Then \( Z_\gamma = B(y_1) \pi_Y(Z_{\Delta_{\pi_Y}}), \) or equivalently by cancellation, \( Z_{\Delta_{\pi_Y}} = B'(y_1) \pi_Y(Z_{\Delta_{\pi_Y}}). \)

---

\(^{21}\) The power series \( B(y_1) \) corresponds to the Taylor expansion of \( \Gamma^{-1}(y_1 + 1). \)
4.2. Structure of polyzetas. Now, via Proposition 4.3, let us draw some consequences about the structure of polyzetas: by local coordinates identification in the assertions of Proposition 4.3, one obtains two families of polynomials, homogenous for the weight, \( \{Q_l\}_{l \in \mathcal{L}^n X - x} \) and \( \{Q_l\}_{l \in \mathcal{L}^n Y - \{y\}} \) (see Example 6.2 in Appendix A), such that \([46, 47]\)

\[
\mathcal{R}_X := (Q\{Q_l\}_{l \in \mathcal{L}^n X - x}, 1_X) = \ker(\zeta),
\]

(resp. \( \mathcal{R}_Y := (Q\{Q_l\}_{l \in \mathcal{L}^n Y - \{y\}}, 1_Y) = \ker(\zeta) \))

describing the kernel of \( \zeta \) (see Example 6.1 in Appendix A), via homogenous polynomial relations for the weight, among the local coordinates of \( Z_{ij} \) (resp. \( Z_{ij} \)), i.e. the convergent values\(^{22} \{\zeta(S_l)\}_{l \in \mathcal{L}^n X - x} \) (resp. \( \{\zeta(S_l)\}_{l \in \mathcal{L}^n Y - \{y\}} \)).

Denoting \( X \) the alphabet \( X \) or \( Y \), this local coordinate identification yields algebraic generator systems (see Example 6.3 in Appendix A) as irreducible\(^{23} \) local coordinates (see Example 6.4 in Appendix A)

\[
\mathcal{L}^\leq_{irr}(X) := \lim_{p \to +\infty} \mathcal{Z}^\leq_{irr}(X) \quad \text{with} \quad \mathcal{Z}^\leq_{irr}(X) \subset \ldots \mathcal{Z}^\leq_{irr}(X) \subset \ldots, \quad (4.8)
\]

such that the restriction of \( \zeta \) on \( \mathcal{Q}[\mathcal{L}^\infty_{irr}(X)] \) is bijective [46, 47], where (see Example 6.4 in Appendix A)

\[
\mathcal{L}^\infty_{irr}(X) := \lim_{p \to +\infty} \mathcal{L}^\leq_{irr}(X) \quad \text{with} \quad \mathcal{L}^\leq_{irr}(X) \subset \ldots \mathcal{L}^\leq_{irr}(X) \subset \ldots, \quad (4.9)
\]

and, for any \( p \geq 2, \mathcal{L}^\leq_{irr}(X) \) is the inverse image of \( \mathcal{Z}^\leq_{irr}(X) \).

Generated by homogenous polynomials for the weight (see Example 6.2 in Appendix A), \( \ker(\zeta) \) is then graded. Moreover, since \( \mathcal{Z} = \text{Im}(\zeta) \), one obtains

**Corollary 4.4** ([46, 47]). — One has

\[
\mathcal{Z} \cong Q1_X + x_0 Q(X)x_1 / \ker(\zeta),
\]

(resp. \( \mathcal{Z} \cong Q1_Y + (Y - \{y\})Q(Y)/ \ker(\zeta) \)).

Hence, \( \mathcal{Z} \) is graded as the quotient of a graded algebra by a graded ideal :

\[
\mathcal{Z} = Q1 + \bigoplus_{p \geq 2} \mathcal{Z}^p,
\]

where for any \( p \geq 2,

\[
\mathcal{Z}^p = \text{span}_Q\{\zeta(w)|w \in x_0 X^*x_1, |w| = p\},
\]

(resp. \( \mathcal{Z}^p = \text{span}_Q\{\zeta(w)|w \in (Y - \{y\})Y^*, (w) = p\} \)).

**Remark 4.5.** — Note that \( \mathcal{L}^n X \) is totally ordered, and so is \( \mathcal{L}^\infty_{irr}(X) \), as being extracted from \( \mathcal{L}^n X \). Hence, for any fixed integer \( n \geq 1 \), it is immediate that

1. letting \( l \in \mathcal{L}^n X \) such that \( (l) = n \), one has \( y_n \leq l \) (resp. \( x_0^{n-1}x_1 \leq l \)),
2. \( \Sigma y_n = y_n \in \mathcal{L}^n Y \) and \( S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \mathcal{L}^n X \) (see Lemma 2.2),
3. \( \Sigma y_{2n+1} = y_{2n+1} = \mathcal{L}^\infty_{irr}(Y) \) and \( S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}^\infty_{irr}(X) \),
4. \( \zeta(2) = \zeta(S_{x_0x_1}) = \zeta(S_{x_0x_1}) \) is irreducible and, by Euler’s identity about the ratio \( \zeta(2k)/\pi^{2k} \), one deduces that \( \Sigma y_{2k} = y_{2k} \notin \mathcal{L}^\infty_{irr}(Y) \) and \( S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}^\infty_{irr}(X) \).

\(^{22}\)Identification allows to obtain homogenous polynomial relations up to weights \( 12 [5] \).

\(^{23}\)by means of rewriting the system.
Note also that for any \( l_1 \in \mathcal{L}_Y - \{y_1\} \) and \( l_2 \in \mathcal{L}_n X - X \) one has in general \([46]\)
\[
\zeta(\pi_X(S_{l_1})) \neq \zeta(S_{\pi_X(l_1)}) \quad \text{and} \quad \zeta(\pi_Y(S_{l_2})) \neq \zeta(S_{\pi_Y(l_2)}),
\]
while this does not occur, due again to Lemma 2.2, for the values \( \{\zeta(l)\}_{l \in \mathcal{L}_Y Y - \{y_1\}} \) (or \( \{\zeta(l)\}_{l \in \mathcal{L}_n X - X} \)) \([2, 40, 38, 28]\).

With the first assertion of Proposition 4.3, we compute the generalized Euler constants, \(i.e.\) the finite parts of divergent harmonic sums \( \{H_w\}_{w \in Y^*} \):

**COROLLARY 4.6** \([14, 16, 45]\). — For any \( k \geq 1 \) and \( w \in Y^* - y_1 Y^* \), one has
\[
\gamma_w y_1^k = \sum_{s_1 + \ldots + s_k > 0} \frac{(-1)^k}{s_1! \ldots s_k!} (-\gamma)^{s_1} \left( -\frac{\zeta(2)}{2} \right)^{s_2} \ldots \left( -\frac{\zeta(k)}{2} \right)^{s_k},
\]
\[
\gamma_w y_k^k = \sum_{i=0}^k \frac{\zeta(x_0 \{ (\pi_X w) \})}{i!} \left( \sum_{j=1}^i b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots) \right),
\]
where the \( b_{n,k}(t_1, \ldots, t_k) \)'s are Bell polynomials.

See also Corollary 5.7, for the independence of \( \gamma \) with respect to the convergent polyzetas.

4.3. **The case of negative multi-indices.** Similarly, asymptotic behaviors of \( \{Li_w\}_{w \in Y^*}, \{H_w\}_{w \in Y^*} \) are analyzed by

**PROPOSITION 4.7** \([21]\). — For any \( n \in \mathbb{N}_+, \ z \in \mathbb{C} \) with \( |z| < 1 \) and \( w \in Y^*_0 \), \( H_w \) and \( Li_w \) are polynomial, of degree \( (w) + |w| \) in \( \mathbb{Q}[n] \) and \( \mathbb{Z}[1 - z]^{-1} \), respectively. Hence, for any \( w \in Y^*_0 \), there exists \( C_w \in \mathbb{Q} \) and \( B_w \in \mathbb{N} \), such that
\[
H_w(n) \sim_{+, \infty} n^{(w) + |w|} C_w^* \quad \text{and} \quad Li_w(z) \sim_{1} (1 - z)^{-(w) - |w|} B_w^*.
\]
Moreover, one has
\[
C_w^* = \prod_{w \in X^*_0, \ |w| \neq 1} ((w) + |v|)^{-1} \quad \text{and} \quad B_w^* = ((w) + |w|)! C_w^*.
\]

**PROPOSITION 4.8** \([21]\). — Let us consider the following generating series
\[
L^- := \sum_{w \in Y^*_0} Li_w^- w, \quad H^- := \sum_{w \in Y^*_0} H_w^- w, \quad C^- := \sum_{w \in Y^*_0} C_w^- w.
\]
Then
\[
\langle H^- | 1_{Y^*_0} \rangle = \langle C^- | 1_{Y^*_0} \rangle = 1, \Delta_{\text{all}} (H^-) = H^- \otimes H^- \quad \text{and} \quad \Delta_{\text{all}} (C^-) = C^- \otimes C^-.
\]
Moreover, analysis of singularities leads to the following global renormalization.

**THEOREM 4.9** (Second Abel like theorem, \([21]\)). — One has
\[
\lim_{z \to 1} h_{-1} (1 - z)^{-1} \otimes L^- (z) = \lim_{n \to +\infty} g_{-1} (n) \otimes H^- (n) = C^-,
\]

\(24\)for which polynomial relations homogenous for the weight are obtained via double shuffle, up to weights 10 \([40, 38], 12 [2] \) and \(16 [28] \).

