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ON THE SOLUTIONS OF THE UNIVERSAL DIFFERENTIAL EQUATION WITH THREE REGULAR SINGULARITIES (ON SOLUTIONS OF KZ_3)

VINCEL HOANG NGOC MINH

Abstract. This review concerns the resolution of a special case of Knizhnik-Zamolodchikov equations (KZ_3) and our recent results on combinatorial aspects of zeta functions on several variables.

In particular, we describe the action of the differential Galois group of KZ_3 on the asymptotic expansions of its solutions leading to a group of associators which contains the unique Drinfel'd associator (or Drinfel'd series). Non trivial expressions of an associator with rational coefficients are also explicitly provided, based on the algebraic structure and the singularity analysis of the multi-indexed polylogarithms and harmonic sums.

Contents

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1. Knizhnik-Zamolodchikov equations and Drinfel'd series

In this paper, we survey our recent results which pertain to an in-depth combinatorial study of the several *complex* variables *zeta* functions defined as follows

$$\forall r \geqslant 1, \qquad \zeta_r : \mathcal{H}_r \longrightarrow \mathbb{R}, \qquad (s_1, \ldots, s_r) \longmapsto \sum_{n_1 > \ldots > n_k > 0} n_1^{-s_1} \ldots n_k^{-s_r},$$

where $\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}$ [29, 30]. They appear in the *regularization* of solutions of the following fuchsian first order differential equation *without initial condition*, with regular singularities in $\{0, 1, +\infty\}$ and noncommutative indeterminates in $X = \{x_0, x_1\}$:

$$(DE) \qquad dG(z) = \left(x_0 \frac{dz}{z} + x_1 \frac{dz}{1-z}\right) G(z). \tag{1.1}$$

Let us denote by $\mathcal{H}(\Omega)$ the ring of holomorphic functions over the simply connected domain $\Omega := \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[$, with $1_\Omega : \Omega \to \mathcal{H}(\Omega)$ as the neutral element $(z \mapsto 1)$. Let us also introduce the following differential forms

$$\omega_0(z) := \frac{dz}{z} \quad \text{and} \quad \omega_1(z) := \frac{dz}{1-z}.$$

This equation can be considered as the universal fuchsian first order differential equation with three regular singularities. Here, the notation has become essentially classical since Drinfel'd's papers [24, 25] which emphasized the importance of (1.1). After some elementary transformations [24, 25] one also finds that (1.1) is (equivalent to) the first non trivial Knizhnik-Zamolodchikov KZ_3. This is connected to the fact that the colored braid group on three strands P_3 is the direct product of its cyclic center with a copy of the free group on two generators. Although this interpretation of (1.1) does not play an explicit role below, it should be kept in mind with a view towards applications.

We may now return to (1.1) for which a solution can be obtained, as already pointed out by Poincaré, and done for the systems of ordinary *linear* differential equations with *regular* singularities in [18, 26, 37, 50], via Picard's iterative *approximation*. The differential Galois group of (1.1) is nothing else than the Hausdorff group, set of exponentials of Lie series in $\mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle$ (see Section 5). In this way, on the completion of $\mathcal{H}(\Omega)\langle X \rangle$, one obtains the so-called *Chen series*, over ω_0 and ω_1 along the path $z_0 \rightsquigarrow z$ on Ω, defined by [9, 33] :

$$C_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \alpha_{z_0}^z(w) w \; \in \; \widehat{\mathcal{H}(\Omega)\langle X \rangle}, \tag{1.2}$$

where X^* is the free monoid, generated by X [1, 58] (1_{X^*} is the neutral element), $\alpha_{z_0}^z(1_{X^*})$ equals 1_Ω and, for subdivisions $(z_0, z_1 \ldots, z_k, z)$ of $z_0 \rightsquigarrow z$ and for $w = x_{i_1} \cdots x_{i_k} \in X^*X$, the coefficient $\alpha_{z_0}^z(x_{i_1} \cdots x_{i_k})$ is defined by

$$\alpha_{z_0}^z(x_{i_1} \cdots x_{i_k}) := \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k) \; \in \; \mathcal{H}(\Omega) \tag{1.3}$$

and satisfies the *shuffle* relation $\alpha_{z_0}^z(u \sqcup\!\sqcup\, v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v)$, for $u, v \in X^*$ [11].

By termwise differentiation, the power series $C_{z_0 \rightsquigarrow z}$ satisfies (1.1), with *initial condition* $C_{z_0 \rightsquigarrow z_0} = 1_{X^*}$. From a theorem due to Ree [56], there exists a primitive series $L_{z_0\rightsquigarrow z} \in \widehat{\mathcal{H}(\Omega)\langle X\rangle}$ such that $e^{L_{z_0\rightsquigarrow z}} = C_{z_0\rightsquigarrow z}$, meaning that $C_{z_0\rightsquigarrow z}$ is group-like. The challenge is then to determine explicitly $L_{z_0\rightsquigarrow z}$, via the Magnus' Lie-integral-functional expansion [54] and to regularize, effectively, $C_{0\rightsquigarrow 1}$ and $L_{0\rightsquigarrow 1}$ (although a lot of iterated integrals be *divergent*). On the other hand, essentially interested in the solutions of (1.1) over the interval $]0,1[$ and using the involution $z \mapsto 1-z$, Drinfel'd stated that (1.1) admits a unique solution G_0 (resp. G_1) satisfying the following asymptotic behaviors [24, 25] :

$$G_0(z) \sim_0 z^{x_0} \quad \text{and} \quad G_1(z) \sim_1 (1-z)^{-x_1}. \tag{1.4}$$

In particular, since G_0 and G_1 are group-like, there is a unique group-like series $\Phi_{KZ} \in \mathbb{R}\langle\langle X\rangle\rangle$, called the *Drinfel'd associator* [55] (or *Drinfel'd series* [34]), such that [24, 25]

$$G_0 = G_1 \Phi_{KZ}. \tag{1.5}$$

Drinfel'd proved also the existence of group-like series in $\mathbb{Q}\langle\langle X\rangle\rangle$ satisfying similar properties of Φ_{KZ}, but he neither constructed such an expression nor made explicit G_0 and G_1 (similarly for $\log(G_0), \log(G_1)$ and $\log(\Phi_{KZ})$).

After that, Lê and Murakami expressed, in particular, the divergent coefficients of Φ_{KZ} as linear combinations of $\{\zeta_r(s_1,\ldots,s_r)\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{N}^r_{\geqslant 1}, s_1\geqslant 2}$, via a regularization based on representation of the chord diagram algebras [52].

One has two ways of considering, for any $(s_1,\ldots,s_r) \in \mathcal{H}_r$, the quantities $\zeta_r(s_1,\ldots,s_r)$ as limits fulfilling identities (see Section 3) [13, 16, 46, 47]. Firstly, they are limits at $z=1$ of *polylogarithms*, and secondly, as truncated sums, they are limits of *harmonic sums* when the upper bound tends to $+\infty$:

$$\mathrm{Li}_{s_1,\ldots,s_k}(z) := \sum_{n_1>\ldots>n_k>0} n_1^{-s_1}\ldots n_k^{-s_k}\, z^{n_1}, \qquad \text{for } z\in\mathbb{C}, |z|<1, \tag{1.6}$$

$$\mathrm{H}_{s_1,\ldots,s_k}(n) := \sum_{n_1>\ldots>n_k>0}^{n} n_1^{-s_1}\ldots n_k^{-s_k}, \qquad \text{for } n\in\mathbb{N}_+. \tag{1.7}$$

More precisely, if $(s_1,\ldots,s_r) \in \mathcal{H}_r$ then[1], after a theorem by Abel, one has

$$\lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_k}(z) = \lim_{n\to\infty} \mathrm{H}_{s_1,\ldots,s_k}(n) = \zeta_r(s_1,\ldots,s_k). \tag{1.8}$$

This does not hold for $(s_1,\ldots,s_r) \notin \mathcal{H}_r$, while (1.6) is well defined over $\{z\in\mathbb{C}, |z| < 1\}$ and so are (1.7) as Taylor coefficients of the following function

$$\mathrm{P}_{s_1,\ldots,s_k}(z) := \frac{\mathrm{Li}_{s_1,\ldots,s_k}(z)}{1-z} = \sum_{n\geqslant 1} \mathrm{H}_{s_1,\ldots,s_k}(n) z^n, \quad \text{for } z\in\mathbb{C}, |z|<1. \tag{1.9}$$

The coefficients in (1.3) are single valued over Ω ; alternatively they can be analytically continued and appear as multivalued functions over $B := \mathbb{C} - \{0,1\}$. In fact, we have mappings from the universal cover of B, denoted by $\tilde{B}$, *i.e.* we choose a universal covering $(B, \tilde{B}, p)$, where $p : \tilde{B} \to B$ is the covering map [9].

This second point of view will be adopted in the sequel. In this respect, let $\mathcal{H}(B)$ (resp. $\mathcal{H}(\tilde{B})$) denote the ring of holomorphic functions over B (resp. $\tilde{B}$), with $1_B : B \to \mathbb{C}$ (resp. $1_{\tilde{B}} : \tilde{B} \to \mathbb{C}$) as the neutral element ($z \mapsto 1$).

[1] $\zeta_1(s_1)$ is nothing else than the Riemann zeta function. It is convenient to set ζ_0 to $1_{\mathbb{R}}$.

Let $s : \Omega \to \tilde{B}$ be a lifting of the canonical embedding $j : \Omega \hookrightarrow B$

$$\begin{array}{ccc} & & \tilde{B} \\ & \overset{s}{\nearrow} & \downarrow p \\ \Omega & \overset{j}{\hookrightarrow} & B \end{array}$$

In particular, for any $g : B \to B$ and $x, y \in \tilde{B}$ such that $g(p(x)) = p(y)$ there exists a unique lifting $\tilde{g}$ (depending on (x, y)) such that $\tilde{g}(x) = y$) and the following commutes [9]

$$\begin{array}{ccc} \tilde{B} & \overset{\tilde{g}}{\dashrightarrow} & \tilde{B} \\ \downarrow p & & \downarrow p \\ B & \overset{g}{\longrightarrow} & B \end{array}$$

The work presented in this survey will concern our recent results about polylogarithms, harmonic sums and zeta values, involved in the coefficients of $C_{z_0 \rightsquigarrow z}$ and $L_{z_0 \rightsquigarrow z}$ belonging to $\mathcal{H}(B)\langle\langle X \rangle\rangle$.

We will base our work essentially on

(1) The isomorphisms of the Cauchy and Hadamard algebras of polylogarithmic functions, as defined in (1.6) and (1.9), respectively, with the shuffle $(\mathbb{C}\langle X \rangle, ш, 1_{X^*})$ and the quasi-shuffle algebras $(\mathbb{C}\langle Y \rangle, ⧢, 1_{Y^*})$ admitting Lyndon words as pure transcendence bases (recalled in Section 2),

(2) The isomorphisms of the bialgebras

$$(A\langle X \rangle, ., 1_{X^*}, \Delta_{ш}, \mathtt{e}) \quad \text{and} \quad (A\langle Y \rangle, ., 1_{Y^*}, \Delta_{⧢}, \mathtt{e})$$

with, respectively, the enveloping algebras of their primitive elements, leading to the constructions of the pairs of bases in duality to factorize the diagonal series thanks to the Cartier-Quillen-Milnor-Moore (CQMM, in short) and Poincaré-Birkhoff-Witt (PBW, in short) theorems (recalled in Section 2),

(3) The use of commutative and noncommutative generating series to establish combinatorial algebraic and analytical aspects of the polylogarithms $\{\mathrm{Li}_{s_1,\ldots,s_r}\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}$, the harmonic sums $\{\mathrm{H}_{s_1,\ldots,s_r}\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}$, and the zeta functions $\{\zeta_r(s_1,\ldots,s_r)\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}$ (recalled in Sections 3–5).

In the sequel, for simplification, we will adopt the notation ζ for $\zeta_r, r \in \mathbb{N}$.