\(25\)The series \( C^- \) is group-like in \( (Q\langle Y_0 \rangle, \text{conc}, 1_{Y^*_0}, \Delta_{\text{all}}, \otimes) \).
where the noncommutative generating series\(^{26}\) \(h\) and \(g\) are defined as follows
\[
h(t) = \sum_{w \in Y_0^*} ((w)+ |w|)t^{(w)+|w|} \quad \text{and} \quad g(t) = \left( \sum_{y \in Y_0} t^{(y)+1}y \right)^*.
\]

Now, by Proposition 4.7 and the Taylor expansion, one deduces

**Corollary 4.10.** — For any \(w \in Y_0^*\) there exists a unique polynomial \(p \in (Z[t], \times, 1)\) of degree \((w)+ |w|\) such that\(^{27}\), via (3.5), for any \(n \in \mathbb{N}_+\) and \(z \in \mathbb{C}\) with \(|z| < 1\) one has
\[
\begin{align*}
\text{Li}_w^{-}(z) &= \sum_{k=0}^{(w)+|w|} \frac{p_k}{(1-z)^k} = \sum_{k=0}^{(w)+|w|} p_k e^{-k \log(1-z)} \in (Z[e^{- \log(1-z)}], \times, 1_B), \\
H_w^{-}(n) &= \sum_{k=0}^{(w)+|w|} p_k \binom{n+k}{k} = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} (n+k)_n \in (Q[(n+\bullet)_n], \times, 1),
\end{align*}
\]

where \((n+\bullet)_n : \mathbb{N} \rightarrow Q\) maps \(i\) to \((n+i)_n = (n+i)!/n!\) and \(Q[(n+\bullet)_n]\) denotes the set of polynomials in \(n\) expanded as follows
\[
\forall \pi \in Q[(n+\bullet)_n], \quad \text{deg}(\pi) = d, \quad \pi = \sum_{i=0}^{d} \pi_k (n+i)_n = \sum_{i=0}^{d} \pi_k \frac{(n+i)!}{n!}.
\]
By Corollary 4.10, denoting by \(\hat{p}\) the exponential transform of \(p\), one also has
\[
\begin{align*}
\text{Li}_w^{-}(z) &= p(e^{- \log(1-z)}), \quad \text{with} \quad p(t) = \sum_{k=0}^{(w)+|w|} p_k t^k \in (Z[t], \times, 1), \quad (4.10) \\
H_w^{-}(n) &= \hat{p}((n+\bullet)_n), \quad \text{with} \quad \hat{p}(t) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} t^k \in (Q[t], \times, 1). \quad (4.11)
\end{align*}
\]
Let us then associate also \(p\) and \(\hat{p}\) with the polynomial\(^{28}\) \(\hat{p}\) obtained as follows
\[
\hat{p}(t) = \sum_{k=0}^{(w)+|w|} k! p_k t^k = \sum_{k=0}^{(w)+|w|} p_k t^{k+1} \in (Z[t], \omega, 1). \quad (4.12)
\]
Next, the previous polynomials \(p, \hat{p}\) and \(\hat{p}\) given in (4.10)–(4.12) can be determined explicitly thanks to Lemma 3.4 and to

**Proposition 4.11 ([21]).** —

\(1\) The following morphism of algebras is bijective
\[
\chi : (Q[y^*], {}^\omega, 1_Y^\omega) \rightarrow (Q[(n+\bullet)_n], \times, 1), \quad S \mapsto H_S.
\]

---

\(^{26}\)Note that \(g\) can be viewed as an “exponential transform” of \(h\) :
\[
g(t) = \sum_{w \in Y_0^*} t^{(w)+|w|}w = \sum_{w \in Y_0^*} \frac{\langle h \mid w \rangle}{((w)+ |w|)!}w.
\]

\(^{27}\)In other terms, for any word \(w\) belonging to \(Y_0^*\) and integer \(k\) verifying \(0 \leq k \leq (w)+ |w|\), such that \(\langle \text{Li}_w^{-} \mid (1-z)^{-k} \rangle = k!(\text{H}_w^{-} \mid (n)_n)\). One verifies in particular, for Proposition 4.7, that \(\langle \text{H}_w^{-} \mid (n)_n \rangle = C_w^{-}\) and \(\langle \text{Li}_w^{-} \mid (1-z)^{-|w|-k} \rangle = ((w)+ |w|)!C_w^{-}\).

\(^{28}\)In other words, \(p\) is the exponential transform of \(\hat{p}\) and, for any integer \(k\) with \(0 \leq k \leq (w)+ |w|\) one has \(\langle \hat{p} \mid z^k \rangle = k!(\langle p \mid z^k \rangle = (k!)^2 \langle \hat{p} \mid z^k \rangle\).
For any \( w = y_{s_1}, \ldots, y_{s_r} \in Y_0^* \), there exists a unique polynomial \( R_w \) belonging to \((\mathbb{Z}[x_1^*], \omega, 1_{X^*})\) of degree \( (w)+|w| \), such that\(^{29}\)
\[
\Lambda R_w(z) = \Lambda w(z) = p(e^{-\log(1-z)}) \in (\mathbb{Z}[e^{-\log(1-z)}], \times, 1_B),
\]
\[
H_{\pi_Y(R_w)}(n) = H_w(-n) = \hat{p}(n + \bullet)_n \quad \in (\mathbb{Q}[(n + \bullet)_n], \times, 1).
\]
In particular, via the extension by linearity of \( R \) on \( \mathbb{Q}[Y_0] \) and Theorem 3.2, \( \{\Lambda R_{y_k}\}_{k \geq 0} \) is linear independent in \( \mathbb{Q}\{\Lambda R_w\}_{w \in Y_0^*} \) and for all \( k, l \in \mathbb{N} \)
\[
\Lambda R_{y_k} \sqcup \Lambda R_{y_l} = \Lambda R_{y_k} \sqcup \Lambda R_{y_l} = \Lambda R_{y_k \sqcup y_l} = \Lambda R_{y_k \sqcup y_l}.
\]
(3) For any \( w \in Y_0^* \), there exists a unique polynomial \( R_w \in (\mathbb{Z}[x_1^*], \omega, 1_{X^*}) \) of degree \( (w)+|w| \) such that \( \hat{p}(x_1^*) = R_w \).
(4) More explicitly, for any \( w = y_{s_1}, \ldots, y_{s_r} \in Y_0^* \), there exists a unique polynomial \( R_w \) belonging to \((\mathbb{Z}[x_1^*], \omega, 1_{X^*})\) of degree \( (w)+|w| \), given by
\[
R_{y_{s_1}, \ldots, y_{s_r}} = \sum_{k_1 = 0}^{s_1} \sum_{k_2 = 0}^{s_2-k_1} \cdots \sum_{k_r = 0}^{(s_1+\cdots+s_r)-(k_1+\cdots+k_{r-1})} \binom{s_1+k_1}{k_2} \binom{s_1+s_2-k_1}{k_2} \cdots \binom{s_1+\cdots+s_r-k_1-\cdots-k_{r-1}}{k_r} \rho_{k_1 \sqcup \cdots \sqcup \rho_{k_r}},
\]
where, for any \( i = 1, \ldots, r \), one has, if \( k_i = 0 \) then \( \rho_{k_i} = x_1^* - 1_{X^*} \) else
\[
\rho_{k_i} = \frac{S_2(k_i, j)(j!)^2}{(j!)^2} \left( \sum_{k=0}^{j-l} (-1)^l \binom{j-l+1}{l} \right),
\]
and the \( S_2(k, j)'s \) denote the Stirling numbers of second kind.

Using Proposition 4.11 and Lemma 3.4 (in particular, the bijectivity of the restriction \( \Lambda_1 : (\mathbb{Z}[x_1^*], \omega, 1_{X^*}) \to (\mathbb{Z}[e^{-\log(1-z)}], \omega, 1_{X^*}) \)) and also the Stirling numbers (of first and second kinds), one obtains

**Corollary 4.12. — The morphism of algebras**

\[
R : (\mathbb{Z}[Y_0], \sqcup, 1_{Y_0^*}) \to (\mathbb{Z}[x_1^*], \omega, 1_{X^*})
\]
is bijective, mapping \( y_0 \mapsto x_1^* - 1_{X^*} \) and \( y_k \mapsto x_1^* \sqcup R'_{y_k} \) \( (k \geq 1) \), where
\[
R'_{y_k} = \sum_{i=0}^{k} i!S_2(k, i)(x_1^* - 1)^{\omega \cdot i} = \sum_{i=0}^{k} \sum_{j=0}^{i} i!S_2(k, i) \binom{i}{j} (-1)^{i-j}(x_1^*)^{\omega \cdot j}
\]
and \( R' \) is extended over \( \mathbb{Z}[Y] \) by linearity. Conversely, one has for any \( k \geq 1 \),
\[
(kx_1^*)^* = 1_{X^*} + R_{y_0} + \sum_{j=2}^{k} S_1(k, j) \frac{R_{y_{j+1}}}{(k-1)!}.
\]
It follows that \( \Lambda R_{y_k} \sqcup \Lambda R_{y_l} = \Lambda S \) \( (k, l \geq 1) \), where
\[
S = x_1^* \sqcup R'_{y_k \sqcup y_l} = (1_{X^*} + R_{y_0}) \sqcup (R'_{y_{k+1}} + R'_{y_k \sqcup y_l}).
\]

\(^{29}\)Recall also that the map \( \pi_Y \) is multiplicative on \( \mathbb{Q} \oplus \mathbb{Q}(X)x_1 \) but not on \( \mathbb{Q}(X) \).
To end this section, let us recall also that, for any \( c \in \mathbb{C} \), one has
\[
(n)_c \sim_{+\infty} n^c = e^{c \log(n)}
\]
and, with the respective scales of comparison (on the right hand side), one has the following finite parts
\[
\begin{align*}
\text{f.p.}_{z \to 1} c \log(1 - z) &= 0, \quad \{(1 - z)^a \log^b((1 - z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \\
\text{f.p.}_{n \to +\infty} c \log n &= 0, \quad \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\end{align*}
\]

5. A GROUP OF ASSOCIATORS

5.1. The action of the Galois differential group.

**Lemma 5.1 ([43, 44]).** — Let \( G \) and \( H \) be solutions of (1.1) which are group-like for \( \Delta_{\pm} \). Then there exists \( C \in \mathcal{L}ie_{\mathbb{C}}\langle X \rangle \), independent of \( z \), such that \( G = H e^C \).

Typically, with the notations of (1.2) and Definition 3.7, the power series \( C_{z_0 \to z} \) and \( L(z) \) satisfy the differential equation (1.1) and have the same value at \( z = z_0 \). Then \( C_{z_0 \to z} = L(z)(L(z_0))^{-1} \) [38, 41]. Since \( C_{z_0 \to z} \) and \( L(z) \) are group-like, so is \( L(z_0) \). It follows that the Hausdorff group, i.e., \( \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle X \rangle\} \), plays the rôle of the differential Galois group of the equation (1.1). More precisely,

**Theorem 5.2 ([43, 44]).** — \( \text{Gal}_{\mathbb{C}}(DE) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle X \rangle\} \).

**Definition 5.3 ([46, 47]).** — Let \( A \) be a subring of \( \mathbb{C} \), containing \( \mathbb{Q} \). We put
\[
dm(A) := \{Z_{\pm} e^C \mid C \in \mathcal{L}ie_A\langle X \rangle, \langle e^C \mid x_0 \rangle = \langle e^C \mid x_1 \rangle = 0\}.
\]

Then \( \text{dm}(A) = \text{Gal}_{\mathbb{C}}^{\geq 2}(DE) \) is a strict normal subgroup of \( \text{Gal}_{\mathbb{C}}(DE) \).

Now, for any \( e^C \in \text{Gal}_{\mathbb{C}}(DE) \), let
\[
\tilde{L} := L e^C \quad \text{and} \quad \tilde{Z}_{\pm} := Z_{\pm} e^C.
\]

Then, by the global analysis of singularities in (4.1), the action of \( e^C \) on \( L \) on the right yields the asymptotic behavior of \( \tilde{L} \) [46, 47]
\[
\tilde{L}(z) \sim_0 e^{x_0 \log z} e^C \quad \text{and} \quad \tilde{L}(z) \sim_1 e^{-x_1 \log(1 - z)} \tilde{Z}_{\pm}
\]
and, via an identity of type Newton-Girard again [51], one also gets :
\[
\Pi(n) \sim_{+\infty} e^{-\sum_{k \geq 1} H_{y_k}(n)(-y_k)^k/k} \pi_Y(Z_{\pm}).
\]

In other words, we obtain the extended Abel like theorem [46, 47]
\[
\lim_{z \to 1} e^{y_1 \log(1 - z)} \pi_Y(\tilde{L}(z)) = \lim_{n \to +\infty} e^{-\sum_{k \geq 1} H_{y_k}(n)(-y_k)^k/k} \Pi(n) = \pi_Y(Z_{\pm}).
\]

By (4.1) and (5.2), one then deduces

**Corollary 5.4.** — \( L \) is the unique solution of (DE) satisfying \( L(z) \sim_0 e^{x_0 \log(z)} \) (i.e., for \( e^C = 1_X^* \)). It follows that \( \Phi_{KZ} = Z_{\pm} \) is unique.