We will examine the following problems :

P1. The renormalization which consists of finding counter terms to eliminate the divergence of the polylogarithms $\{\mathrm{Li}_{s_1,\ldots,s_r}\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{Z}^r}$ at $z = 1$, and of the harmonic sums $\{\mathrm{H}_{s_1,\ldots,s_r}\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{Z}^r}$ for $n \to +\infty$ (see Theorems 4.1 and 4.9 below).

For this, a theorem due to Abel is extended to treat, simultaneously, all convergent cases as well as all divergent cases via their generating series.

P2. The regularization which consists of evaluating *analytically* the finite parts (involved in the coefficients of $C_{0\rightsquigarrow 1}$ and $L_{0\rightsquigarrow 1}$) of the singular expansions of the polylogarithms $\{\mathrm{Li}_{s_1,\ldots,s_r}\}^{r\geqslant 1}_{(s_1,\ldots,s_r)\in\mathbb{N}^r_{\geqslant 1}}$ at $z = 1$ with respect to the comparison scale $\{(1-z)^{-a}\log^b(1-z)\}_{a,b\in\mathbb{N}}$, and the asymptotic expansions of the harmonic sums $\{\mathrm{H}_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}_{\geqslant 1}^r}^{r\geqslant 1}$ for $n\to+\infty$ in the scales $\{n^{-a}\log^b(n)\}_{a,b\in\mathbb{N}}$ and $\{n^{-a}\mathrm{H}_1^b(n)\}_{a,b\in\mathbb{N}}$, via *combinatorial* aspects of their noncommutative generating series (see Proposition 5.9 below).

For this, the definition of the regularization characters over the *algebraic bases* of noncommutative polynomial algebras have to be reduced to match with their analytical meanings.

P3. For any multiindex $(-s_1,\ldots,-s_k)$ in $\mathbb{N}_-^r$, since the polylogarithms (resp. harmonic sums) are polynomial in $e^{-\log(1-z)}$ for $|\,z\,|<1$ (resp. in $n\in\mathbb{N}$) with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$) (see Propositions 4.7 and 4.11 below) :

$$\mathrm{Li}_{-s_1,\ldots,-s_k}(z)=\sum_{k=0}^{r+s_1+\ldots+s_k}p_k e^{-k\log(1-z)}=p(e^{-\log(1-z)}),\tag{1.10}$$

$$\mathrm{H}_{-s_1,\ldots,-s_k}(n)=\sum_{k=0}^{r+s_1+\ldots+s_k}\frac{p_k}{k!}(n+k)_n\quad=\hat{p}(n).\tag{1.11}$$

Hence, $\mathrm{Li}_{-s_1,\ldots,-s_k}(1)$ (resp. $\mathrm{H}_{-s_1,\ldots,-s_k}(+\infty)$), as divergent sums, can be regularized (see Lemma 5.12 below) by the value $p(1)\in\mathbb{Z}$ (resp. $\hat{p}(1)\in\mathbb{Q}$) admitting generating series as *rational* associators (see Theorem 5.15 below).

This way, the previous regularizations are extended algebraically (*i.e.* by transcendent extension over a subalgebra of noncommutative rational series, see Proposition 5.11 below) and analytically (*i.e.* by evaluation of their finite parts within the comparison scales $\{(1-z)^{-a}\log^b(1-z)\}_{a,b\in\mathbb{N}}$ and $\{n^{-a}\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, see Lemma 5.12 below), allowing to regularize, in particular, the iterated integrals and their Taylor coefficients associated with the rational series in $(\mathbb{C}[x_1^*],\sqcup\!\sqcup,1_{X^*})$ and[2] $(\mathbb{C}[y_1^*],\,\text{⧢}\!\!\!-\,,1_{Y^*})$, *i.e.* the following sums with divergent coefficients (see Theorem 5.15 below)

$$\sum_{n\geqslant 0}\mathrm{Li}_{\underbrace{1,\ldots,1}_{n\text{ times}}}(1)\,t^n\quad\text{and}\quad\sum_{n\geqslant 0}\mathrm{H}_{\underbrace{1,\ldots,1}_{n\text{ times}}}(+\infty)\,t^n.$$

P4. For any multiindex $(s_1,\ldots,s_r)$ in $\mathbb{N}_{\geqslant 1}^r$, by expanding $(1-z)^{-1}$ the polylogarithms as in (1.6) can be obtained as iterated integrals over the differential forms ω_0 and ω_1 along the path $0\rightsquigarrow z$ associated with the words $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ over X^*x_1, as in (1.3). They induce *shuffle* relations while the Taylor coefficients as in (1.7) induce *quasi-shuffle* relations among convergent zeta values, as obtained in (1.8) (see Theorem 3.1 below).

In fact, the polynomial relations (*homogenous* in weight) over a commutative $\mathbb{Q}$-extension, denoted by A, among convergent zeta values, are *relations obtained at singularities* among elements of a transcendence basis of the algebra of polylogarithms (or harmonic sums, see Proposition 4.3 below). These relations are not due to but imply the *double-shuffle relations* and do not need any regularization. Moreover, if Euler's constant $\gamma\notin A$, then they are algebraically independent of γ (see Corollary 5.7 below).

[2]Here, $Y=\{y_k\}_{k\geqslant 1}$ and ⧢- is the quasi-shuffle (or *stuffle*, for *sticky shuffle*) product.

The organization of this paper is as follows :

- In Section 2, the algebraic combinatorial framework is introduced. In particular, we will give an explicit isomorphism φ_{π_1} from the shuffle bialgebra to the quasi-shuffle bialgebra (Theorem 2.1).

  Working with φ_{π_1}, the construction by Mélançon-Reutenauer-Schützenberger (MRS, in short), initially elaborated in the shuffle bialgebra and useful to factorize the group-like series and then rational power series (Theorem 2.3), will be extended in the quasi-shuffle bialgebra for the similar factorizations via the constructions of pairs of bases in duality (see (2.6)–(2.7)).
- In Section 3, to study their structure via generating series, polylogarithms and harmonic sums at integral multiindices will be encoded by words over various alphabets (Theorems 3.1, 3.2 and Lemmas 2.4–3.5). In particular, the bi-integro differential algebra of polylogarithms will be examined (Proposition 3.6) and their noncommutative generating series will be put in the MRS form (their logarithms will be also provided, Proposition 3.8).

  Concerning the polylogarithms at positive indices, we will insist on the fact that their noncommutative generating series is the actual solution of (1.1), and the noncommutative generating series of the finite parts of their singular expansions corresponds to the associator Φ_{KZ} which will be also put in MRS form without divergent zeta values as local coordinates.
- In Section 4, with noncommutative generating series, the global renormalizations of polylogarithms and harmonic sums will provide *associators* (Theorems 4.1 and 4.9). In particular, using the bridge equations connecting shuffle structures (Propositions 4.2 and 4.3), the enumerable families of *irreducible* zetas values will be implemented (see (4.8)–(4.9)) and Euler's γ constant will be generalized as finite parts of harmonic sums (Corollary 4.6). This will be achieved by identifying the local coordinates in *infinite* dimension and by obtaining algebraic relations among zeta values.

  With commutative generating series, many functions (algebraic functions with singularities in $\{0, 1, +\infty\}$, see Example 3.3) forgotten in the straight algebra of polylogarithms, at positive indices, will be recovered.
- In Section 5, the elements of the differential Galois group $\mathrm{Gal}_{\mathbb{C}}(DE)$ containing the groups of monodromy and of associators will be considered as *regularized solutions* of (1.1). The actions of $\mathrm{Gal}_{\mathbb{C}}(DE)$ on the singular expansions of the solutions of (1.1) will be then discussed (Theorem 5.2) : on the one hand, since the group of associators contains itself Φ_{KZ} and the local coordinates of each associator are homogenous in weight polynomials on zeta values over A, the independence of the convergent zeta values with respect to γ will be discussed according to A (Corollary 5.7), and $\log(\Phi_{KZ})$ will be also expressed (Proposition 5.9); on the other hand, since the polylogarithms at negative indices are polynomial in $(1-z)^{-1}$ with coefficients in $\mathbb{Z}$ (Propositions 4.7–4.11), the generating series of the finite parts of their singular expansions will specify the *regularization characters* (Propositions 5.6–5.11) and give examples of *rational* associators (Theorem 5.15).

2. Combinatorial framework

2.1. **Shuffle and quasi-shuffle algebras.** Let A be a commutative and associative $\mathbb{Q}$-algebra with unit.

Let $X = \{x_0, x_1\}$ (resp. $Y_0 = \{y_s\}_{s\geqslant 0}$) be an alphabet equipped with the total order $x_0 < x_1$ (resp. $y_0 > y_1 > y_2 > \ldots$) and let $Y = Y_0 - \{y_0\}$. The free monoid generated by X (resp. Y, or Y_0) is denoted by X^* (resp. Y^*, or Y_0^*) and admits the empty word, 1_{X^*} (resp. 1_{Y^*} and $1_{Y_0^*}$) as unit [1].

The sets of polynomials and formal power series over X^* (resp. Y^* or Y_0^*) with coefficients in A are denoted respectively by $A\langle X\rangle$ (resp. $A\langle Y\rangle$ or $A\langle Y_0\rangle$) and $A\langle\langle X\rangle\rangle$ (resp. $A\langle\langle Y\rangle\rangle$ or $A\langle\langle Y_0\rangle\rangle$) [1]. The sets of polynomials are A-modules admitting $\{w\}_{w\in X^*}$ (resp. $\{w\}_{w\in Y^*}$ and $\{w\}_{w\in Y_0^*}$) as linear bases, *i.e.*

$$A\langle X\rangle \cong A[X^*],\quad A\langle Y\rangle \cong A[Y^*],\quad A\langle Y_0\rangle \cong A[Y_0^*]. \tag{2.1}$$

Therefore, their full duals are

$$A\langle\langle X\rangle\rangle = A^{X^*},\quad A\langle\langle Y\rangle\rangle = A^{Y^*},\quad A\langle\langle Y_0\rangle\rangle = A^{Y_0^*}$$

and the natural pairing is given by the scalar product

$$\langle S \mid P\rangle = \sum_{u\in Z^*} S(u)P(u) \quad \text{with} \quad Z \in \{X, Y, Y_0\},$$

where, $S(u)$ and $P(u)$ are the coefficients[3] of u in the series S and the polynomial P, respectively.

As algebras (see (2.1)) the A-modules $A\langle X\rangle$ (resp. $A\langle Y\rangle$ and $A\langle Y_0\rangle$) come equipped with the associative concatenation product and

(1) in $A\langle X\rangle$, the associative commutative shuffle product [11, 27, 56] is defined, for any $u, v, w \in X^*$ and $x, y \in X$, as follows [33]

$$w \sqcup\!\sqcup 1_{X^*} = 1_{X^*} \sqcup\!\sqcup w = w \quad \text{and} \quad xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v),$$

(2) in $A\langle Y\rangle$ and $A\langle Y_0\rangle$, the associative commutative quasi-shuffle product [49] is defined for all $y_i, y_j \in Y_0$ and $u, v, w \in Y_0^*$ as follows [48]

$$\begin{aligned} w \uplus 1_{Y_0^*} &= 1_{Y^*} \uplus w = w,\\ y_i u \uplus y_j v &= y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v). \end{aligned}$$

Their associated coproducts, $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\uplus}$, are defined for $u_1, v_1, w_1 \in X^*$ and $u_2, v_2, w_2 \in Y_0^*$ as follows

$$\begin{aligned} \langle u_1 \sqcup\!\sqcup v_1 \mid w_1\rangle &= \langle u_1 \otimes v_1 \mid \Delta_{\sqcup\!\sqcup}(w_1)\rangle,\\ \langle u_2 \uplus v_2 \mid w_2\rangle &= \langle u_2 \otimes v_2 \mid \Delta_{\uplus}(w_2)\rangle. \end{aligned}$$

These operators are morphisms for the concatenation defined on the letters $x \in X$ and $y_k \in Y_0$ by

$$\begin{aligned} \Delta_{\sqcup\!\sqcup}(x) &= 1 \otimes x + x \otimes 1,\\ \Delta_{\uplus}(y_k) &= 1 \otimes y_k + y_k \otimes 1 + \sum_{i+j=k} y_i \otimes y_j. \end{aligned}$$

The algebras $(A\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ and $(A\langle Y\rangle, \uplus, 1_{Y^*})$ admit the sets of Lyndon words denoted, respectively, by $\mathcal{L}ynX$ and $\mathcal{L}ynY$, as pure transcendence bases [57]

[3]This coefficient is then $\langle S \mid u\rangle$ and $\langle P \mid u\rangle$.

(resp. [46, 47]). A pair of Lyndon words (l_1, l_2) is called the standard factorization of l if $l = l_1 l_2$ and l_2 is the smallest nontrivial proper right factor of l (for the lexicographic order) or, equivalently, its (Lyndon) longest such [53].

2.2. **Diagonal series on bialgebras.** Let $\mathcal{L}ie_A\langle X\rangle$ and $\mathcal{L}ie_A\langle\langle X\rangle\rangle$ denote the sets of, respectively, Lie polynomials and Lie series over X with coefficients in A [53, 57].

The CQMM theorem [7] guarantees that the connected $\mathbb{N}$-graded, co-commutative Hopf algebra[4] is the enveloping algebra of its primitive elements (hence, $A\langle X\rangle = \mathcal{U}(\mathcal{L}ie_A\langle X\rangle))$. Classically, the pair of dual bases, $\{P_w\}_{w\in X^*}$ expanded over the basis $\{P_l\}_{l\in\mathcal{L}ynX}$ of $\mathcal{L}ie_A\langle X\rangle$ and $\{S_w\}_{w\in X^*}$ containing the pure transcendence basis of the shuffle algebra denoted by $\{S_l\}_{l\in\mathcal{L}ynX}$, permits an expression of the diagonal series as follows [57]

$$\mathcal{D}_X := \sum_{w\in X^*} w\otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{S_l\otimes P_l}. \tag{2.2}$$

We also get two other connected $\mathbb{N}$-graded, co-commutative Hopf algebras isomorphic to the enveloping algebras of their Lie algebras of their primitive elements :

$$\mathcal{H}_{\sqcup\!\sqcup} := (A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \mathsf{e}) \cong \mathcal{U}(\mathcal{L}ie_A\langle Y\rangle),$$
$$\mathcal{H}_{\uplus} := (A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\uplus}, \mathsf{e}) \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\uplus})),$$

where $\mathrm{Prim}(\mathcal{H}_{\uplus}) = \mathrm{Im}(\pi_1) = \mathrm{span}_A\{\pi_1(w) \mid w \in Y^*\}$ and π_1 is the extended eulerian projector defined, for any $w \in Y^*$, by [46, 47]

$$\pi_1(w) = w + \sum_{k=2}^{(w)} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 \uplus \ldots \uplus u_k\rangle u_1\ldots u_k. \tag{2.3}$$

Denoting by (l_1, l_2) the standard factorization of $l \in \mathcal{L}ynY - Y$, let us consider

(1) The PBW basis $\{p_w\}_{w\in Y^*}$ of $\mathcal{U}(\mathcal{L}ie_A\langle Y\rangle)$ constructed recursively as follows [57]

$$\begin{cases} p_{y_n} = y_n, & \text{for } y_n \in Y,\\ p_l = [p_{l_1}, p_{l_2}], & \text{for } l \in \mathcal{L}ynY - Y,\ st(l) = (l_1, l_2),\\ p_w = p_{l_1}^{i_1}\ldots p_{l_k}^{i_k}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k} \text{ with } l_1,\ldots,l_k \in \mathcal{L}ynY,\ l_1 > \ldots > l_k. \end{cases} \tag{2.4}$$

(2) and, by duality[5], the basis $\{s_w\}_{w\in Y^*}$ of $(A\langle Y\rangle, \sqcup\!\sqcup, 1_{Y^*})$, *i.e.*

$$\langle p_u \mid s_v\rangle = \delta_{u,v} \quad \text{for all } u, v \in Y^*.$$

This linear basis can be computed recursively as follows [57].

$$\begin{cases} s_{y_n} = y_s, & \text{for } y_n \in Y,\\ s_l = y_n s_u, & \text{for } l = y_n u \in \mathcal{L}ynY,\\ s_w = \dfrac{s_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup s_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1!\ldots i_k!}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k} \text{ with}\\ & l_1,\ldots,l_k \in \mathcal{L}ynY,\ l_1 > \ldots > l_k. \end{cases} \tag{2.5}$$

[4]Here, e denotes the counit defined by $\mathsf{e}(P) = \langle P \mid 1_{X^*}\rangle$ (for any $P \in A\langle Y\rangle$).

[5]The dual family, *i.e.* the set of coordinates forms, is linearly free (but not a basis in general) in the algebraic dual which is the space of noncommutative series, but as the enveloping algebra under consideration is graded in finite dimension by multidegree. In Fact it consists of multihomogeneous polynomials.

As in (2.2), let $\mathcal{D}_{\shuffle}$ be the diagonal series on $\mathcal{H}_{\shuffle}$. Then [57]

$$\mathcal{D}_{\shuffle} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} s_w \otimes p_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{s_l\otimes p_l}.$$

THEOREM 2.1 ([47]). — *Let $\varphi_{\pi_1} : (A\langle Y\rangle, ., 1_{Y^*}) \to (A\langle Y\rangle, ., 1_{Y^*})$ be the endomorphism of algebras mapping y_k to $\pi_1(y_k)$. Then φ_{π_1} is an automorphism of $A\langle Y\rangle$ and it realizes an isomorphism from the bialgebra $\mathcal{H}_{\shuffle}$ to the bialgebra $\mathcal{H}_{\stuffle}$. In particular, the following diagram is commutative*

$$\begin{array}{ccc}
\mathbb{Q}\langle \bar{Y}\rangle & \xrightarrow{\ \Delta_{\shuffle}\ } & \mathbb{Q}\langle \bar{Y}\rangle \otimes \mathbb{Q}\langle \bar{Y}\rangle \\
{\scriptstyle \varphi_{\pi_1}}\downarrow & & \downarrow{\scriptstyle \varphi_{\pi_1}\otimes\varphi_{\pi_1}} \\
\mathbb{Q}\langle Y\rangle & \xrightarrow[\ \Delta_{\stuffle}\ ]{} & \mathbb{Q}\langle Y\rangle \otimes \mathbb{Q}\langle Y\rangle
\end{array}.$$

and

$$\mathcal{H}_{\stuffle} \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\stuffle})) \quad \textit{and} \quad \mathcal{H}_{\stuffle}^{\vee} \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\stuffle}))^{\vee}.$$

Moreover, the bases $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ of, respectively, $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\stuffle}))$ and $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\stuffle}))^{\vee}$, are images by φ_{π_1} and $\check{\varphi}_{\pi_1}^{-1}$ of $\{p_w\}_{w\in Y^*}$ and $\{s_w\}_{w\in Y^*}$.

Algorithmically[6], the families $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ of *polynomials homogenous for the weight* can be constructed directly and recursively as follows [3, 46, 47]

(1) The PBW basis $\{\Pi_w\}_{w\in Y^*}$ of $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\stuffle}))$:

$$\begin{cases}
\Pi_{y_s} = \pi_1(y_s), & \text{for } y_s\in Y,\\
\Pi_l \;\, = [\Pi_{l_1},\Pi_{l_2}], & \text{for } l\in\mathcal{L}ynY - Y,\ st(l)=(l_1,l_2),\\
\Pi_w = \Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k} \text{ with } l_1,\ldots,l_k\in\mathcal{L}ynY,\ l_1>\ldots>l_k.
\end{cases} \tag{2.6}$$

(2) and, by duality, *i.e.*

$$\langle \Pi_u \mid \Sigma_v\rangle = \delta_{u,v} \quad \text{for all } u,v\in Y^*,$$

the basis $\{\Sigma_w\}_{w\in Y^*}$ of $(\mathbb{Q}\langle Y\rangle, \stuffle, 1_{Y^*})$:

$$\begin{cases}
\Sigma_{y_s} = y_s, & \text{for } y_s\in Y,\\
\Sigma_l \;\, = \displaystyle\sum_{(*)} \frac{1}{i!} y_{s_{k_1}+\ldots+s_{k_i}} \Sigma_{l_1\ldots l_n}, & \text{for } l = y_{s_1}\ldots y_{s_k}\in\mathcal{L}ynY,\\
\Sigma_w = \dfrac{\Sigma_{l_1}^{\stuffle i_1} \stuffle \ldots \stuffle \Sigma_{l_k}^{\stuffle i_k}}{i_1!\ldots i_k!}, & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, \text{ with } \\ & l_1,\ldots,l_k\in\mathcal{L}ynY,\ l_1>\ldots>l_k.
\end{cases} \tag{2.7}$$

In $(*)$, the sum is taken over all $\{k_1,\ldots,k_i\}\subset\{1,\ldots,k\}$ and $l_1 \geqslant \ldots \geqslant l_n$ such that $(y_{s_1},\ldots,y_{s_k}) \overset{*}{\Leftarrow} (y_{s_{k_1}},\ldots,y_{s_{k_i}}, l_1,\ldots,l_n)$, where $\overset{*}{\Leftarrow}$ denotes the transitive closure of the relation on standard sequences, denoted by $\Leftarrow$ [3].

[6] In [4], other pairs of bases in duality for $\mathcal{H}_{\stuffle}$ are also proposed.

Let $\mathcal{D}_{\uplus} = \mathcal{D}_{\sqcup\!\sqcup}$ be the diagonal series[7] on Y. One has [46, 47]

$$\mathcal{D}_{\uplus} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\Sigma_l\otimes\Pi_l}. \tag{2.8}$$

More generally, under suitable conditions[8] these factorizations still hold for the φ-deformed shuffle product, thanks to an extension of Theorem 2.1 [6, 7, 31].

Now, let us consider the following morphism

$$\begin{aligned}\pi_Y^\circ : (A\,1_{X^*} \oplus A\langle X\rangle x_1, .) &\longrightarrow (A\langle Y\rangle, .),\\ x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 &\longmapsto y_{s_1}\ldots y_{s_r}, \quad \text{for } r\geqslant 1,\end{aligned}$$

and $\pi_Y^\circ(a) = a$ for any $a\in A$. The extension of π_Y° over $A\langle X\rangle$ is the map $\pi_Y : (A\langle X\rangle, .) \to (A\langle Y\rangle, .)$ satisfying $\pi_Y(p) = 0$ for any $p\in A\langle X\rangle x_0$. Hence, $\ker \pi_Y = A\langle X\rangle x_0$ and $\mathrm{Im}(\pi_Y) = A\langle Y\rangle$. Let π_X be the inverse of π_Y° :

$$\begin{aligned}\pi_X : (A\langle Y\rangle, .) &\longrightarrow (A\oplus A\langle X\rangle x_1, .),\\ y_{s_1}\ldots y_{s_r} &\longmapsto x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1, \quad \text{for } r\geqslant 1.\end{aligned}$$

For the scalar products, the projectors π_X and π_Y° are then mutually adjoints :

$$\forall p\in A\oplus A\langle X\rangle x_1, \quad \forall q\in A\langle Y\rangle, \quad \langle \pi_Y^\circ(p) \mid q\rangle = \langle p\mid \pi_X(q)\rangle.$$

We have $\pi_Y\circ\pi_X = \mathrm{Id}_X$. But $\pi_X\circ\pi_Y \neq \mathrm{Id}_Y$. It is an orthogonal projector of $A\langle X\rangle$ on $A\oplus A\langle X\rangle x_1$ parallel to $A\oplus A\langle X\rangle x_0$. Indeed $\ker(\pi_X\circ\pi_Y) = A\langle X\rangle x_0$ and $\mathrm{Im}(\pi_X\circ\pi_Y) = A\langle Y\rangle$.