---

30 This group contains the group \( DM(A) \) introduced in [10, 55] (\( DM \) for double mélange).

31 Note that since (see [38, 41]) \( (Z_{\pm} \mid x_0) = (Z_{\pm} \mid x_1) = 0 \), by identification of the coefficients one has \( (Z_{\pm} \mid x_1) = (e^C \mid x_1) \) and \( (Z_{\pm} \mid x_0) = (e^C \mid x_0) \) which are not 0.
\textbf{Proposition 5.5} ([46, 47]). — Let \( \{ \tau_w \}_{w \in Y^*} \) be the finite parts of the asymptotic expansions of \( \{ \Pi_w \}_{w \in Y^*} \) in \( \{ n^{-a} \log^b (n) \}_{a,b \in \mathbb{N}} \), and let \( \mathcal{Z}_\gamma \) be their noncommutative generating series. Then

\[
\mathcal{Z}_\gamma := \sum_{w \in Y^*} \tau_w w, \quad \Delta_{\|} (\mathcal{Z}_\gamma) = \mathcal{Z}_\gamma \otimes \mathcal{Z}_\gamma, \quad \langle \mathcal{Z}_\gamma | 1_{Y^*} \rangle = 1.
\]

In other words, the following map is a character

\[
\tau_{\bullet} : (\mathbb{Q}(Y), \|, 1_{Y^*}) \to (\mathcal{Z}_\gamma, \times, 1), \quad w \mapsto \tau_w.
\]

\textbf{Proposition 5.6} (Extended bridge equation, [46, 47]). — Under the action of the group \( \text{Gal}_C(\mathbb{D}E) \), one gets\textsuperscript{32}

\[
\mathcal{Z}_{\|} = F.P. z \to \mathcal{L}(z), \quad \{(1 - z)^a \log^b ((1 - z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}},
\]

\[
\mathcal{Z}_{\|} = F.P. n \to \mathcal{H}(n), \quad \{ n^a \mathcal{H}^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}},
\]

\[
\mathcal{Z}_\gamma = F.P. n \to \mathcal{H}(n), \quad \{ n^a \log^b (n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\]

Moreover, by Proposition 5.5, the extension of MRS factorization and the extended Abel like theorem lead to \( \mathcal{Z}_\gamma = e^{\gamma_1} \mathcal{Z}_{\|} \). Hence, for any \( \mathcal{Z}_{\|} \in \text{dm}(A) \), by cancellation and with expressions of \( B, B' \) given in Proposition 4.3, one obtains

\[
\mathcal{Z}_\gamma = B(y_1) \pi_Y (\mathcal{Z}_{\|}) \iff \mathcal{Z}_{\|} = B'(y_1) \pi_Y (\mathcal{Z}_{\|}).
\]

Elements of the group \( \text{dm}(A) \) satisfying similar properties as \( \Phi_{KZ} \) are called associators\textsuperscript{33}, as regularized solutions of \( \mathbb{D}E \) [46, 47]. Moreover, by the identification of local coordinates in the second point of Proposition 5.6, one gets

\textbf{Corollary 5.7} ([46, 47]). — If \( \gamma \notin A \) then \( \gamma \) is transcendent over the \( A \)-algebra generated by convergent zeta values.

\textbf{Remark 5.8}. — As example of the action of the differential Galois group on the singular expansions, we are interested in the action of their monodromy group\textsuperscript{34} [46] generated by \( e^{2 \pi i m_0} \) and \( e^{2 \pi i m_1} \), where [41, 38]

\[
m_0 = x_0 \quad \text{and} \quad m_1 = Z_{\|} e^{-2 \pi i x_1} Z_{\|}^{-1} = \prod_{l \in \mathbb{C} \frac{\mathbb{Z}}{\mathbb{Z}}} e^{-\zeta(S_l) \alpha_1 P_l (-x_1)}.
\]

By Proposition 4.3 and (5.1), the actions of the monodromy group on the right of \( Z_{\|} \) and \( Z_\gamma \) are the following

\textsuperscript{32} Note that, once the scales of comparison are fixed, the coefficients \( \{(\mathcal{Z}_{\|} | w)\}_{w \in x_0 X^* x_1}, \{(\mathcal{Z}_{\|} | w)\}_{w \in (Y^* - \{y_1\}) Y^*} \) and \( \{(\mathcal{Z}_\gamma | w)\}_{w \in (Y^* \setminus \{y_1\}) Y^*} \), as finite parts of the asymptotic expansions of \( \{(\mathcal{L} | w)\}_{w \in x_0 X^* x_1}, \{(\mathcal{L} | w)\}_{w \in (Y^* - \{y_1\}) Y^*} \) and \( \{(\Pi | w)\}_{w \in (Y^* \setminus \{y_1\}) Y^*} \), are determined, by the extended Abel like theorem.

\textsuperscript{33} In [34, 55], associators (or Drinfel’d series) are defined as group-like series in \( \mathbb{R}[\langle X \rangle] \) satisfying a system of algebraic relations (duality, pentagonal and hexagonal), but the authors do not produce any associator other than \( \Phi_{KZ} \), which was completely determined earlier in [40, 38] (without divergent zeta values as local coordinates).

\textsuperscript{34} A proof of linear independence of multi-valued polylogarithms is obtained via this monodromy group. It can be also proved by use of the differential Galois group [12, 43, 44].

An other proof for mono-valued polylogarithm functions, as a special case of hyperlogarithms, can be also obtained over functions field [17, 19].
(1) If \( e^C = e^{2i\pi m_0} \) then \( Z_{\omega} = Z_{\omega} e^{2i\pi x_0} \) and
\[
Z_\gamma = \exp \left( \gamma y_1 - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right) \pi y Z_{\omega} = Z_\gamma.
\]

This means that the monodromy at 0 of \( L \) consists of the multiplication on the right of \( Z_{\omega} \) by \( e^{2i\pi x_0} \) and does not modify \( Z_\gamma \).

(2) If \( e^C = e^{2i\pi m_0} \) then \( \bar{Z}_{\omega} = e^{-2i\pi x_1} Z_{\omega} \) and
\[
\bar{Z}_\gamma = \exp \left( \gamma - 2i\pi y_1 - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right) \pi y Z_{\omega} = e^{-2i\pi y_1} Z_\gamma.
\]

This means that the monodromy at 1 of \( L \) consists of the multiplication on the left of \( Z_{\omega} \) and \( Z_\gamma \) by, respectively, \( e^{-2i\pi x_1} \) and \( e^{-2i\pi y_1} \).

Finally, the actions of the monodromy group on \( L \) does not allow, in this case, neither to introduce \( e^{\gamma x_1} \) on the left of \( Z_{\omega} \) nor to eliminate the left factor \( e^{\gamma y_1} \) of \( Z_\gamma \) [46, 47].

5.2. **Associated \( \Phi_{KZ} \).** Now, let us examine some properties of the noncommutative generating series \( Z_{\omega} \) and \( Z_{\omega} \), *i.e.* \( \Phi_{KZ} \) (see Corollary 5.4).

In a way similar to what was said of the character \( \gamma \) (see Proposition 4.2), Definition 3.9 and Proposition 3.10 lead to

**Proposition 5.9** ([14, 16, 46, 47]). — One has
\[
\langle Z_{\omega} | 1_{Y^{-}} \rangle = \langle Z_{\omega} | 1_{X^{-}} \rangle = 1
\]
and
\[
\Delta_{\omega}(Z_{\omega}) = Z_{\omega} \otimes Z_{\omega}, \quad \Delta_{\omega} \left( \log(Z_{\omega}) \right) = \log(Z_{\omega}) \otimes 1_{Y^{-}} + 1_{Y^{-}} \otimes \log(Z_{\omega}),
\]
\[
\Delta_{\omega}(Z_{\omega}) = Z_{\omega} \otimes Z_{\omega}, \quad \Delta_{\omega} \left( \log(Z_{\omega}) \right) = \log(Z_{\omega}) \otimes 1_{X^{-}} + 1_{X^{-}} \otimes \log(Z_{\omega}),
\]
and
\[
\log(Z_{\omega}) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \zeta_{\omega}(u_1 \omega \cdots \omega u_k) u_1 \cdots u_k,
\]
\[
\log(Z_{\omega}) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in X^+} \zeta_{\omega}(u_1 \omega \cdots \omega u_k) u_1 \cdots u_k.
\]

Moreover, the polymorphism \( \zeta \) can be extended as follows
\[
\zeta_{\omega} : (Q(X), \omega, 1_{X^{-}}) \rightarrow (Z, x, 1), \quad \zeta_{\omega} : (Q(Y), \omega, 1_{Y^{-}}) \rightarrow (Z, x, 1),
\]
according to its products and satisfying, for any \( l \in \mathcal{L}yn Y - \{ y_1 \} \),
\[
\zeta_{\omega}(\pi_X(l)) = \zeta_{\omega}(l) = \gamma_l = \zeta(l).
\]

and, for the generators of length (resp. weight) one, for \( X^* \) (resp. \( Y^* \))
\[
\zeta_{\omega}(x_0) = 0 = f.p_{x \rightarrow z} \text{Li}_x(z), \quad \{ (1 - z)^a \log^b ((1 - z)^{-1}) \}_{a \in \mathbb{Z}, b \in \mathbb{N}},
\]
\[
\zeta_{\omega}(y_1) = 0 = f.p_{n \rightarrow + \infty} \text{H}_y(n), \quad \{ n^a \text{H}_y^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\]