The map π_X is a morphism of associative algebras with unity (AAU) and the map π_Y is multiplicative on $A.1_{X^*}\oplus A\langle X\rangle x_1$ but not on $A\langle X\rangle$. For example,

$$0 = \pi_Y(x_0)\pi_Y(x_1) \neq \pi_Y(x_0x_1) = \pi_Y^\circ(x_0x_1) = y_2.$$

These can be extended by linearity and continuity over $A\langle\!\langle X\rangle\!\rangle$ and $A\langle\!\langle Y\rangle\!\rangle$, respectively.

Lemma 2.2 ([53, 58]). — *$l\in\mathcal{L}ynX - \{x_0\}$ if and only if $\pi_Y(l)\in\mathcal{L}ynY$.*

2.3. **Exchangeable and noncommutative rational series.** Recall that a formal power series R is *exchangeable* if and only if two words have the same coefficient in $R\in A\langle\!\langle X\rangle\!\rangle$ whenever they have the same commutative image, *i.e.* for any $u, v\in X^*$, if $|u|_x = |v|_x$ for any $x\in X$ then $\langle R\mid u\rangle = \langle R\mid v\rangle$ [33]. It follows that an exchangeable series R takes the following form [33]

$$R = \sum_{i_0,i_1\geqslant 0} r_{i_0,i_1} x_0^{i_0} \sqcup\!\sqcup\, x_1^{i_1} = \sum_{i_0,i_1\geqslant 0} r_{i_0,i_1} \frac{x_0^{\sqcup\!\sqcup\, i_0}}{i_0!} \sqcup\!\sqcup\, \frac{x_1^{\sqcup\!\sqcup\, i_1}}{i_1!}. \tag{2.9}$$

The set of exchangeable series is denoted by $A_{\mathrm{exc}}\langle\!\langle X\rangle\!\rangle$.

Let $A^{\mathrm{rat}}\langle\!\langle X\rangle\!\rangle$ denote the closure, of $A\langle X\rangle$ in $A\langle\!\langle X\rangle\!\rangle$ under[9] $\{+, ., {}^*\}$. It is closed under shuffle [1]. A power series $R\in A^{\mathrm{rat}}\langle\!\langle X\rangle\!\rangle$ is said to be *rational*.

---

[7] The set-theoretical object is the same, but the different indexing here expresses the fact that they will be considered as living in different algebras.

[8] In fact associative commutative dualizable and moderate, see [6, 7, 31].

[9] Let $R\in A\langle\!\langle X\rangle\!\rangle$ be such that $\langle R\mid 1_{X^*}\rangle = 0$. Then $R^* = 1_{X^*} + R + R^2 + \cdots$..

Let $R \in A^{\mathrm{rat}}\langle\langle X\rangle\rangle$. By the Kleene-Schützenberger theorem [1] there exists a *linear representation* (β,μ,η) of dimension $n \geqslant 1$, where

$$\beta \in \mathcal{M}_{n,1}(A), \quad \mu : X^* \longrightarrow \mathcal{M}_{n,n}(A), \quad \eta \in \mathcal{M}_{1,n}(A) \tag{2.10}$$

such that

$$R = \sum_{w \in X^*} (\beta\mu(w)\eta)\, w = \beta\Big(\sum_{x\in X} \mu(x)x\Big)^* \eta.$$

Hence, letting $M(x) := \mu(x)x$ for $x \in X$, one has $M(X) = M(x_0) + M(x_1)$ as morphism of monoids, and, using Lazard's elimination [53, 58], one gets

$$M(X^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*).$$

Via the diagonal series $\mathcal{D}_X$ given in (2.2), the Kleene-Schützenberger theorem [1] can also be extended as follows

Theorem 2.3 ([36, 37, 43]). — *A series $R \in A\langle\langle X\rangle\rangle$ is rational if and only if there exists a linear representation (β,μ,η), of dimension $n \geqslant 1$, where*

$$\beta \in \mathcal{M}_{n,1}(A), \quad \mu : X^* \longrightarrow \mathcal{M}_{n,n}(A), \quad \eta \in \mathcal{M}_{1,n}(A)$$

such that

$$R = \beta((\mathrm{Id}\otimes\mu)\mathcal{D}_X)\eta = \beta\Big(\prod_{l\in\mathcal{L}ynX}^{\searrow} e^{S_l\mu(P_l)}\Big)\eta.$$

Now, let (β,μ,η) be a *minimal*[10] linear representation of $R \in A^{\mathrm{rat}}\langle\langle X\rangle\rangle$ [1], and let $\mathcal{L}(\mu)$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$. Moreover, if the matrices $\{\mu(x)\}_{x\in X}$ are triangular, then there are diagonal and nilpotent matrices, $\{D(x)\}_{x\in X}$ and $\{N(x)\}_{x\in X}$ in $\mathcal{M}_{n,n}(AX)$ such that $M(X) = D(X) + N(X)$. Hence, again by Lazard's elimination, one also gets

$$M(X^*) = ((D(X^*)T(X))^* D(X^*)). \tag{2.11}$$

The set of exchangeable rational series, *i.e.* $A^{\mathrm{rat}}\langle\langle X\rangle\rangle \cap A_{\mathrm{exc}}\langle\langle X\rangle\rangle$, is denoted by $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$. As examples, one can consider the following forms (F_0), (F_1) and (F_2) of rational series in $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ [35, 36, 37] :

(F_0) $E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}$, where $x_{i_1},\ldots,x_{i_j} \in X$ and $E_1,\ldots,E_j \in \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle$,

(F_1) $E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}$, where $x_{i_1},\ldots,x_{i_j} \in X$ and $E_1,\ldots,E_j \in \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$,

(F_2) $E_1x_{i_1}\ldots E_jx_{i_j}E_{j+1}$, where $x_{i_1},\ldots,x_{i_j} \in X$ and $E_1,\ldots,E_j \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$.

One has

Lemma 2.4. — (1) *Let $k \in \mathbb{N}_+, t_0, t_1 \in \mathbb{C}$. Then $(x_i^*)^{\sqcup\!\sqcup\, k} = (kx_i)^*$,*

$$(t_0x_0 + t_1x_1)^* = (t_0x_0)^* \sqcup\!\sqcup (t_1x_1)^* \quad \text{and} \quad (t_ix_i)^{*k} = (t_ix_i)^* \sqcup\!\sqcup (1 - t_ix_i)^{k-1}.$$

(2) *The series of form (F_0), (F_1) and (F_2) generate sub-bialgebras of $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, $\sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathtt{e})$.*

(3) *Let (β,μ,η) be a minimal linear representation of $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathcal{L}(\mu)$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$. Since $R = \beta M(X^*)\eta$,*

[10] Now, A is supposed to be a field.

(a) *R is a linear combination of expressions of the form (F_0) (resp. (F_1)) if and only if $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent*[11]*. Hence, if $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$ (resp. $\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$) then $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent.*

(b) *R is a linear combination of expressions of the form (F_2) if and only if $\mathcal{L}(\mu)$ is solvable*[12]*. Hence, if $R \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$ then $\mathcal{L}(\mu)$ is solvable.*

(c) *$R \in \mathbb{C}\langle X\rangle$ if and only if for any $P \in \mathcal{L}ie_{\mathbb{C}}\langle X\rangle$ the matrix $\mu(P)$, belonging to $\mathcal{L}(\mu)$, is nilpotent.*

(d) *$R \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \Leftrightarrow [\mu(x_0), \mu(x_1)] = 0 \Leftrightarrow R \in \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$.*

To end this section, let us note that for any $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal linear representation (β, μ, η) of dimension n and, for any $x, y \in X$ one has

$$\langle S \mid xy\rangle = \beta\mu(x)\mu(y)\eta = \sum_{i=1}^{n}(\beta\mu(x)e_i)(e_i^T\mu(y)\eta) = \sum_{i=1}^{n}\langle S_i^{(1)} \mid x\rangle\langle S_i^{(2)} \mid y\rangle,$$

where e_i is the vector such that $e_i^T = \begin{pmatrix}0 & \ldots & 0 & 1 & 0 & \ldots & 0\end{pmatrix}$. Hence $S_i^{(1)}$ (resp. $S_i^{(2)}$) admits (β, μ, e_i) (resp. (e_i^T, μ, η)) as a linear representation, and

$$(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathtt{e})$$

is nothing but the Sweedler dual of the bialgebra $(\mathbb{C}\langle X\rangle, \mathtt{conc}, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \mathtt{e})$ [57].

3. Indexation by words and generating series

3.1. **Indexation by words.** For any $r \in \mathbb{N}$, any multiindex$(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ can be associated with the words $x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1 \sqcup \{1_{X^*}\}$. Similarly, any[13] $(s_1, \ldots, s_r) \in \mathbb{N}^r$ can be associated with the word $y_{s_1} \ldots y_{s_r} \in Y_0^*$. Put $\mathrm{Li}_{x_0^r}(z) := (\log(z))^r/r!$.

(1) Let $\mathrm{Li}_{s_1,\ldots,s_k}$ and $\mathrm{H}_{s_1,\ldots,s_k}$ be indexed by words [38, 39] :

$$\mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} := \mathrm{Li}_{s_1,\ldots,s_r} \quad\text{and}\quad \mathrm{H}_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{s_1,\ldots,s_r}.$$

(2) Let $\mathrm{Li}_{-s_1,\ldots,-s_k}$ and $\mathrm{H}_{-s_1,\ldots,-s_k}$ be indexed by words [21, 22] :

$$\mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{Li}_{-s_1,\ldots,-s_r} \quad\text{and}\quad \mathrm{H}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{-s_1,\ldots,-s_r}.$$

In particular, $\mathrm{Li}^-_{y_0^r}(z) := (z/(1-z))^r$ and $\mathrm{H}^-_{y_0^r}(n) := \binom{n}{r} = (n)_r/r!$, where $(n)_r = (n+r)\ldots(n)$.

All of $\{\mathrm{Li}^-_w\}_{w\in Y_0^*}$ and $\{\mathrm{H}^-_w\}_{w\in Y_0^*}$ are divergent at their singularities.

Theorem 3.1 ([41, 38, 42]). — (1) *The following morphisms of algebras are* injective *(and* surjective *by definition)*

$$\mathrm{H}_\bullet : (\mathbb{Q}\langle Y\rangle, \text{⧢}, 1_{Y^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, \times, 1), \qquad w \longmapsto \mathrm{H}_w,$$

$$\mathrm{Li}_\bullet : (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1_{\tilde{B}}), \qquad w \longmapsto \mathrm{Li}_w$$

(2) *The families $\{\mathrm{H}_w\}_{w\in Y^*}$ and $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathbb{Q}$-linearly independent.*

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[11] Using (2.10), one gets the expected expression for R.

[12] By Lie's theorem [15], using (2.11), one gets the expected expression for R.

[13] The weight of $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ (resp. $\mathbb{N}^r$) is defined as the integer $s_1 + \ldots + s_r$ which corresponds to the weight, denoted (w), of its associated word $w \in Y^*$ (resp. Y_0^*) and, if $w \in Y^*$, it corresponds also to the length, denoted by $|u|$, of its associated word $u \in X^*$.

(3) *The families* $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ *and* $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ *are* $\mathbb{Q}$*-algebraically independent.*

But at the singularities $\{1,+\infty\}$, for any $u\in x_0X^*x_1$ (resp. $u\in Y^*-y_1Y^*$) Li_u (resp. H_u) receives the value $\zeta(v):=\mathrm{Li}_v(1)$ (resp. $\zeta(u):=\mathrm{H}_u(+\infty)$) and are no more linearly independent (and then the values $\{\mathrm{H}_l(+\infty)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (resp. $\{\mathrm{Li}_l(1)\}_{l\in\mathcal{L}ynX-X}$) are no longer algebraically independent) [38, 40, 59].

There also exists a law of algebra, denoted by $\top$, in $\mathbb{Q}\langle\langle Y_0\rangle\rangle$ (which is not dualizable) [6, 31] such that

Theorem 3.2 ([21]). — *Let us consider the following morphisms of algebras (which, by definition, are* surjective*)*

$$\begin{aligned}\mathrm{H}^-_\bullet &: (\mathbb{Q}\langle Y_0\rangle,\ ⧢\ ,1_{Y_0^*})\longrightarrow(\mathbb{Q}\{\mathrm{H}^-_w\}_{w\in Y_0^*},\times,1), && w\longmapsto\mathrm{H}^-_w,\\ \mathrm{Li}^-_\bullet &: (\mathbb{Q}\langle Y_0\rangle,\top,1_{Y_0^*})\longrightarrow(\mathbb{Q}\{\mathrm{Li}^-_w\}_{w\in Y_0^*},\times,1_{\tilde{B}}), && w\longmapsto\mathrm{Li}^-_w\,.\end{aligned}$$

Then $\ker\mathrm{H}^-_\bullet=\ker\mathrm{Li}^-_\bullet=\mathbb{Q}\langle\{w-w\top 1_{Y_0^*}|w\in Y_0^*\}\rangle$ *and the families* $\{\mathrm{H}^-_{y_k}\}_{k\geqslant 0}$ *and* $\{\mathrm{Li}^-_{y_k}\}_{k\geqslant 0}$ *are* $\mathbb{Q}$*-linearly independent.*

Moreover, let $\top':\mathbb{Q}\langle Y_0\rangle\times\mathbb{Q}\langle Y_0\rangle\to\mathbb{Q}\langle Y_0\rangle$ *be a law such that* $\mathrm{Li}^-_\bullet$ *is a morphism for* $\top'$ *and* $(1_{Y_0^*}\top'\mathbb{Q}\langle Y_0\rangle)\cap\ker(\mathrm{Li}^-_\bullet)=\{0\}$*. Then* $\top'=g\circ\top$*, where* $g\in GL(\mathbb{Q}\langle Y_0\rangle)$ *is such that* $\mathrm{Li}^-_\bullet\circ g=\mathrm{Li}^-_\bullet$*.*

Now, for any $i\in\mathbb{N}$ let $t_i\in\mathbb{C}$ be such that $|t_i|<1$ and $z\in\mathbb{C}$ satisfying $|z|<1$. Then [35] (to be compared with (1.4) and (1.5))

$$\sum_{n\geqslant 0}\mathrm{Li}_{x_0^n}(z)\,t_0^n=z^{t_0}\quad\text{and}\quad\sum_{n\geqslant 0}\mathrm{Li}_{x_1^n}(z)\,t_1^n=(1-z)^{-t_1}.\tag{3.1}$$

What precedes suggests to extend the domain of $\mathrm{Li}_\bullet$ which is, up to now and through linear extension, restricted to $\mathbb{C}\langle X\rangle$, to some rational series as follows.

3.2. **Indexation by noncommutative rational series.** Let us call $\mathrm{Dom}(\mathrm{Li}_\bullet)$ the set of series of $\mathbb{C}\langle\langle X\rangle\rangle$

$$S=\sum_{n\geqslant 0}S_n\quad\text{with}\quad S_n:=\sum_{|w|=n}\langle S\mid w\rangle w$$

such that the following sum converges uniformly on all compacts of $\tilde{B}$

$$\sum_{n\geqslant 0}\mathrm{Li}_{S_n}\,.\tag{3.2}$$

One can check easily that [22] :

- The set $\mathrm{Dom}(\mathrm{Li}_\bullet)$ is closed under shuffle products.
- For any $S,T\in\mathrm{Dom}(\mathrm{Li}_\bullet)$ one has $\mathrm{Li}_{S ⧢ T}=\mathrm{Li}_S\,\mathrm{Li}_T$.
- One has $\mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle\subset\mathrm{Dom}(\mathrm{Li}_\bullet)$.

This extension is compatible with identities between rational series such as Lazard's elimination [53, 58], for instance (see Appendix C) :

$$\mathrm{Li}_S(z)=\sum_{n\geqslant 0}\langle S\mid x_0^n\rangle\frac{\log^n(z)}{n!}+\sum_{k\geqslant 1}\sum_{w\in x_0^* ⧢ x_1^k}\langle S\mid w\rangle\,\mathrm{Li}_w(z),$$

and explains that, for R as given in (2.9), Li_R is expressible as analytic composition of $\log(z)$ and $\log(1-z)$:

$$\mathrm{Li}_R(z)=\sum_{i_0,i_1\geqslant 0}\frac{r_{i_0,i_1}}{i_0!i_1!}\log^{i_0}(z)(-\log(1-z))^{i_1}.$$

Example 3.3. — Consider the extension of $\mathrm{Li}_\bullet$ defined in (3.2). Then [35, 36, 37]

(1) By (3.1), $\mathrm{Li}_{(t_0x_0)^*}(z)=z^{t_0}$ and $\mathrm{Li}_{(t_1x_1)^*}(z)=(1-z)^{-t_1}$. More generally, for any $i,j\in\mathbb{N}_+$, one has by Lemma 2.4

$$\mathrm{Li}_{((t_0x_0)^*)^{ш\, i}\, ш\, ((t_1x_1)^*)^{ш\, j}}(z)=z^{it_0}(1-z)^{-jt_1},$$
$$\mathrm{Li}_{(t_0x_0+t_1x_1)^*\, ш\, x_0^i\, ш\, x_1^j}(z)=\frac{z^{t_0}}{(1-z)^{t_1}}\frac{\log^i(z)\log^j((1-z)^{-1})}{i!j!}.$$

(2) For $a\in\mathbb{C}$ and $i\in\mathbb{N}_+$, one has by Lemma 2.4

$$\mathrm{Li}_{(ax_0)^{*i}}(z)=z^a\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{(a\log(z))^k}{k!},\tag{3.3}$$

$$\mathrm{Li}_{(ax_1)^{*i}}(z)=\frac{1}{(1-z)^a}\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{(-a\log(1-z))^k}{k!}.\tag{3.4}$$

(3) From the previous points, one has (see Lemma 2.4)

$$\begin{aligned}
\{\mathrm{Li}_S\}_{S\in\mathbb{C}[x_0^*]\, ш\, \mathbb{C}[(-x_0^*)]\, ш\, \mathbb{C}[x_1^*]}&=\mathrm{span}_\mathbb{C}\{z^a(1-z)^{-b}\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\{\mathrm{Li}_S\}_{S\in\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\, ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle}&=\mathrm{span}_\mathbb{C}\{z^a(1-z)^{b}\}_{a,b\in\mathbb{C}},\\
\{\mathrm{Li}_S\}_{S\in\mathbb{C}\langle X\rangle\, ш\, \mathbb{C}[x_0^*]\, ш\, \mathbb{C}[(-x_0^*)]\, ш\, \mathbb{C}[x_1^*]}&=\mathrm{span}_\mathbb{C}\left\{\frac{z^a}{(1-z)^b}\mathrm{Li}_w(z)\right\}_{a\in\mathbb{Z},b\in\mathbb{N}}^{w\in X^*}\\
&\subset\mathrm{span}_\mathbb{C}\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{Z}^r}^{r\geqslant 1}\\
&\quad\oplus\mathrm{span}_\mathbb{C}\{z^a|a\in\mathbb{Z}\},\\
\{\mathrm{Li}_S\}_{S\in\mathbb{C}\langle X\rangle\, ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\, ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle}&=\mathrm{span}_\mathbb{C}\left\{\frac{z^a}{(1-z)^b}\mathrm{Li}_w(z)\right\}_{a,b\in\mathbb{C}}^{w\in X^*}\\
&\subset\mathrm{span}_\mathbb{C}\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{C}^r}^{r\geqslant 1}\\
&\quad\oplus\mathrm{span}_\mathbb{C}\{z^a|a\in\mathbb{C}\},
\end{aligned}$$

(4) For any $(s_1,\ldots,s_r)\in\mathbb{N}_+^r$ and $|t_i|<1$, let

$$W=(t_1x_0)^{*s_1}x_0^{s_1-1}x_1\ldots(t_rx_0)^{*s_r}x_0^{s_r-1}x_1$$

(which is of the form[14] (F_0) of Lemma 2.4). Then[15]

$$\mathrm{Li}_W(z)=\sum_{n_1>\ldots>n_r>0}\frac{z^{n_1}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}.$$

[14]For the form (F_0) one can apply Theorems 2.3 and 2.4 of [35].
[15]This holds for $t_i\in\mathbb{C}-\mathbb{N}_+$, $i\in\mathbb{N}$, by analytic continuation [39].

In particular, for $s_1 = \ldots = s_r = 1$ one has

$$\mathrm{Li}_{(t_1x_0)^*x_1\ldots(t_rx_0)^*x_1} = \sum_{n_1,\ldots,n_r>0} \mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}\, t_0^{n_1-1}\ldots t_r^{n_r-1}$$
$$= \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{(n_1-t_1)\ldots(n_r-t_r)}.$$

Let $\partial_z := d/dz$ and let us recall that, for any $k \geqslant 1$,

$$\frac{1}{(1-z)^k} = \frac{\partial_z^{k-1}}{(k-1)!}\left(\frac{1}{1-z}\right) \quad \text{and} \quad \frac{1}{z^k} = (-1)^{k-1}\frac{\partial_z^{k-1}}{(k-1)!}\left(\frac{1}{z}\right)$$

and the Taylor coefficients of $(1-z)^{-k}$ are expressed as follows for all $n \geqslant 1$

$$\langle (1-z)^{-k} \mid z^n\rangle = \Gamma^{-1}(k)(n+k-1)_{k-1}. \tag{3.5}$$

Let $\mathcal{G}$ denote the group of transformations of[16] B generated by $\{z \mapsto 1-z, z \mapsto 1/z\}$, permuting the singularities in $\{0, 1, +\infty\}$ as a copy of $\mathfrak{S}_3$.

Let us also consider the differential rings

$$\mathcal{C}_0' = \mathbb{C}[z^{-1}], \qquad \mathcal{C}_1' = \mathbb{C}[(1-z)^{-1}], \qquad \mathcal{C}_0 = \mathbb{C}[z, z^{-1}],$$
$$\mathcal{C}_1 = \mathbb{C}[z, (1-z)^{-1}], \quad \mathcal{C}' = \mathbb{C}[z^{-1}, (1-z)^{-1}], \quad \mathcal{C} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}]$$

(considered as subrings of $\mathcal{H}(B)$). It follows that

Lemma 3.4. — (1) *The differential ring $\mathcal{C}$ is closed under the action of $\mathcal{G}$, i.e. $G(g(z)) \in \mathcal{C}$ for all $G \in \mathcal{C}$ and $g \in \mathcal{G}$.*

(2) *For any $G = p_1(z) + p_2(z^{-1}) + p_3((1-z)^{-1}) \in \mathcal{C}$, with $p_1, p_2, p_3 \in \mathbb{C}[z]$, $p_2(0) = p_3(0) = 0$ and $p_2, p_3 \neq 0$. Letting $G_0(z) := P_2(z^{-1}) \in \mathcal{C}_0'$ and $G_1(z) := P_3((1-z)^{-1}) \in \mathcal{C}_1'$, one has $G(z) \sim_0 G_0(z)$ and $G(z) \sim_1 G_1(z)$.*

(3) *The following morphism of algebras is* surjective

$$\lambda : (\mathbb{C}[x_0^*, (-x_0)^*, x_1^*], ш, 1_{X^*}) \longrightarrow (\mathcal{C}, \times, 1_B), \quad R \longmapsto \mathrm{Li}_R\,.$$

Moreover, $\ker(\lambda)$ *is the shuffle-ideal generated by $x_0^* ш x_1^* - x_1^* + 1$.*

(4) *The following morphisms of algebras are* bijective

$$\lambda' : \quad (\mathbb{C}[x_0^*, x_1^*], ш, 1_{X^*}) \longrightarrow (\mathcal{C}', \times, 1_B), \qquad R \longmapsto \mathrm{Li}_R,$$
$$\lambda_i' : \quad (\mathbb{C}[x_i^*], ш, 1_{X^*}) \longrightarrow (\mathcal{C}_i', \times, 1_B), \qquad R \longmapsto \mathrm{Li}_R \qquad \text{for } i = 0, 1.$$

In fact, one has

Lemma 3.5 ([21]). — (1) *The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ in $(\mathbb{C}\langle\!\langle X\rangle\!\rangle, ш, 1_{X^*})$. In particular, the power series x_0^* and x_1^* are transcendent over $\mathbb{C}\langle X\rangle$.*

(2) *The module $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is $\mathbb{C}\langle X\rangle$-free and the family $\{(x_0^*)^{ш k} ш (x_1^*)^{ш l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ forms a $\mathbb{C}\langle X\rangle$-basis of it.*

Hence, $\{w ш (x_0^*)^{ш k} ш (x_1^*)^{ш l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}_{w\in X^*}$ is a $\mathbb{C}$-basis of it.