By Lazard’s elimination, the free Lie algebra \( \mathfrak{Lie}_A(X) \), as an \( A \)-module, is the direct sum of \( A x_0 \) and of a Lie ideal, denoted by \( J \) and freely generated by \( \{ \text{ad}^{l_0}_{x_0} x_1 \} \). Then, by the calculations in Appendix B and by the identities
\[
(x_0 \cup x_1)^* = (x_0^* x_1)^* x_0^* \quad \text{and} \quad e^{x_0} x_1 e^{-x_0} = e^{\text{ad}_{x_0} x_1},
\]
one has
Proposition 5.10 (Gradation of $L$ and $Z_{\sqcup \sqcup}$, [43, 44]). — Let the operation $\circ$ be defined, for any $l \in \mathbb{N}$ and $P \in C \langle X \rangle$, by $x_1 x_0^l \circ P = x_1 (x_0^l \circ P)$. Then

$$L(z) = \sum_{k \geq 0} \sum_{w \in z^* \cup x_1^k} L_i w(z)$$

$$= e^{x_0 \log(z)} \left( 1 X^* + \sum_{k \geq 1} \sum_{l_1, \ldots, l_k \geq 0} L_i x_1 x_0^l \cdots x_1 x_0^k (z) \prod_{i=1}^k \text{ad}^l_{x_0} x_1 \right)$$

$$= \sum_{k \geq 0} \int_0^z \omega_1(t) \cdots \int_0^{t_{k-1}} \omega_1(t_1) \kappa_k(z, t_1, \ldots, t_k),$$

$$Z_{\sqcup \sqcup} = \sum_{k \geq 0} \sum_{l_1, \ldots, l_k \geq 0} \zeta_{\sqcup \sqcup} (x_1 x_0^l \circ \cdots \circ x_1 x_0^k) \prod_{i=0}^k \text{ad}^l_{x_0} x_1,$$

where $\text{supp}(x_1 x_0^l \circ \cdots \circ x_1 x_0^k) = \{ w \in x_1 X^* \mid |w|_{x_1} = k, |w|_{x_0} = l_1 + \cdots + l_k \}$ and $\kappa_k(z, t_1, \ldots, t_k)$ for any $k \geq 0$ is the formal power series given by

$$\kappa_k(z, t_1, \ldots, t_k) = e^{x_0 \log(z) - \log(t_1)} x_1 \cdots e^{x_0 \log(t_{k-1}) - \log(t_k)} x_1 e^{x_0 \log(t_k)}$$

$$= e^{x_0 \log(z)} e^{x_0 \log(t_1)} x_1 \cdots e^{x_0 \log(t_k)} x_1$$

$$= e^{x_0 \log(z)} \sum_{l_1, \ldots, l_k \geq 0} \prod_{i=1}^k \frac{\log^l(t_i)}{l!} \text{ad}^l_{x_0} x_1.$$

On the one hand, by Theorem 3.1 the morphism $L_i$ is injective and the two families $\{ \text{ad}^l_{x_0} x_1 \cdots \text{ad}^k_{x_0} x_1 \}_{k \geq 0}$ and $\{ x_1 x_0^l \circ \cdots \circ x_1 x_0^k \}_{k \geq 0}$ are dual bases of, respectively, $\mathcal{U}(J)$ and $\mathcal{U}(J)^\vee$.

On the other hand, by Proposition 5.9 it turns out that $\zeta_{\sqcup \sqcup}$ corresponds to the adjoint of the regularization proposed in [34, 52].

5.3. Associators with rational coefficients. Since for any $t \in \mathbb{C}$ with $|t| < 1$ one has $L_{(tx_1)^r}(z) = (1 - z)^{-t}$, and by [16]

$$H_{\pi_Y (tx_1)} = \sum_{k \geq 0} H_{y_1^k} t^k = \exp \left( - \sum_{k \geq 1} H_{y_k} \frac{(-t)^k}{k} \right), \quad (5.4)$$

by Lemma 3.5 and Proposition 5.9 we can extend the characters $\zeta_{\sqcup \sqcup}$ and $\gamma_*$, over $C \langle X \rangle \cup C [x_1^*]$ and $C \langle Y \rangle \cup C [y_*]$, respectively, by using the Euler beta and gamma functions\(^{35}\) and also the incomplete beta function, i.e. for any $a, b \in \mathbb{C}$ such that $|z| < 1$, $\Re a > 0$ and $\Re b > 0$,

$$B(z; a, b) := \int_0^z \frac{dt}{t^a (1 - t)^{b-1}}$$

and

$$B(1; a, b) = \Gamma(a) \Gamma(b) \Gamma(a+b).$$

\(^{35}\)Following [20], for any $z \in \mathbb{C}$ the function $\Gamma(z)$ is meromorphic, admitting simple poles in $-\mathbb{N}$ and satisfying $\Gamma(z) = \Gamma(z)$. The function $\Gamma^{-1}(z)$ is entire and admits simple zeros in $-\mathbb{N}$.
It is immediate that\(^{36}\)
\[ B(z; a, b) = \text{Li}_{x_0}[(ax_0)^\ast \omega (1-b)x_1)](z) = \text{Li}_{x_1}[(a-1)x_0)^\ast \omega (-bx_1)](z). \]

**Proposition 5.11.** — The characters \(\zeta_{\omega 1}\) and \(\gamma_1\) can be extended algebraically as follows for \(t \in \mathbb{C}\) with \(|t| < 1:\)
\[ \zeta_{\omega 1} : (\mathbb{C}X) \mapsto \mathbb{C}[x_1^\ast, \omega, 1x^\ast) \rightarrow (\mathbb{C}, x, 1\mathbb{C}), \]
\[ (tx_1)^\ast \rightarrow 1\mathbb{C}, \]
\[ \gamma_1 : (\mathbb{C}Y) \mapsto \mathbb{C}[y_1^\ast, \omega, 1y^\ast) \rightarrow (\mathbb{C}, x, 1\mathbb{C}), \]
\[ (ty_1)^\ast \rightarrow \exp \left( \gamma t - \sum_{n \geq 2} \zeta(n)(-t)^n \right) = \frac{1}{\Gamma(1+t)}. \]

It follows then that
\[ B(a, b) = \zeta_{\omega 1}(x_0[(ax_0)^\ast \omega ((1-b)x_1)] = \zeta_{\omega 1}(x_1)((1-a)x_0)^\ast \omega (-bx_1)). \]

Moreover, for any \(u, v \in \mathbb{C}\) such that \(|u| < 1, |v| < 1\) and \(|u + v| < 1\), one has\(^{37}\)
\[ \exp \left( \sum_{n \geq 2} \zeta(n)((u+v)^n - (u^n + v^n)) \right) = \frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-u-v)} = \frac{\gamma(-uy_1)^\ast \gamma(-vy_1)^\ast}{\gamma(-uy_1)^\ast \omega \gamma(-vy_1)^\ast} = \zeta_{\omega 1}(x_0[(-ux_0)^\ast \omega (-1+v)x_1)^\ast]) \]
\[ = \zeta_{\omega 1}(x_1[(-1+u)x_0)^\ast \omega (-vx_1)^\ast]) \]
and
\[ \zeta_{\omega 1}((-u+v)x_1)^\ast = \zeta_{\omega 1}((-ux_1)^\ast \omega (-vx_1)^\ast) = \zeta_{\omega 1}((-ux_1)^\ast)\zeta_{\omega 1}((-vx_1)^\ast) = 1. \]

With the notations in Corollary 4.10, the values \(p(1)\) and \(\hat{p}(1)\) obtained by (4.10) and (4.11), respectively, represent the following finite parts:

\(^{36}\)see the form of rational series given in (\(P_2\)) and Lemma 2.4.

\(^{37}\)The first equality is already presented in [25]. Moreover, since \((-uy_1)^\ast \omega (uy_1)^\ast = (-u^2y_2)^\ast\), letting \(v = -u\) it follows that
\[ \exp \left( -\sum_{n \geq 1} \zeta(2n)\frac{u^{2n}}{n} \right) = \Gamma(1-u)\Gamma(1+u) = \frac{1}{\gamma(-uy_1)^\ast \omega (uy_1)^\ast} = \frac{1}{\gamma(-u^2y_2)^\ast}. \]

It is also a consequence obtained by expanding identities like (5.4), for any \(y_r \in Y, [14, 16] \)
\[ y_r^k = \frac{(-1)^k}{k!} \sum_{s_1 + \ldots + s_k > 0 \atop s_1 + \ldots + k s_k = k} \frac{(-y_r)^\omega s_1 \omega \ldots \omega (y_{kr}) \omega s_k}{k^{s_k}}. \]
LEMMA 5.12. — (1) Put \( P_Q(z) := e^{-\log(1-z)} \log Q(z) \) for any
\[ Q \in \{ Z(x_0, -x_0)^*, x_1^*, \omega, 1_X^* \}/\{ x_0^*, \omega x_1^* - x_1^* + 1 \}. \]
Then \( P_Q = \operatorname{Li}_k \circ \omega, P_Q \in \mathbb{Z}[z, z^{-1}, e^{-\log(1-z)}]. \)
(2) By Lemma 3.4 the converse holds. Moreover, by (4.13) and (4.14) one has
\[ f_\cdot P_{z \to 1} Q (z) = f_\cdot P_{z \to 1} Q (z) \in \mathbb{Z}, \quad \{ (1 - z)^a \log b(1 - z)^{-1} \} \in \mathbb{Z}, \quad \{ n^a \log b(n) \} \in \mathbb{Z}, \]
(3) For any \( w \in Y^* \), let \( R_w \) be explicitly determined as in Proposition 4.11. There exists a unique polynomial \( p \in \mathbb{Z}[t] \) of valuation 1 and of degree \( (w+|w|) \) such that \( R_w = \hat{p}(x_1^*) \) and
\[ f_\cdot P_{z \to 1} Q (z) = p(1) \in \mathbb{Z}, \quad \{ (1 - z)^a \log b(1 - z)^{-1} \} \in \mathbb{Z}, \quad \{ n^a \log b(n) \} \in \mathbb{Z}, \]
where \( \hat{p} \in \mathbb{Q}[t] \) is the exponential transform of \( p \).

As determined in Proposition 4.7, \( B_w \) and \( C_w \) do not realize characters for \( (\mathbb{Q}(X), \omega, 1_X^*) \) and \( (\mathbb{Q}(Y), \omega, 1_Y^*) \), respectively [21]. Hence, instead of regularizing the divergent sums \( \zeta_{\omega}(R_w) \) and \( \zeta_{\omega}(\pi_Y(R_w)) \) by \( B_w \) and \( C_w \), one can use, respectively, \( p(1) \) and \( \hat{p}(1) \) (depending on \( w \)) as shown in Theorem 5.15 below which is a consequence of Lemma 5.12, Propositions 4.11, 5.11 and Corollary 4.12 :

DEFINITION 5.13. — Let \( Y \) and \( \Lambda \) be the noncommutative generating series of, respectively, \( \{ H_{\pi_Y(R_w)} \}_{w \in Y^*} \) and \( \{ \operatorname{Li}_{R_{\pi_Y(w)}} \}_{w \in X^*} \) (with \( \langle \Lambda(z) \mid x_0 \rangle = \log(z) \)) :
\[ Y := \sum_{w \in Y^*} H_{\pi_Y(R_w)} w \in \mathbb{Q}[\langle n + \bullet \rangle] \langle Y \rangle, \]
\[ \Lambda := \sum_{w \in X^*} \operatorname{Li}_{R_{\pi_Y(w)}} w \in \mathbb{Q}[\langle e^{-\log(1-z)} \rangle \langle \log(z) \rangle \langle X \rangle]. \]