(3) *One has $\mathbb{C}^{\mathrm{rat}}\langle\!\langle x_i\rangle\!\rangle = \mathrm{span}_{\mathbb{C}}\{(tx_i)^* ш \mathbb{C}\langle x_i\rangle \mid t \in \mathbb{C}\}$ for any $x_i \in X$.*

[16]Any $g \in \mathcal{G}$ maps bijectively B to itself, one can apply the Monodromy Principle to lift $\mathcal{G}$ as a group of transformations of $\tilde{B}$.

Now, let us also consider the following differential integration operators acting on $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ [46] :

$$\theta_0 := z\partial_z \qquad \text{and} \qquad \theta_1 := (1-z)\partial_z,$$

$$\forall f\in\mathcal{C},\quad \iota_0(f)=\int_{z_0}^{z} f(s)\omega_0(s) \qquad \text{and} \qquad \iota_1(f)=\int_0^z f(s)\omega_1(s).$$

The operator ι_0 is well-defined on $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (see Definition 6.5 in Appendix D, where the choice of z_0 is recalled). One can check easily

Proposition 3.6 ([22, 38, 41]). — (1) *The operators* $\{\theta_0,\theta_1,\iota_0,\iota_1\}$ *satisfy*

$$\theta_1+\theta_0=\big[\theta_1,\theta_0\big]=\partial_z \qquad \textit{and} \qquad \theta_k\iota_k=\mathrm{Id}\ \textit{for}\ k=0,1,$$
$$[\theta_0\iota_1,\theta_1\iota_0]=0 \qquad \textit{and} \qquad (\theta_0\iota_1)(\theta_1\iota_0)=(\theta_1\iota_0)(\theta_0\iota_1)=\mathrm{Id}.$$

(2) *The subspace* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *is closed under the action of* $\{\theta_0,\theta_1\}$ *and* $\{\iota_0,\iota_1\}$. *Thus, for any* $w=y_{s_1}\dots y_{s_r}\in Y^*$ *(whence* $\pi_X(w)=x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1$*) and* $u=y_{t_1}\dots y_{t_r}\in Y_0^*$, *the functions* Li_w *and* Li^-_u *satisfy*

$$\mathrm{Li}_w=(\iota_0^{s_1-1}\iota_1\dots\iota_0^{s_r-1}\iota_1)1_\Omega \qquad \textit{and} \qquad \mathrm{Li}^-_u=(\theta_0^{t_1+1}\iota_1\dots\theta_0^{t_r+1}\iota_1)1_\Omega,$$
$$\iota_0\,\mathrm{Li}_{\pi_X(w)}=\mathrm{Li}_{x_0\pi_X(w)} \qquad \textit{and} \qquad \iota_1\,\mathrm{Li}_w=\mathrm{Li}_{x_1\pi_X(w)},$$
$$\theta_0\,\mathrm{Li}_{x_0\pi_X(w)}=\mathrm{Li}_{\pi_X(w)} \qquad \textit{and} \qquad \theta_1\,\mathrm{Li}_{x_1\pi_X(w)}=\mathrm{Li}_{\pi_X(w)}\,.$$

(3) *The bi-integro differential ring* $(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*},\theta_0,\iota_0,\theta_1,\iota_1)$ *is stable under the action of* $\mathcal{G}$, *i.e. for all* $h\in\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *and* $g\in\mathcal{G}$

$$h(g(z))\in\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

(4) $\theta_0\iota_1$ *and* $\theta_1\iota_0$ *are scalar operators in* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, *respectively with eigenvalues* $\lambda:=z\to z(1-z)$ *and* $1/\lambda$. *I.e. for all* $f\in\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *one has*

$$(\theta_0\iota_1)f=\lambda f \quad \textit{and} \quad (\theta_1\iota_0)f=(1/\lambda)f.$$

3.3. **Noncommutative generating series.** The graphs (typed as series) of the isomorphisms of algebras, $\mathrm{Li}_\bullet$ and $\mathrm{H}_\bullet$, defined in Theorem 3.1, then become

Definition 3.7 ([13, 38, 40, 42]). — Let us consider the following power series

$$\mathrm{L}:=\sum_{w\in X^*}\mathrm{Li}_w\,w \quad \text{and} \quad \mathrm{H}:=\sum_{w\in Y^*}\mathrm{H}_w w.$$

With suitable structures (topological ring [8]), by (2.2) and (2.8), one can write $\mathrm{H}=(\mathrm{H}_\bullet\otimes\mathrm{Id}_Y)\mathcal{D}_{⧢}$ and $\mathrm{L}=(\mathrm{Li}_\bullet\otimes\mathrm{Id}_X)\mathcal{D}_X$. Thus, by Theorem 3.1, one obtains

Proposition 3.8 ([38, 40, 46, 47]). — *One has*

$$\Delta_{⧢}(\mathrm{H})=\mathrm{H}\otimes\mathrm{H} \qquad \textit{and} \qquad \langle\mathrm{H}\mid 1_{Y^*}\rangle=1,$$
$$\Delta_{⧢}(\mathrm{L})=\mathrm{L}\otimes\mathrm{L} \qquad \textit{and} \qquad \langle\mathrm{L}\mid 1_{X^*}\rangle=1,$$
$$\mathrm{H}=\prod_{l\in\mathcal{L}ynY}^{\searrow}e^{\mathrm{H}_{\Sigma_l}\Pi_l} \qquad \textit{and} \qquad \mathrm{L}=\prod_{l\in\mathcal{L}ynX}^{\searrow}e^{\mathrm{Li}_{S_l}\,P_l}.$$

Hence[17], *their logarithms are primitive, for the corresponding co-products, and*[18]

$$\log(\mathrm{H}) = \sum_{k\geqslant 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \mathrm{H}_{u_1 \,⧢\!\!-\, \ldots \,⧢\!\!-\, u_k}\, u_1\ldots u_k,$$
$$\log(\mathrm{L}) = \sum_{k\geqslant 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in X^+} \mathrm{Li}_{u_1 \,⧢\, \ldots \,⧢\, u_k}\, u_1\ldots u_k.$$

One can then set the following :

Definition 3.9. — Let us consider the following power series

$$Z_{⧢\!\!-} := \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}(+\infty)\Pi_l} \quad\text{and}\quad Z_{⧢} := \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\mathrm{Li}_{S_l}(1)P_l}.$$

By termwise differentiation, the power series L given in Definition 3.7 satisfies the noncommutative differential equation (1.1) and, via the factorization form given in Proposition 3.8, it also satisfies the boundary condition [38, 41]

$$\mathrm{L}(z) \sim_0 e^{x_0\log(z)} \quad\text{and}\quad \mathrm{L}(z) \sim_1 e^{-x_1\log(1-z)}.$$

Equation (1.8) and Theorem 3.1 lead to

Definition 3.10. — We define ζ to be the following polymorphism (which is surjective by definition):

$$\zeta : \begin{array}{c}(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1, ⧢, 1_{X^*})\\ (\mathbb{Q}1_{Y^*}\oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle, ⧢\!\!-, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z},\times,1),$$
$$\begin{array}{c} x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1}\\ y_{s_1}\ldots y_{s_k}\end{array} \longmapsto \sum_{n_1>\ldots>n_k>0} n_1^{-s_1}\ldots n_k^{-s_k},$$

where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra (algebraically) generated by[19] $\{\zeta(l)\}_{l\in\mathcal{L}ynX-X}$ (resp. $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$), or, equivalently, $\{\zeta(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (resp. $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$).

4. Global asymptotic behaviors at singularities

4.1. **The case of positive multi-indices.**

The analysis of singularities on the coefficients of the noncommutative generating series of $\{\mathrm{Li}_w\}_{w\in X^*}$, put in the factorized form (see Proposition 3.8) leads to[20] [38, 41]

$$\lim_{z\to 0} \mathrm{L}(z)e^{-x_0\log z} = 1 \quad\text{and}\quad \lim_{z\to 1} e^{x_1\log(1-z)}\mathrm{L}(z) = Z_{⧢}. \tag{4.1}$$

Knowing that G_0 and G_1, as interpreted in (1.4), are unique and by (1.5), it turns out that, through the interpretation given, $Z_{⧢}$ coincides with Φ_{KZ} [34, 55] and, via an identity of type Newton-Girard [51], we obtain [14, 16, 45]

$$\mathrm{H}(n) \sim_{+\infty} \sum_{k\geqslant 0} \mathrm{H}_{y_1^k} y_1^k \pi_Y(Z_{⧢}) \quad\text{and}\quad \sum_{k\geqslant 0} \mathrm{H}_{y_1^k} y_1^k = e^{\sum_{k\geqslant 1}\mathrm{H}_{y_k}(n)(-y_1)^k/k}. \tag{4.2}$$

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[17] via Friedrich's criterion [57] and its extension [46, 47]

[18] From log(L), one can extract the expression of the euleurian projector on $\mathcal{H}_{⧢}$ [57] and similarly, from log(H), for the extended euleurian projector, as given in (2.3), on $\mathcal{H}_{⧢\!\!-}$ [46, 47].

[19] We will describe relations among $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$ (resp. $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$) by local coordinate identification in Section 4.2.

[20] *i.e.* $\mathrm{L}(z) \sim_0 z^{x_0}$ and $\mathrm{L}(z) \sim_1 (1-z)^{-x_1} Z_{⧢}$.

In other terms, we have the following global renormalization

THEOREM 4.1 (First Abel like theorem, [14, 16, 45]). —

$$\lim_{z\to 1} e^{y_1 \log(1-z)}\pi_Y(\mathrm{L}(z)) = \lim_{n\to\infty} e^{\sum_{k\geqslant 1}\mathrm{H}_{y_k}(n)(-y_1)^k/k}\mathrm{H}(n) = \pi_Y(Z_{\sqcup\!\sqcup}).$$

Thus, the coefficients $\{\langle Z_{\sqcup\!\sqcup}|u\rangle\}_{u\in X^*}$ (*i.e.* $\{\zeta_{\sqcup\!\sqcup}(u)\}_{u\in X^*}$) and $\{\langle Z_{\uplus}|v\rangle\}_{v\in Y^*}$ (*i.e.* $\{\zeta_{\uplus}(v)\}_{v\in Y^*}$) represent, respectively, the finite part of the singular expansion, in the comparison scale $\{(1-z)^{-a}\log^b(1-z)\}_{a,b\in\mathbb{N}}$, of Li_w at $z=1$

$$\mathrm{f.p.}_{z\to 1}\,\mathrm{Li}_w(z) = \zeta_{\sqcup\!\sqcup}(w), \quad \{(1-z)^a\log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{4.3}$$

and the asymptotic expansion, in $\{n^{-a}\mathrm{H}_1^b(n)\}_{a,b\in\mathbb{N}}$, of H_w for $n\to+\infty$:

$$\mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n) = \zeta_{\uplus}(w), \quad \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \tag{4.4}$$

For commodity, we will denote

$$\mathrm{F.P.}_{z\to 1}\mathrm{L}(z) = Z_{\sqcup\!\sqcup}, \quad \{(1-z)^a\log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{4.5}$$

$$\mathrm{F.P.}_{n\to+\infty}\mathrm{H}(n) = Z_{\uplus}, \quad \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \tag{4.6}$$

On the other hand, by a transfer theorem [32], let $\{\gamma_w\}_{w\in Y^*}$ be the finite part of an asymptotic expansion, in $\{n^{-a}\log^b(n)\}_{a,b\in\mathbb{N}}$, of $\{\mathrm{H}_w\}_{w\in Y^*}$ for $n\to+\infty$:

$$\mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n) = \gamma_w, \quad \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.$$