Let \( Z_{\gamma} \) and \( Z_{\zeta_{\omega}} \) be the noncommutative generating series of \(^{38}\), respectively, \( \{ \gamma_{\pi_Y(R_w)} \}_{w \in Y^*} \) and \( \{ \zeta_{\omega}(R_{\pi_Y(w)}) \}_{w \in X^*} \) :
\[ Z_{\gamma} := \sum_{w \in Y^*} \gamma_{\pi_Y(R_w)} w \in \mathbb{Q}[\langle Y \rangle] \quad \text{and} \quad Z_{\zeta_{\omega}} := \sum_{w \in X^*} \zeta_{\omega}(R_{\pi_Y(w)}) w \in \mathbb{Q}[\langle X \rangle]. \]

Via the diagonal series \( \mathcal{D}_{\omega}, \mathcal{D}_{\omega} \) given in (2.2)-(2.8), one has

LEMMA 5.14. — The extension \( \pi_Y : (\mathbb{C}[x_0] [Y_0], \top, 1_Y^*) \rightarrow (\mathbb{C}[x_0] [x_1^*], \omega, 1_X^*) \) is bijective. Hence :

(1) Let \( \hat{\pi}_Y \) be the morphism of algebras defined, over an algebraic basis, by
\[ \hat{\pi}_YS_l = \pi_Y S_l \] for any \( l \in \mathcal{L}_{\mathcal{Y}X} - \{ x_0 \} \), and \( \hat{\pi}_Y(\pi_Y) = \pi_Y \) (such that \( \operatorname{Li}_{R_{\pi_Y(x_0)}} (z) = \log(z) \), whence \( \zeta(\hat{\pi}_Y(x_0)) = 0 \)).
Then
\[ Y = ((H_{\pi_Y} \circ \pi_Y \circ R_{\pi_Y} \otimes \text{Id}) \mathcal{D}_{\pi_Y} \quad \text{and} \quad \Lambda = ((\operatorname{Li}_k \circ R_{\pi_Y} \otimes \hat{\pi}_Y \otimes \text{Id}) \mathcal{D}_{\pi_Y}, \]
\[ Z_{\gamma} = ((\gamma_{\pi_Y} \otimes \pi_Y \circ R_{\pi_Y} \otimes \text{Id}) \mathcal{D}_{\pi_Y} \quad \text{and} \quad Z_{\zeta_{\omega}} = ((\zeta_{\omega} \otimes R_{\pi_Y} \otimes \pi_Y \otimes \text{Id}) \mathcal{D}_{\pi_Y}. \]

\(^{38}\)Note that, on the one hand, by Proposition 5.9 one has \( \langle Z_{\zeta_{\omega}} \mid x_0 \rangle = \zeta_{\omega}(x_0) = 0 \). On the other hand, since \( R_{\pi_Y(2x_1^*)} = -(2x_1^*)^* - x_1^* \), one has \( \operatorname{Li}_{R_{\pi_Y(2x_1^*)}} (z) = (1 - z)^{-2} - (1 - z)^{-1} \) and \( H_{\pi_Y(\pi_Y(2x_1^*))} (n) = \binom{-2}{n} - \binom{-1}{n} \). Hence, \( \langle Z_{\zeta_{\omega}} \mid x_1 \rangle = \zeta_{\omega}(R_{\pi_Y(2x_1^*)}) = 0 \), and \( \langle Z_{\gamma} \mid x_1 \rangle = \pi_{\gamma}(R_{\pi_Y(2x_1^*)}) = -1/2 \).
(2) For any $u \in X^*$ and $v \in Y^*$ one has
\[
\text{f.p.}_{z \to 1} \langle A(z) \mid u \rangle = \langle Z^{-\mu} \mid u \rangle, \quad \{(1 - z)^a \log^b((1 - z)^{-1})\}_a \in \mathbb{Z}, b \in \mathbb{N},
\]
\[
\text{f.p.}_{n \to +\infty} \langle \Upsilon(n) \mid v \rangle = \langle Z^{-\gamma} \mid v \rangle, \quad \{n^a \log^b(n)\}_a \in \mathbb{Z}, b \in \mathbb{N},
\]
which means that (see also (4.5), (4.6) and (4.7))
\[
Z^{-\gamma} = \text{F.P.}_{n \to +\infty} \Upsilon(n) \quad \text{and} \quad Z^{-\mu} = \text{F.P.}_{z \to 1} A(z).
\]
Hence, by Propositions 4.11 and 5.11, Lemmas 2.4–3.5 and 5.12, one derives

**Theorem 5.15.** —

1. For any $(s_1, \ldots, s_r) \in \mathbb{N}^+_\cup$ associated with $l \in \mathcal{Lyn} Y$ there exists a unique $p \in \mathbb{Z}[t]$ of valuation 1 and of degree $(l)+|l|$ such that
\[
\hat{p}(x^*_l) = R_l \quad \in (\mathbb{Z}[x^*_l], \omega, 1_{X^*}),
\]
\[
p(e^{-\log((1-z)}) = \text{Li}_{R_l}(z) \quad \in (\mathbb{Z}[e^{-\log(1-z)}], 1_{B}),
\]
\[
\hat{p}(n + \bullet)_n = H_{\pi_Y(R_l)}(n) \quad \in (\mathbb{Q}(n + \bullet)_n, 1),
\]
\[
\zeta(-s_1, \ldots, -s_r) = p(1) = \zeta_{\omega}(R_l) \quad \in (\mathbb{Z}, 1),
\]
\[
\gamma_{-s_1, \ldots, -s_r} = \hat{p}(1) = \gamma_{\pi_Y(R_l)} \quad \in (\mathbb{Q}, 1),
\]
where $\hat{p} \in \mathbb{Q}[t]$ is the exponential transform of $p$, and $p$ is obtained as the exponential transform of $\hat{p} \in \mathbb{Z}[t]$.

2. One has $\langle Z^{-\gamma} \mid 1_{Y^*} \rangle = \langle Z^{-\mu} \mid 1_{X^*} \rangle = 1$ and
\[
\Delta_{\omega}(Z^{-\gamma}) = Z^{-\gamma} \otimes Z^{-\gamma} \quad \text{and} \quad \Delta_{\omega}(Z^{-\mu}) = Z^{-\mu} \otimes Z^{-\mu},
\]
\[
Z^{-\gamma} = \prod_{l \in \mathcal{Lyn}Y} e^{\pi_{\gamma}(R_l) H_l} \quad \text{and} \quad Z^{-\mu} = \prod_{l \in \mathcal{Lyn}X} e^{\pi_{\mu}(\pi_Y(S_l)) P_l},
\]

3. Similarly, $\langle \Upsilon \mid 1_{Y^*} \rangle = \langle \Lambda \mid 1_{X^*} \rangle = 1$ and
\[
\Delta_{\omega}(\Upsilon) = \Upsilon \otimes \Upsilon \quad \text{and} \quad \Delta_{\omega}(\Lambda) = \Lambda \otimes \Lambda,
\]
\[
\Upsilon = \prod_{l \in \mathcal{Lyn}Y} e^{H_{\pi_Y(R_l) S_l}} \quad \text{and} \quad \Lambda = \prod_{l \in \mathcal{Lyn}X} e^{\text{Li}_{\pi_Y(S_l)} P_l},
\]

4. Under the action of $\mathcal{G}$ [35], as for $L$ [38, 41], for any $g \in \mathcal{G}$ there exists a letter substitution $\sigma_g$ and a primitive series $C$ such that
\[
\Lambda(g(z)) = \sigma_g(\Lambda(z)) e^C \quad \text{and} \quad \Lambda(z) \sim_0 e^{x_0 \log(z)}.
\]

**Remark 5.16.** — By Corollary 5.4, $\Lambda$ does not satisfy (DE) while $Z^{-\omega}$ and $Z^{-\gamma}$, regularizing $\Lambda$ and $\Upsilon$ respectively, satisfy similar properties as $Z^{-\omega}$ and $Z^{-\gamma}$.

The series $Z^{-\omega}$ (or $Z^{-\gamma}$) is not unique because in Theorem 5.15 the elements of the family $\{\text{Li}_{R_l}\}_{l \in \mathcal{Lyn}Y}$ are polylogarithms with negative multiindices which are polynomial in $e^{-\log((1-z))}$.

Indeed, for any $l \in \mathcal{Lyn}Y$ one has $R_l \in \mathbb{Z}[x^*_l]$. Then, letting $\rho_l$ be a monomial in $\mathbb{Z}[x^*_0, (x_0)^*]$ with $\rho_l \neq 0$ and using Lemma 5.12, one gets the same regularized values $\zeta_{\omega}(R_l)$ and $\gamma_{\pi_Y(R_l)}$ for the series $R_l \omega \rho_l \in \mathbb{Z}^{\text{rat}} \llangle x_1 \rrangle \omega \mathbb{Z}^{\text{rat}} \llangle x_0 \rrangle = \mathbb{Z}^{\text{exc}} \llangle X \rrangle$, i.e. (see Appendix C)
\[
\text{f.p.}_{z \to 1} \text{Li}_{R_l \omega \rho_l}(z) = \zeta_{\omega}(R_l), \quad \{(1 - z)^a \log^b((1 - z)^{-1})\}_a \in \mathbb{Z}, b \in \mathbb{N},
\]
\[
\text{f.p.}_{n \to +\infty} H_{\pi_Y(R_l \omega \rho_l)}(n) = \gamma_{\pi_Y(R_l)}, \quad \{n^a \log^b(n)\}_a \in \mathbb{Z}, b \in \mathbb{N}.
\]
For example, one can take $\rho_l$, by substituting each letter $x_1$ by $x_0$ in $R_l$. 

6. Conclusion

In this paper, we have surveyed our recent results concerning the resolution of $KZ_3$ via a noncommutative symbolic computation, and the algebraic combinatorial aspects of the polylogarithms $\{\text{Li}_{s_1, \ldots, s_r}\}_{(s_1, \ldots, s_r) \in \mathbb{C}^r}$, the harmonic sums $\{H_{s_1, \ldots, s_r}\}_{(s_1, \ldots, s_r) \in \mathbb{C}^r}$, and the zeta functions $\{\zeta(s_1, \ldots, s_r)\}_{(s_1, \ldots, s_r) \in \mathbb{C}^r}$ with the help of their commutative and noncommutative generating series.

This review is mainly based on the combinatorics on the shuffle bialgebras and their diagonal series, i.e. $\mathcal{D}_{\text{shuffle}}, \mathcal{D}_{\text{shuffle}^\perp}$ and $\mathcal{D}_X$. In particular, it used

1. The construction of pairs of bases (Lie algebra bases and transcendence bases) in duality (Theorem 2.1) to factorize the noncommutative rational power series (Theorem 2.3) and to obtain the algebraic structure of $\{\zeta(s_1, \ldots, s_r)\}_{(s_1, \ldots, s_r) \in \mathbb{N}_1^r}$ (polynomial relations homogenous in weight, and independence over a commutative extension of $\mathbb{Q}$, denoted by $A$) by identification of local coordinates, in infinite dimension (Corollary 4.4).