Then let Z_γ be the noncommutative generating series of $\{\gamma_w\}_{w\in Y^*}$. One has

$$\mathrm{F.P.}_{n\to+\infty}\mathrm{H}(n) = Z_\gamma := \sum_{w\in Y^*}\gamma_w w, \quad \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \tag{4.7}$$

PROPOSITION 4.2 ([46, 47]). — (1) *The following map is a character*

$$\gamma_\bullet : (\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*}) \longrightarrow (\mathcal{Z}[\gamma], \times, 1), \quad w \longmapsto \gamma_w.$$

(2) *Equivalently, one has* $\Delta_{\uplus}(Z_\gamma) = Z_\gamma\otimes Z_\gamma$ *and* $\langle Z_\gamma \mid 1_{Y^*}\rangle = 1$. *Hence,*

$$Z_\gamma = e^{\gamma y_1}\prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\zeta(\Sigma_l)\Pi_l} = e^{\gamma y_1}Z_{\uplus}$$

and $\Delta_{\uplus}(\log(Z_\gamma)) = \log(Z_\gamma)\otimes 1_{Y^*} + 1_{Y^*}\otimes\log(Z_\gamma)$. *It follows then*

$$\log(Z_\gamma) = \sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\gamma_{u_1\uplus\ldots\uplus u_k}u_1\ldots u_k.$$

The asymptotic behaviors on (4.2) and Proposition 4.2 lead to

PROPOSITION 4.3 (Bridge equation, [14, 16, 45, 46, 47]). — *Put*[21]

$$B(y_1) = \exp\left(\gamma y_1 - \sum_{k\geqslant 2}\frac{\zeta(k)}{k}(-y_1)^k\right) \quad \textit{and} \quad B'(y_1) = \exp\left(-\sum_{k\geqslant 2}\frac{\zeta(k)}{k}(-y_1)^k\right).$$

Then $Z_\gamma = B(y_1)\pi_Y(Z_{\sqcup\!\sqcup})$, *or equivalenty by cancellation,* $Z_{\sqcup\!\sqcup} = B'(y_1)\pi_Y(Z_{\sqcup\!\sqcup})$.

[21] The power series $B(y_1)$ corresponds to the Taylor expansion of $\Gamma^{-1}(y_1+1)$.

4.2. **Structure of polyzetas.** Now, via Proposition 4.3, let us draw some consequences about the structure of polyzetas : by local coordinates identification in the assertions of Proposition 4.3, one obtains two families of polynomials, homogenous for the weight, $\{Q_l\}_{l\in\mathcal{L}ynX-X}$ and $\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (see Example 6.2 in Appendix A), such that [46, 47]

$$\mathcal{R}_X := (\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynX-X}, ш, 1_{X^*}) \quad = \ker(\zeta),$$
$$(\text{resp. } \mathcal{R}_Y := (\mathbb{Q}\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}, ⧢, 1_{Y^*}) = \ker(\zeta))$$

describing the kernel of ζ (see Example 6.1 in Appendix A), via homogenous polynomial relations for the weight, among the local coordinates of $Z_{ш}$ (resp. $Z_{⧢}$), *i.e.* the convergent values[22] $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$ (resp. $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$).

Denoting $\mathcal{X}$ the alphabet X or Y, this local coordinate identification yields algebraic generator systems (see Example 6.3 in Appendix A) as irreducible[23] local coordinates (see Example 6.4 in Appendix A)

$$\mathcal{Z}_{irr}^{\infty}(\mathcal{X}) := \lim_{p\to+\infty} \mathcal{Z}_{irr}^{\leqslant p}(\mathcal{X}) \quad \text{with} \quad \mathcal{Z}_{irr}^{\leqslant 2}(\mathcal{X}) \subset \ldots \mathcal{Z}_{irr}^{\leqslant p}(\mathcal{X}) \subset \ldots, \tag{4.8}$$

such that the restriction of ζ on $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$ is bijective [46, 47], where (see Example 6.4 in Appendix A)

$$\mathcal{L}_{irr}^{\infty}(\mathcal{X}) := \lim_{p\to+\infty} \mathcal{L}_{irr}^{\leqslant p}(\mathcal{X}) \quad \text{with} \quad \mathcal{L}_{irr}^{\leqslant 2}(\mathcal{X}) \subset \ldots \mathcal{L}_{irr}^{\leqslant p}(\mathcal{X}) \subset \ldots, \tag{4.9}$$

and, for any $p \geqslant 2$, $\mathcal{L}_{irr}^{\leqslant p}(\mathcal{X})$ is the inverse image of $\mathcal{Z}_{irr}^{\leqslant p}(\mathcal{X})$.

Generated by homogenous polynomials for the weight (see Example 6.2 in Appendix A), $\ker(\zeta)$ is then graded. Moreover, since $\mathcal{Z} = \mathrm{Im}(\zeta)$, one obtains

Corollary 4.4 ([46, 47]). — *One has*

$$\mathcal{Z} \cong \mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker(\zeta),$$
$$(\textit{resp. } \mathcal{Z} \cong \mathbb{Q}1_{Y^*} \oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle/\ker(\zeta)).$$

Hence, $\mathcal{Z}$ is graded as the quotient of a graded algebra by a graded ideal :

$$\mathcal{Z} = \mathbb{Q}1 \oplus \bigoplus_{p\geqslant 2} \mathcal{Z}^p,$$

where for any $p \geqslant 2$,

$$\mathcal{Z}^p = \mathrm{span}_{\mathbb{Q}}\{\zeta(w) | w \in x_0X^*x_1, |w| = p\},$$
$$(\textit{resp. } \mathcal{Z}^p = \mathrm{span}_{\mathbb{Q}}\{\zeta(w) | w \in (Y-\{y_1\})Y^*, (w) = p\}).$$

Remark 4.5. — Note that $\mathcal{L}yn\mathcal{X}$ is totally ordered, and so is $\mathcal{L}_{irr}^{\infty}(\mathcal{X})$, as being extracted from $\mathcal{L}yn\mathcal{X}$. Hence, for any fixed integer $n \geqslant 1$, it is immediate that

(1) letting $l \in \mathcal{L}yn\mathcal{X}$ such that $(l) = n$, one has $y_n \preceq l$ (resp. $x_0^{n-1}x_1 \preceq l$),
(2) $\Sigma_{y_n} = y_n \in \mathcal{L}ynY$ and $S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \mathcal{L}ynX$ (see Lemma 2.2),
(3) $\Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}_{irr}^{\infty}(X)$,
(4) $\zeta(2) = \zeta(\Sigma_{y_2}) = \zeta(S_{x_0x_1})$ is irreducible and, by Euler's identity about the ratio $\zeta(2k)/\pi^{2k}$, one deduces that $\Sigma_{y_{2k}} = y_{2k} \notin \mathcal{L}_{irr}^{\infty}(Y)$ and $S_{x_0^{2k-1}x_1} = x_0^{2k-1}x_1 \notin \mathcal{L}_{irr}^{\infty}(X)$.

[22] Identification allows to obtain homogenous polynomial relations up to weights 12 [5].
[23] by means of rewriting the system.

Note also that for any $l_1 \in \mathcal{L}ynY-\{y_1\}$ and $l_2 \in \mathcal{L}ynX-X$ one has in general [46] $\zeta(\pi_X(\Sigma_{l_1})) \neq \zeta(S_{\pi_X(l_1)})$ and $\zeta(\pi_Y(S_{l_2})) \neq \zeta(\Sigma_{\pi_Y(l_2)})$, while this does not occur, due again to Lemma 2.2, for the values[24] $\{\zeta(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (or $\{\zeta(l)\}_{l\in\mathcal{L}ynX-X}$) [2, 40, 38, 28].

With the first assertion of Proposition 4.3, we compute the *generalized Euler constants*, *i.e.* the finite parts of divergent harmonic sums $\{\mathrm{H}_w\}_{w\in y_1Y^*}$:

Corollary 4.6 ([14, 16, 45]). — *For any $k \geqslant 1$ and $w \in Y^* - y_1Y^*$, one has*

$$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k},$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} \sqcup\!\sqcup\, \pi_X w])}{i!}\left(\sum_{j=1}^{i} b_{i,j}(\gamma,-\zeta(2),2\zeta(3),\ldots)\right),$$

where the $b_{n,k}(t_1,\ldots,t_k)$'s are Bell polynomials.

See also Corollary 5.7, for the independence of γ with respect to the convergent polyzetas.

4.3. **The case of negative multi-indices.** Similarly, asymptotic behaviors of $\{\mathrm{Li}^-_w\}_{w\in Y_0^*}, \{\mathrm{H}^-_w\}_{w\in Y_0^*}$ are analyzed by

Proposition 4.7 ([21]). — *For any $n \in \mathbb{N}_+$, $z \in \mathbb{C}$ with $|z| < 1$ and $w \in Y_0^*$, H^-_w and Li^-_w are polynomial, of degree $(w)+|w|$ in $\mathbb{Q}[n]$ and $\mathbb{Z}[(1-z)^{-1}]$, respectively. Hence, for any $w \in Y_0^*$, there exists $C^-_w \in \mathbb{Q}$ and $B^-_w \in \mathbb{N}$, such that*

$$\mathrm{H}^-_w(n) \sim_{+\infty} n^{(w)+|w|}C^-_w \quad \text{and} \quad \mathrm{Li}^-_w(z) \sim_1 (1-z)^{-(w)-|w|}B^-_w.$$

Moreover, one has

$$C^-_w = \prod_{\substack{w=uv\\ v\neq 1_{Y_0^*}}} ((v)+|v|)^{-1} \quad \text{and} \quad B^-_w = ((w)+|w|)!C^-_w.$$

Proposition 4.8 ([21]). — *Let us consider the following generating series*

$$\mathrm{L}^- := \sum_{w\in Y_0^*} \mathrm{Li}^-_w\, w, \quad \mathrm{H}^- := \sum_{w\in Y_0^*} \mathrm{H}^-_w w, \quad C^- := \sum_{w\in Y_0^*} C^-_w w.$$

Then[25]

$$\langle \mathrm{H}^- \mid 1_{Y_0^*}\rangle = \langle C^- \mid 1_{Y_0^*}\rangle = 1, \Delta_{\,\text{⧢}\!\!-}(\mathrm{H}^-) = \mathrm{H}^- \otimes \mathrm{H}^-$$

and $\Delta_{\sqcup\!\sqcup}(C^-) = C^- \otimes C^-$.

Moreover, analysis of singularities leads to the following global renormalization.

Theorem 4.9 (Second Abel like theorem, [21]). — *One has*

$$\lim_{z\to 1} h^{\odot -1}((1-z)^{-1}) \odot \mathrm{L}^-(z) = \lim_{n\to+\infty} g^{\odot-1}(n) \odot \mathrm{H}^-(n) = C^-,$$

[24] for which polynomial relations homogenous for the weight are obtained via double shuffle, up to weights 10 [40, 38], 12 [2] and 16 [28].