2. The algebraic structures (Theorems 3.1 and 3.2) and the analysis of singularities (Theorems 4.1 and 4.9) of the polylogarithms and the harmonic sums, for which the global renormalizations has been obtained via Abel like theorems for the pairs of generating series $L, H$ and $L^-, H^-$. In particular, the series $L$ corresponds to the actual solution of (1.1) satisfying the standard asymptotic behaviors as given in (1.4) (Corollary 5.4).

3. The paper culminates with the action (Theorem 5.2) of the differential Galois group $\text{Gal}_C(DE)$ (containing the group of associators $dm(A)$) on the asymptotic expansions of solutions of the equation (1.1) (see (5.2)–(5.3)).

The group $dm(A)$ contains the unique associator $\Phi_{KZ}$, i.e. the series $Z_{\perp}$ determined by asymptotic conditions (Corollary 5.4), which is also associated with series $Z_{\perp}$ and $Z_{\gamma}$. All of them are, for the corresponding co-products, group-like series and their logarithms are also provided (Propositions 4.2, 4.3 and 5.9).

4. Non trivial expressions for associators with rational coefficients, i.e. $Z_{\perp}$ and $Z_{\gamma}$, are also explicitly provided thanks to various processes of regularization via the noncommutative generating series $\Lambda$ and $\Upsilon$, which are group-like, respectively, for $\Delta_{\perp}$ and $\Delta_{\perp}$ (Theorem 5.15).

5. Via the local coordinates of the power series $Z_{\perp}, Z_{\gamma}, Z_{\gamma}$ and $Z_{\perp}$, regularization maps for divergent zeta are constructed (Propositions 5.9, 5.11) over algebraic bases matching with analytical meaning: on the one hand, the character $\zeta_{\perp}$ corresponds to the regularization, obtained as the finite parts of the singular expansions of $\{\text{Li}_{s_1, \ldots, s_r}\}_{(s_1, \ldots, s_r) \in \mathbb{Z}^r}$; on the other hand, the characters $\zeta_{\gamma}$ and $\gamma_*$ correspond to the regularizations obtained as the finite parts of the asymptotic expansions of $\{H_{s_1, \ldots, s_r}\}_{(s_1, \ldots, s_r) \in \mathbb{Z}^r}$, in different comparison scales.

In particular, the character $\gamma_*$ furnished a generalization of the Euler’s $\gamma$ constant, $\{\gamma_{s_1, \ldots, s_r}\}_{(s_1, \ldots, s_r) \in \mathbb{N}_1^r}$ (Corollary 4.6), and moreover, if $\gamma \notin A$ then $\gamma$ is transcendent over the $\mathcal{A}$-algebra generated by the convergent zeta values $\{\zeta(s_1, \ldots, s_r)\}_{(s_1, \ldots, s_r) \in \mathbb{N}_1^r, s_1 \geq 2}$ (Corollary 5.7).
APPENDIX A

By Proposition 4.3, identification of local coordinates, one obtains homogeneous polynomials relations among the local coordinates \{ζ(Σ_i)\}_{i ∈ ℒ} and \{ζ(Σ_i)\}_{i ∈ ℒ} (see Example 6.1).

Example 6.1 (Homogenous polynomials relations among local coordinates\(^3\)).

<table>
<thead>
<tr>
<th>Relations on {ζ(Σ_i)}_{i ∈ ℒ} {y_1}</th>
<th>Relations on {ζ(Σ_i)}_{i ∈ ℒ} {y_1}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ζ(Σ_{y_2 y_1}) = \frac{3}{2}ζ(Σ_{y_3}))</td>
<td>(ζ(S_{x_0 x_1}^2) = ζ(S_{x_0 x_1}^2))</td>
</tr>
<tr>
<td>(ζ(Σ_{y_1}) = \frac{3}{10}ζ(Σ_{y_2})^2)</td>
<td>(ζ(S_{x_0 x_1}^2) = \frac{2}{5}ζ(Σ_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>(ζ(Σ_{y_2 y_1}) = \frac{3}{2}ζ(Σ_{y_2})^2)</td>
<td>(ζ(S_{x_0 x_1}^2) = \frac{1}{10}ζ(Σ_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>(ζ(Σ_{y_3 y_2 y_1}) = \frac{3}{2}ζ(Σ_{y_3})ζ(Σ_{y_2}) - \frac{5}{12}ζ(Σ_{y_3}))</td>
<td>(ζ(S_{x_0 x_1}^2) = \frac{2}{5}ζ(Σ_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>(ζ(Σ_{y_3 y_1} y_{y_2}) = 3ζ(Σ_{y_3})ζ(Σ_{y_2}) - 5ζ(Σ_{y_3}))</td>
<td>(ζ(S_{x_0 x_1}^2) = \frac{1}{2}ζ(Σ_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>(ζ(Σ_{y_3 y_1} y_{y_2}) = \frac{5}{12}ζ(Σ_{y_3}))</td>
<td>(ζ(S_{x_0 x_1}^2) = ζ(S_{x_0 x_1}^2))</td>
</tr>
<tr>
<td>(ζ(Σ_{y_3 y_2 y_1}) = \frac{5}{12}ζ(Σ_{y_3})ζ(Σ_{y_2}) + \frac{5}{4}ζ(Σ_{y_3}))</td>
<td>(ζ(S_{x_0 x_1}^2) = ζ(S_{x_0 x_1}^2))</td>
</tr>
</tbody>
</table>

One obtains also two families of polynomials homogenous for the weight, describing the kernel of the polymorphism ζ (see Example 6.2, \{Q_i\}_{i ∈ ℒ}).

\(^3\)These relations are sorted by weight and are ordered by Lyndon words.
Example 6.2 (Homogenous polynomials\textsuperscript{40} generating \(\ker(\zeta)\)).

<table>
<thead>
<tr>
<th>({Q_l}_{l \in \mathcal{L}}) (Y-{y_1})</th>
<th>({Q_l}_{l \in \mathcal{L}}) (X-{x_1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>[\zeta(\Sigma y_2 y_3) - \frac{3}{2} \Sigma y_3 = 0]</td>
</tr>
<tr>
<td>4</td>
<td>[\zeta(\Sigma y_4 - \frac{2}{5} \Sigma y_3 y_2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_3 y_1 - \frac{3}{10} \Sigma y_2 y_1^2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_2 y_1 - \frac{4}{5} \Sigma y_1 y_2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_3 y_1 - \frac{1}{2} \Sigma y_1 y_2) = 0]</td>
</tr>
<tr>
<td>5</td>
<td>[\zeta(\Sigma y_3 y_2 - 3 \Sigma y_3 y_1 \Sigma y_2 - 5 \Sigma y_5) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_4 y_1 - \Sigma y_3 y_1 \Sigma y_2 + \frac{5}{2} \Sigma y_5) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_2 y_1 - \frac{3}{2} \Sigma y_3 y_2 - \frac{25}{12} \Sigma y_5) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_3 y_1 - \frac{5}{12} \Sigma y_5) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_2 y_1 - \frac{1}{2} \Sigma y_3 y_1 y_2 + \frac{3}{4} \Sigma y_5) = 0]</td>
</tr>
<tr>
<td>6</td>
<td>[\zeta(\Sigma y_6 - \frac{8}{35} \Sigma y_3 y_2 y_3) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_4 y_2 - \frac{4}{21} \Sigma y_2 y_3) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_5 y_1 - \frac{2}{7} \Sigma y_3 y_2 y_3) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_3 y_1 y_2 - \frac{17}{36} \Sigma y_2 y_3 + \frac{9}{2} \Sigma y_2 y_2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_3 y_2 y_2 - \frac{3}{5} \Sigma y_3 y_2 + \frac{9}{5} \Sigma y_3 y_2 y_2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_4 y_1 - \frac{3}{10} \Sigma y_3 y_2 y_2 - \frac{3}{10} \Sigma y_2 y_2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(S_{y_2 y_1} - \frac{16}{65} \Sigma y_3 y_2 y_2 - \frac{4}{5} \Sigma y_2 y_2) = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(\Sigma y_3 y_1 - \frac{1}{10} \Sigma y_2 y_2 = 0]</td>
</tr>
<tr>
<td></td>
<td>[\zeta(S_{y_2 y_1} - \frac{17}{50} \Sigma y_3 y_2 + \frac{2}{5} \Sigma y_2 y_2) = 0]</td>
</tr>
</tbody>
</table>

By substituting “\(=\)” by “\(\rightarrow\)” in the previous homogenous polynomial relations one obtains a Noetherian rewriting system without critical pairs among local coordinates \(\{\zeta(\Sigma l)\}_{l \in \mathcal{L}}\) \((\text{resp.} \{\zeta(S l)\}_{l \in \mathcal{L}})\) (see Example 6.3).

\textsuperscript{40}These polynomials are sorted by weight and are ordered by Lyndon words.
Example 6.3 (Noetherian homogenous rewriting system among local coordinates\(^{41}\)). —

<table>
<thead>
<tr>
<th>Rewriting on ({\zeta(S)}_{1 \in \mathcal{L}yn Y - {y_1}})</th>
<th>Rewriting on ({\zeta(S)}_{1 \in \mathcal{L}yn X - X})</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 (\zeta(S_{y_2 y_3 y_4}) \rightarrow \frac{2}{3} \zeta(S_{y_3}))</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow \zeta(S_{x_0 x_1}))</td>
</tr>
<tr>
<td>4 (\zeta(S_{y_4}) \rightarrow \frac{2}{3} \zeta(S_{y_2})^2)</td>
<td>(\zeta(S_{x_0 x_1^3}) \rightarrow \frac{2}{5} \zeta(S_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>(\zeta(S_{y_4}) \rightarrow \frac{3}{5} \zeta(S_{y_2})^2)</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow \frac{1}{10} \zeta(S_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>(\zeta(S_{y_4}) \rightarrow \frac{2}{3} \zeta(S_{y_2})^2)</td>
<td>(\zeta(S_{x_0 x_1^3}) \rightarrow \frac{2}{5} \zeta(S_{x_0 x_1})^2)</td>
</tr>
<tr>
<td>5 (\zeta(S_{y_3 y_4}) \rightarrow 3 \zeta(S_{y_3}) \zeta(S_{y_2}) - 5 \zeta(S_{y_6}))</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow -\zeta(S_{x_0 x_1}) \zeta(S_{x_0 x_1}) + 2 \zeta(S_{x_0 x_1}))</td>
</tr>
<tr>
<td>(\zeta(S_{y_4}) \rightarrow -\zeta(S_{y_3}) \zeta(S_{y_2}) + \frac{5}{2} \zeta(S_{y_6}))</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow -\frac{3}{2} \zeta(S_{x_0 x_1}) + \zeta(S_{x_0 x_1}) \zeta(S_{x_0 x_1}))</td>
</tr>
<tr>
<td>(\zeta(S_{y_4}) \rightarrow \frac{3}{5} \zeta(S_{y_3}) \zeta(S_{y_2}) - \frac{25}{12} \zeta(S_{y_5}))</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow -\zeta(S_{x_0 x_1}) \zeta(S_{x_0 x_1}) + 2 \zeta(S_{x_0 x_1}))</td>
</tr>
<tr>
<td>(\zeta(S_{y_4}) \rightarrow \frac{\sqrt{2}}{12} \zeta(S_{y_5}))</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow \frac{1}{2} \zeta(S_{x_0 x_1}))</td>
</tr>
<tr>
<td>(\zeta(S_{y_4}) \rightarrow \frac{1}{2} \zeta(S_{y_3}) \zeta(S_{y_2}) + \frac{3}{4} \zeta(S_{y_5}))</td>
<td>(\zeta(S_{x_0 x_1^2}) \rightarrow \zeta(S_{x_0 x_1}))</td>
</tr>
</tbody>
</table>

This means that for any \(l \in \mathcal{L}yn Y - \{y_1\}\) (resp. \(l \in \mathcal{L}yn X - X\), the element \(\zeta(S_l)\) (resp. \(\zeta(S_l)\)) is rewritten in a unique way as polynomials (normal forms) with coefficients in \(\mathbb{Q}\) in irreducible local coordinates \(\mathcal{Z}_{\text{irr}}^{\infty}(Y)\) (resp. \(\mathcal{Z}_{\text{irr}}^{\infty}(X)\)) forming an algebraic generator system for \(\mathcal{Z}\) (see Example 6.4).

Example 6.4. — At weight 12 one has

\[
\Sigma_{y_2} = y_2, \quad \Sigma_{y_3} = y_3, \quad \Sigma_{y_5} = y_5, \quad \Sigma_{y_7} = y_7, \quad \Sigma_{y_9} = y_9, \quad \Sigma_{y_{11}} = y_{11}
\]

\(^{41}\)These rules are sorted by weight and are ordered by Lyndon words.
and
\[ S_{x_0x_1} = x_0x_1, \quad S_{0^2x_1} = x_0^2x_1, \quad S_{0^4x_1} = x_0^4x_1, \]
\[ S_{x_0^6x_1} = x_0^6x_1, \quad S_{x_0^8x_1} = x_0^8x_1, \quad S_{x_0^{10}x_1} = x_0^{10}x_1. \]

The identification of local coordinates leads to the irreducible polyzetas (see [47] for a short discussion)
\[ Z_{irr}^{\leq 12}(Y) = \{ \zeta(S_{y_2}), \zeta(S_{y_3}), \zeta(S_{y_4}), \zeta(S_{y_5}), \zeta(S_{y_6}), \zeta(S_{y_7}), \zeta(S_{y_8}), \zeta(S_{y_9}), \zeta(S_{y_{10}}), \zeta(S_{y_{11}}) \}, \]
\[ L_{irr}^{\leq 12}(Y) = \{ \Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_4}, \Sigma_{y_5}, \Sigma_{y_6}, \Sigma_{y_7}, \Sigma_{y_8}, \Sigma_{y_9}, \Sigma_{y_{10}}, \Sigma_{y_{11}}, \Sigma_{y_{12}}, \Sigma_{y_{13}}, \Sigma_{y_{14}} \}. \]
\[ Z_{irr}^{\leq 12}(X) = \{ \zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^3x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^5x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0^7x_1}), \zeta(S_{x_0^8x_1}), \zeta(S_{x_0^9x_1}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0^{11}x_1}), \zeta(S_{x_0^{12}x_1}), \zeta(S_{x_0^{13}x_1}), \zeta(S_{x_0^{14}x_1}) \}, \]
\[ L_{irr}^{\leq 12}(X) = \{ S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^3x_1}, S_{x_0^4x_1}, S_{x_0^5x_1}, S_{x_0^6x_1}, S_{x_0^7x_1}, S_{x_0^8x_1}, S_{x_0^9x_1}, S_{x_0^{10}x_1}, S_{x_0^{11}x_1}, S_{x_0^{12}x_1}, S_{x_0^{13}x_1}, S_{x_0^{14}x_1} \}.

**APPENDIX B**

\[ \sum_{w \in x_0^z} \text{Li}_w(z) = e^{x_0 \log(z)}, \]
\[ \sum_{w \in x_0^2 \cup x_1} \text{Li}_w(z) = \int_0^z e^{x_0[\log(z) - \log(t)]} x_1 \omega_1(t) e^{x_0 \log(t)} = \int_0^z \omega_1(t) \kappa_1(z, t), \]
where
\[ \kappa_1(z, t) = e^{x_0[\log(z) - \log(t)]} x_1 e^{x_0 \log(t)} = e^{x_0 \log(z)} e^{\text{ad}_{-x_0} \log(t)} x_1. \]
\[ \sum_{w \in x_0^2 \cup x_1^2} \text{Li}_w(z) = \int_0^z \int_0^{t_1} e^{x_0[\log(z) - \log(t_1)]} x_1 \omega_1(t_1) \int_0^{t_2} e^{x_0[\log(t_1) - \log(t_2)]} x_1 \omega_1(t_2) e^{x_0 \log(t_2)} \]
\[ = \int_0^z \omega_1(t_1) \int_0^{t_2} \omega_1(t_2) \kappa_2(z, t_1, t_2), \]
\[ \sum_{w \in x_0^3 \cup x_1^3} \text{Li}_w(z) = \int_0^z \int_0^{t_1} \int_0^{t_2} \omega_1(t_1) \int_0^{t_3} \omega_1(t_3) \kappa_3(z, t_1, t_2, t_3), \]
\[ \sum_{w \in x_0^4 \cup x_1^4} \text{Li}_w(z) = \int_0^z \omega_1(t_1) \int_0^{t_2} \omega_1(t_2) \int_0^{t_3} \omega_1(t_3) \kappa_3(z, t_1, t_2, t_3), \]
where
\[ \kappa_3(z, t_1, t_2, t_3) = e^{x_0[\log(z) - \log(t_1)]} x_1 e^{x_0[\log(t_1) - \log(t_2)]} x_1 e^{x_0[\log(t_2) - \log(t_3)]} x_1 e^{x_0 \log(t_3)} \]
\[ = e^{x_0 \log(z)} e^{\text{ad}_{-x_0} \log(t_1)} x_1 e^{\text{ad}_{-x_0} \log(t_2)} x_1 e^{\text{ad}_{-x_0} \log(t_3)} x_1. \]
\[ \vdots \]
\[ \sum_{w \in x_0^k \cup x_1^k} \text{Li}_w(z) = \int_0^z \omega_1(t_1) \cdot \int_0^{k-1} \omega_1(t_k) \kappa_k(z, t_1, \ldots, t_k), \]
where
\begin{align*}
\kappa_k(z, t_1, \ldots, t_k) &= e^{x_0(\log(z) - \log(t_1))}x_1 \cdots e^{x_0(\log(t_{k-1}) - \log(t_k))}x_1 e^{x_0 \log(t_k)} \\
&= e^{x_0 \log(z)} e^{\text{ad}_{-x_0 \log(t_1)} x_1} \cdots e^{\text{ad}_{-x_0 \log(t_k)} x_1} \\
&= e^{x_0 \log(z)} \sum_{l_1, \ldots, l_k \geq 0} \prod_{i=1}^k \frac{\log^i(t_i)}{i!} \text{ad}_{-x_0 x_1}^i.
\end{align*}

Hence (see the notations of Proposition \ref{prop:log}) \cite{43, 44},
\begin{align*}
\sum_{w \in x_0^* \sqcup x_1^k} \text{Li}_w(z)w &= e^{x_0 \log(z)} \sum_{l_1, \ldots, l_k \geq 0} \int_0^z \omega_1(t_1) \frac{\log^i(t_1)}{i!} \cdots \\
&\qquad \int_0^{t_{k-1}} \omega_1(t_k) \frac{\log^k(t_k)}{k!} \prod_{i=1}^k \text{ad}_{-x_0 x_1}^i \\
&= e^{x_0 \log(z)} \sum_{l_1, \ldots, l_k \geq 0} \text{Li}_{x_1 x_0^l \cdots o x_1 x_0^k}(z) \prod_{i=1}^k \text{ad}_{-x_0 x_1}^i.
\end{align*}

See also Example \ref{ex:polylog} and Appendix C, for the commutative generating series of polylogarithms.

\section*{Appendix C}

For \( k \geq 0 \) and \( |t| < 1 \) let us define \( V_k = (tx_0^*) \sqcup x_1^k \) and \( W_k = (tx_1^*) \sqcup x_0^k \). By (3.4) one has \cite{35, 36, 37}
\begin{align*}
\text{Li}_{V_k}(z) &= z^t \frac{(-\log(1-z))^k}{k!} \quad \text{and} \quad \text{Li}_{W_k}(z) = (1-z)^{-t} \frac{\log^k(z)}{k!}.
\end{align*}

Hence \cite{35, 36, 37},
\begin{align*}
\text{Li}_{(tx_0^*) \sqcup x_1^k}(z) &= \sum_{k \geq 0} \text{Li}_{V_k}(z) = \frac{z^t}{1-z}, \\
\text{Li}_{x_0^* \sqcup (tx_1^*)}(z) &= \sum_{k \geq 0} \text{Li}_{W_k}(z) = \frac{z}{(1-z)^t},
\end{align*}

and then (see Remark \ref{rem:gamma})
\begin{align*}
\zeta_{\gamma}(\{(tx_1^*) \sqcup x_0^k\}) &= \sum_{k \geq 0} \zeta_{\gamma}(V_k) = 1, \quad \zeta_{\gamma}(\{(tx_0^*) \sqcup x_1^k\}) = \sum_{k \geq 0} \zeta_{\gamma}(V_k) = 1
\end{align*}

and
\begin{align*}
\gamma_{\gamma}(\{(tx_1^*) \sqcup x_0^k\}) &= \frac{1}{\Gamma(1+t)}.
\end{align*}

By (3.4), for any \( k \geq 1 \) one also has \cite{35, 36, 37}
\begin{align*}
\text{Li}_{(tx_1)^{*i+1}-(tx_1)^{*-i}}(z) &= t(1-z)^i \log(1-z) \sum_{k=0}^{i-1} \binom{i-1}{k} (-t \log(1-z))^k \\
&= t(1-z)^i \log(1-z) \sum_{k=0}^{i-1} \binom{i-1}{k} (-t \log(1-z))^k.
\end{align*}

More generally, as in Theorem 2.3, let \( S \) belong to \( C^{\text{rat}} \langle \langle X \rangle \rangle \) and be of linear representation \((\beta, \mu, \eta)\) of dimension \( n \geq 1 \). Then the following matrix is nothing
else than the “Dyson series” [36, 37]

\[ R(z) = \sum_{w \in \mathcal{X}^*} \text{Li}_w(z) \mu(w) = \prod_{l \in \mathcal{E} \cap n \mathcal{X}} e^{\text{Li}_{S_l}(z) \mu(P_l)}. \]

If \( S \) is exchangeable, \textit{i.e.} \( \mu(x_0), \mu(x_1) = 0 \), then \( R \) reduces to (see Lemma 2.4) [36, 37]

\[ R(z) = e^{\log(z) \mu(x_0) - \log(1-z) \mu(x_1)}. \]

The matrix \( R \) belongs to \( \mathcal{M}_{n,n}(\mathbb{C}[\log(z), \log(1-z)][z^a, (1-z)^b]_{a,b \in \mathbb{C}}) \) and if \( \mu(x_0) \) and \( \mu(x_1) \) are diagonal matrices, then \( R \in \mathcal{M}_{n,n}(\mathbb{C}[z^a, (1-z)^b]_{a,b \in \mathbb{C}}) \) [36, 37]. On the one hand, for \( |t_0| < 1 \) and \( |t_1| < 1 \), let us introduce the concatenation morphism \( \tau_1 \), mapping \( x_0 \) to \( 1 \) and \( x_1 \) to \( t \). Similarly, let \( \tau_0 \) map \( x_1 \) to \( 1 \) and \( x_0 \) to \( t \). It follows then (see Appendix B)

\[ \tau_1(L(z)) = \text{Li}(tx_1^*) \cup \cup x_0^*(z) = \frac{z}{(1-z)^t} \quad \text{and} \quad \tau_0(L(z)) = \text{Li}(tx_0^*) \cup \cup x_1^*(z) = \frac{z^t}{1-z}. \]

On the other hand, let \( \tau \) map \( x_1 \) to \( t_0 \) and \( x_0 \) to \( t_1 \). Then

\[ \tau(L(z)) = \text{Li}(t_0x_0)^* \cup \cup (t_1x_1)^*(z) = \frac{z^{t_0}}{(1-z)^{t_1}}. \]

**APPENDIX D**

The algebra \( \mathcal{H}({\Omega}) \) is endowed with the topology of \textit{compact convergence} whose seminorms are indexed by compact subsets of \( \Omega \), and defined by

\[ p_K(f) := ||f||_K = \sup_{s \in K} |f(s)|. \]

Of course, \( p_{K_1 \cup K_2} = \sup(p_{K_1}, p_{K_2}) \), and therefore the same topology is defined by extracting a \textit{fundamental subset of seminorms}, which here can be chosen denumerable. As \( \mathcal{H}(\Omega) \) is complete in this topology, it is a Frechet space and even, as \( p_K(fg) \leq p_K(f)p_K(g) \), it is a Frechet algebra (even more, as \( p_K(1_{\Omega}) = 1 \), a Frechet algebra with unit).

With the standard topology above, an operator \( \phi \in \text{End}(\mathcal{H}(\Omega)) \) is continuous if and only if, with \( K_i \) compacts of \( \Omega \),

\[ (\forall K_2)(\exists K_1)(\exists M_{21} > 0)(\forall f \in \mathcal{H}(\Omega))(||\phi(f)||_K_2 \leq M_{21} ||f||_{K_1}), \]

the algebra \( C\{\text{Li}_w\}_{w \in \mathcal{X}^*} \) (and \( \mathcal{H}(\Omega) \)) is closed under the operators \( \theta_i \) for \( i = 0, 1 \). We will first build sections of them, then see that they are continuous and, propose (discontinuous) sections more adapted to renormalisation and the computation of associators.

For \( z_0 \in \Omega \), let us define \( \iota_i^{z_0} \in \text{End}(\mathcal{H}(\Omega)) \) by

\[ \iota_i^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_0(s) \quad \text{and} \quad \iota_1^{z_0}(f) = \int_{z_0}^{z} f(s)\omega_1(s). \]

It is easy to check that \( \theta_i \iota_i^{z_0} = \text{Id}_{\mathcal{H}(\Omega)} \) and that they are continuous on \( \mathcal{H}(\Omega) \) (for the topology of compact convergence), because for all \( K \subset \text{compact} \ \Omega \) we have

\[ |p_K(\iota_i^{z_0}(f))| \leq p_K(f) [\sup_{z \in K} |\int_{z_0}^{z} \omega_i(s)|]. \]
and this is sufficient to prove continuity. The operators \( \iota_{i}^{2} \) are also well defined on \( C\{\text{Li}_{w}\}_{w \in X^{*}} \), and it is easy to check that
\[
\iota_{i}^{2}(C\{\text{Li}_{w}\}_{w \in X^{*}}) \subset C\{\text{Li}_{w}\}_{w \in X^{*}}.
\]
Due to the decomposition of \( \mathcal{H}(\Omega) \) into a direct sum of closed subspaces
\[
\mathcal{H}(\Omega) = \mathcal{H}_{20\to 0}(\Omega) \oplus \mathcal{C}\Omega,
\]
it is not hard to see that the graphs of \( \theta_{i} \) are closed. Thus the \( \theta_{i} \) are also continuous. Much more interesting (and adapted to the explicit computation of associators), we have the operator \( \iota_{i} \) (without superscripts), mentioned in the introduction and (more rigorously) defined by means of a \( \mathcal{C} \)-basis of
\[
C\{\text{Li}_{w}\}_{w \in X^{*}} = \mathcal{C} \otimes_{\mathcal{C}} C\{\text{Li}_{w}\}_{w \in X^{*}}.
\]
As \( C\{\text{Li}_{w}\}_{w \in X^{*}} \cong C[\text{Ly}(X)] \), one can partition the alphabet of this polynomial algebra in \( (\text{Ly}(X) \cap X^{*}x_{1}) \cup \{x_{0}\} \) and obtain the decomposition
\[
C\{\text{Li}_{w}\}_{w \in X^{*}} \cong C \otimes_{\mathcal{C}} C\{\text{Li}_{w}\}_{w \in X^{*}x_{1}} \otimes_{\mathcal{C}} C\{\text{Li}_{w}\}_{w \in x_{0}^{*}}.
\]
Due to the following identity \([35] \),
\[
ux_{1}x_{0}^{n} = ux_{1}x_{0}^{n} - \sum_{k=1}^{n}(u \lvert x_{0}^{k})x_{1}x_{0}^{n-k},
\]
we have an algorithm to convert \( \text{Li}_{ux_{1}x_{0}^{n}} \) as
\[
\text{Li}_{ux_{1}x_{0}^{n}}(z) = \sum_{m \leq n} P_{m}(z)\log^{m}(z) = \sum_{m \leq n, w \in X^{*}x_{1}} \langle P_{m}(z) \mid w \rangle \text{Li}_{w}(z)\log^{m}(z).
\]
This means that
\[
\mathcal{B} := (z^{k}\text{Li}_{ux_{1}}(z)\text{Li}_{x_{0}^{n}}(z))_{(k,n,u) \in \mathbb{Z} \times \mathbb{N} \times X^{*}} \cup (z^{k}\text{Li}_{x_{0}^{n}}(z))_{(k,n,u) \in \mathbb{Z} \times \mathbb{N}}
\cup ((1 - z)^{-l}\text{Li}_{ux_{1}}(z)\text{Li}_{x_{0}^{n}}(z))_{(l,n,u) \in \mathbb{N}^{+} \times \mathbb{N} \times X^{*}} \cup ((1 - z)^{-l}\text{Li}_{x_{0}^{n}}(z))_{(l,n,u) \in \mathbb{N}^{+} \times \mathbb{N}};
\]
is a \( \mathcal{C} \)-basis of \( C\{\text{Li}_{w}\}_{w \in X^{*}} \). With this basis, we can define \( \iota_{0} \) as follows.

**Definition 6.5** ([22]). — Define the index map \( \text{ind} : \mathcal{B} \to \mathbb{Z} \) by
\[
\text{ind}(z^{k}(1 - z)^{-l}\text{Li}_{x_{0}^{n}}(z)) = k \quad \text{and} \quad \text{ind}(z^{k}(1 - z)^{-l}\text{Li}_{ux_{1}}(z)\log^{n}(z)) = k + |ux_{1}|.
\]
Then \( \iota_{0} \) is computed as follows
\[
\iota_{0}(b) = \begin{cases} 
\int_{0}^{z} b(s)\omega_{0}(s), & \text{if } \text{ind}(b) \geq 1, \\
\int_{1}^{z} b(s)\omega_{0}(s), & \text{if } \text{ind}(b) \leq 0.
\end{cases}
\]

To show discontinuity of \( \iota_{0} \), one of the possibilities consists in exhibiting two sequences \( f_{n}, g_{n} \in C\{\text{Li}_{w}\}_{w \in X^{*}} \) converging to the same limit but such that
\[
\lim_{n \to \infty} \iota_{0}(f_{n}) \neq \lim_{n \to \infty} \iota_{0}(g_{n}).
\]
Here, we choose the function \( z \) to be approximated in a twofold way, and if \( \iota_{0} \) were continuous, we would have equality of the limits of the image-sequences (which is not the case). We first remark that
\[
z = \sum_{n \geq 0} \frac{\log^{n}(z)}{n!} = \sum_{n \geq 1} (-1)^{n+1} \frac{\log^{n}((1 - z)^{-1}}{n!}
\]
Set
\[ f_n = \sum_{0 \leq m \leq n} \frac{\log^m(z)}{m!} \quad \text{and} \quad g_n = \sum_{1 \leq m \leq n} \frac{(-1)^{m+1} \log^m((1 - z)^{-1})}{m!} \]
(these two sequences are in \( \mathbb{C}\{\text{Li}_w\}_{w \in X^+} \)). It is easily seen that \( \omega_0(f_n) = f_{n+1} - 1 \), and then \( \lim_{n \to +\infty} \omega_0(f_n)(z) = z - 1 \). Now, for any \( s \in [0, z] \) with \( z \in ]0, 1[ \) one has
\[ |g(s)| = \left| \sum_{m=1}^{n} (-1)^{m+1} \frac{\log^n(1 - s)}{m!} \right| \leq \frac{s}{1 - s}. \]

In order to exchange limits, we apply Lebesgue’s dominated convergence theorem to the measure space \( (]0, z], \mathcal{B}, dz/z) \) (\( \mathcal{B} \) is the usual Borel \( \sigma \)-algebra) and the function \( p(x) = s(1 - s)^{-1} \) which is — as are the functions \( g_n \) — integrable on \( ]0, z] \) for every \( z \in ]0, 1[ \). Then
\[ \lim_{n \to +\infty} g_n(s) = \lim_{n \to +\infty} \int_0^z g_n(s) \frac{ds}{s} = \int_0^z \lim_{n \to +\infty} g_n(s) \frac{ds}{s} = \int_0^z s \frac{ds}{s} = z. \]

Hence, for \( z \in ]0, 1[ \) we obtain \( \lim_{n \to +\infty} (\omega_0(f_n)) = z - 1 \neq z = \lim_{n \to +\infty} (\omega_0(g_n)) \) which completes the proof.

Acknowledgements

I would like to thank Gérard Duchamp for the fruitful interactions that we had to carry out this paper. The writing has been improved significantly by the anonymous referees for which I am also grateful.

References


E. Wardi. Mémoire de DEA, Lille, 1999.


Manuscript received February 5, 2017,
revised February 3, 2019,
accepted March 16, 2019.

Vincel HOANG NGOC MINH
LIPN - UMR 7030, CNRS, 93430 Villetaneuse, France. University of Lille, 1 Place Déliot, 59024 Lille, France.
minh@lipn.univ-paris13.fr