[25] The series C^- is group-like in $(\mathbb{Q}\langle\langle Y_0\rangle\rangle, \mathtt{conc}, 1_{Y_0^*}, \Delta_{\sqcup\!\sqcup}, \mathtt{e})$.

where the noncommutative generating series[26] *h and g are defined as follows*

$$h(t) = \sum_{w \in Y_0^*} ((w)+ \mid w \mid)! t^{(w)+|w|} w \quad \text{and} \quad g(t) = \Big(\sum_{y \in Y_0} t^{(y)+1} y \Big)^* .$$

Now, by Proposition 4.7 and the Taylor expansion, one deduces

Corollary 4.10. — *For any $w \in Y_0^*$ there exists a unique polynomial $p \in (\mathbb{Z}[t], \times, 1)$ of degree $(w)+ \mid w \mid$ such that*[27]*, via (3.5), for any $n \in \mathbb{N}_+$ and $z \in \mathbb{C}$ with $\mid z \mid < 1$ one has*

$$\mathrm{Li}_w^-(z) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{(1-z)^k} = \sum_{k=0}^{(w)+|w|} p_k e^{-k \log(1-z)} \in (\mathbb{Z}[e^{-\log(1-z)}], \times, 1_B),$$

$$\mathrm{H}_w^-(n) = \sum_{k=0}^{(w)+|w|} p_k \binom{n+k}{k} = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} (n+k)_n \quad \in (\mathbb{Q}[(n+\bullet)_n], \times, 1),$$

where $(n+\bullet)_n : \mathbb{N} \to \mathbb{Q}$ maps i to $(n+i)_n = (n+i)!/n!$ and $\mathbb{Q}[(n+\bullet)_n]$ denotes the set of polynomials in n expanded as follows

$$\forall \pi \in \mathbb{Q}[(n+\bullet)_n], \quad \deg(\pi) = d, \quad \pi = \sum_{i=0}^{d} \pi_k (n+i)_n = \sum_{i=0}^{d} \pi_k \frac{(n+i)!}{n!}.$$

By Corollary 4.10, denoting by $\hat{p}$ the exponential transform of p, one also has

$$\mathrm{Li}_w^-(z) = p(e^{-\log(1-z)}), \qquad \text{with} \qquad p(t) = \sum_{k=0}^{(w)+|w|} p_k t^k \in (\mathbb{Z}[t], \times, 1), \tag{4.10}$$

$$\mathrm{H}_w^-(n) = \hat{p}((n+\bullet)_n), \qquad \text{with} \qquad \hat{p}(t) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} t^k \in (\mathbb{Q}[t], \times, 1). \tag{4.11}$$

Let us then associate also p and $\hat{p}$ with the polynomial[28] $\check{p}$ obtained as follows

$$\check{p}(t) = \sum_{k=0}^{(w)+|w|} k! p_k t^k = \sum_{k=0}^{(w)+|w|} p_k t^{\sqcup\!\sqcup\, k} \in (\mathbb{Z}[t], \sqcup\!\sqcup, 1). \tag{4.12}$$

Next, the previous polynomials $p, \hat{p}$ and $\check{p}$ given in (4.10)–(4.12) can be determined explicitly thanks to Lemma 3.4 and to

Proposition 4.11 ([21]). — (1) *The following morphism of algebras is* bijective

$$\chi : (\mathbb{Q}[y_1^*], \sqcup\!\sqcup, 1_{Y^*}) \longrightarrow (\mathbb{Q}[(n+\bullet)_n], \times, 1), \quad S \longmapsto \mathrm{H}_S.$$

[26] Note that g can be view as an "exponential transform" of h :

$$g(t) = \sum_{w \in Y_0^*} t^{(w)+|w|} w = \sum_{w \in Y_0^*} \frac{\langle h \mid w \rangle}{((w)+ \mid w \mid)!} w.$$

[27] In other terms, for any word w belonging to Y_0^* and integer k verifying $0 \leqslant k \leqslant (w)+ \mid w \mid$, such that $\langle \mathrm{Li}_w^- \mid (1-z)^{-k} \rangle = k! \langle \mathrm{H}_w^- \mid (n)_k \rangle$. One verifies in particular, for Proposition 4.7, that $\langle \mathrm{H}_w^- \mid (n)_{(w)+|w|} \rangle = C_w^-$ and $\langle \mathrm{Li}_w^- \mid (1-z)^{-(w)-|w|} \rangle = ((w)+ \mid w \mid)! C_w^-$.

[28] In other words, p is the exponential transform of $\check{p}$ and, for any integer k with $0 \leqslant k \leqslant (w)+ \mid w \mid$ one has $\langle \hat{p} \mid z^k \rangle = k! \langle p \mid z^k \rangle = (k!)^2 \langle \check{p} \mid z^k \rangle$.

(2) *For any* $w = y_{s_1}, \dots y_{s_r} \in Y_0^*$, *there exists a unique polynomial* R_w *belonging to* $(\mathbb{Z}[x_1^*], ш, 1_{X^*})$ *of degree* $(w)+ |w|$, *such that*[29]

$$\begin{aligned}\mathrm{Li}_{R_w}(z) = \mathrm{Li}_w^-(z) = p(e^{-\log(1-z)}) &\in (\mathbb{Z}[e^{-\log(1-z)}], \times, 1_B),\\ \mathrm{H}_{\pi_Y(R_w)}(n) = \mathrm{H}_w^-(n) = \hat{p}((n+\bullet)_n) \qquad &\in (\mathbb{Q}[(n+\bullet)_n], \times, 1).\end{aligned}$$

In particular, via the extension by linearity of $R_\bullet$ *over* $\mathbb{Q}\langle Y_0\rangle$ *and Theorem 3.2,* $\{\mathrm{Li}_{R_{y_k}}\}_{k\geqslant 0}$ *is linear independent in* $\mathbb{Q}\{\mathrm{Li}_{R_w}\}_{w\in Y_0^*}$ *and for all* $k,l\in\mathbb{N}$

$$\mathrm{Li}_{R_{y_k} ш R_{y_l}} = \mathrm{Li}_{R_{y_k}} \mathrm{Li}_{R_{y_l}} = \mathrm{Li}_{y_k}^- \mathrm{Li}_{y_l}^- = \mathrm{Li}_{y_k \top y_l}^- = \mathrm{Li}_{R_{y_k\top y_l}}.$$

(3) *For any* $w \in Y_0^*$, *there exists a unique polynomial* $R_w \in (\mathbb{Z}[x_1^*], ш, 1_{X^*})$ *of degree* $(w)+ |w|$ *such that* $\check{p}(x_1^*) = R_w$.

(4) *More explicitly, for any* $w = y_{s_1}, \dots y_{s_r} \in Y_0^*$, *there exists a unique polynomial* R_w *belonging to* $(\mathbb{Z}[x_1^*], ш, 1_{X^*})$ *of degree* $(w)+ |w|$, *given by*

$$\begin{aligned}R_{y_{s_1}\dots y_{s_r}} = \sum_{k_1=0}^{s_1}\sum_{k_2=0}^{s_1+s_2-k_1}\dots\sum_{k_r=0}^{\substack{(s_1+\dots+s_r)-\\(k_1+\dots+k_{r-1})}}\binom{s_1}{k_1}\binom{s_1+s_2-k_1}{k_2}\dots\\ \binom{s_1+\dots+s_r-k_1-\dots-k_{r-1}}{k_r}\rho_{k_1} ш \dots ш \rho_{k_r},\end{aligned}$$

where, for any $i = 1, \dots, r$, *one has, if* $k_i = 0$ *then* $\rho_{k_i} = x_1^* - 1_{X^*}$ *else*

$$\rho_{k_i} = \sum_{j=1}^{k_i} S_2(k_i, j)(j!)^2 \sum_{l=0}^{j} \frac{(-1)^l}{l!}\frac{(x_1^*)^{ш(j-l+1)}}{(j-l)!},$$

and the $S_2(k, j)$*'s denote the Stirling numbers of second kind.*

Using Proposition 4.11 and Lemma 3.4 (in particular, the bijectivity of the restriction $\mathrm{Li}_\bullet : (\mathbb{Z}[x_1^*], ш, 1_{X^*}) \to (\mathbb{Z}[e^{-\log(1-z)}], ., 1_B))$ and also the Stirling numbers (of first and second kinds), one obtains

Corollary 4.12. — *The morphism of algebras*

$$R_\bullet : (\mathbb{Z}\langle Y_0\rangle, \top, 1_{Y_0^*}) \to (\mathbb{Z}[x_1^*], ш, 1_{X^*})$$

is bijective, mapping $y_0 \mapsto x_1^* - 1_{X^*}$ *and* $y_k \mapsto x_1^* ш R'_{y_k}$ $(k \geqslant 1)$, *where*

$$R'_{y_k} = \sum_{i=0}^{k} i! S_2(k,i)(x_1^*-1)^{ш\, i} = \sum_{i=0}^{k}\sum_{j=0}^{i} i! S_2(k,i)\binom{i}{j}(-1)^{i-j}(x_1^*)^{ш\, j}$$

and $R'_\bullet$ *is extended over* $\mathbb{Z}\langle Y\rangle$ *by linearity. Conversely, one has for any* $k \geqslant 1$,

$$(kx_1)^* = 1_{X^*} + R_{y_0} + \sum_{j=2}^{k}\frac{S_1(k,j)}{(k-1)!}R_{y_{j+1}}.$$

It follows that $\mathrm{Li}_{R_{y_k}} \odot \mathrm{Li}_{R_{y_l}} = \mathrm{Li}_S$ *(for* $k, l \geqslant 1$*), where*

$$S = x_1^* ш R'_{y_k \uplus y_l} = (1_{X^*} + R_{y_0}) ш (R'_{y_{k+l}} + R'_{y_k ш y_l}).$$

[29] Recall also that the map π_Y is multiplicative on $\mathbb{Q} \oplus \mathbb{Q}\langle X\rangle x_1$ but not on $\mathbb{Q}\langle X\rangle$.

To end this section, let us recall also that, for any $c \in \mathbb{C}$, one has

$$(n)_c \sim_{+\infty} n^c = e^{c\log(n)}$$

and, with the respective scales of comparison (on the right hand side), one has the following finite parts

$$\text{f.p.}_{z\to 1} c\log(1-z) = 0, \quad \{(1-z)^a \log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \tag{4.13}$$

$$\text{f.p.}_{n\to+\infty} c\log n = 0, \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \tag{4.14}$$

5. A group of associators

5.1. The action of the Galois differential group.

Lemma 5.1 ([43, 44]). — *Let G and H be solutions of* (1.1) *which are group-like for $\Delta_{\sqcup\!\sqcup}$. Then there exists $C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle$, independent of z, such that $G = He^C$.*

Typically, with the notations of (1.2) and Definition 3.7, the power series $C_{z_0\rightsquigarrow z}$ and $\mathrm{L}(z)$ satisfy the differential equation (1.1) and have the same value at $z = z_0$. Then $C_{z_0\rightsquigarrow z} = \mathrm{L}(z)(\mathrm{L}(z_0))^{-1}$ [38, 41]. Since $C_{z_0\rightsquigarrow z}$ and $\mathrm{L}(z)$ are group-like, so is $\mathrm{L}(z_0)$. It follows that the Hausdorff group, *i.e.* $\{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\}$, plays the rôle of the differential Galois group of the equation (1.1). More precisely,

Theorem 5.2 ([43, 44]). — $\mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\}$.

Definition 5.3 ([46, 47]). — Let A be a subring of $\mathbb{C}$, containing $\mathbb{Q}$. We put[30]

$$dm(A) := \{Z_{\sqcup\!\sqcup} e^C \mid C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \langle e^C \mid x_0\rangle = \langle e^C \mid x_1\rangle = 0\}.$$

Then $dm(A) = \mathrm{Gal}_{\mathbb{C}}^{\geqslant 2}(DE)$ is a strict normal subgroup of $\mathrm{Gal}_{\mathbb{C}}(DE)$.

Now, for any $e^C \in \mathrm{Gal}_{\mathbb{C}}(DE)$, let[31]

$$\overline{\mathrm{L}} := \mathrm{L}e^C \quad \text{and} \quad \overline{Z}_{\sqcup\!\sqcup} := Z_{\sqcup\!\sqcup} e^C. \tag{5.1}$$

Then, by the global analysis of singularities in (4.1), the action of e^C on L on the right yields the asymptotic behavior of $\overline{\mathrm{L}}$ [46, 47]

$$\overline{\mathrm{L}}(z) \sim_0 e^{x_0\log z} e^C \quad \text{and} \quad \overline{\mathrm{L}}(z) \sim_1 e^{-x_1\log(1-z)}\overline{Z}_{\sqcup\!\sqcup} \tag{5.2}$$

and, via an identity of type Newton-Girard again [51], one also gets :

$$\overline{\mathrm{H}}(n) \sim_{+\infty} e^{-\sum_{k\geqslant 1} \mathrm{H}_{y_k}(n)(-y_1)^k/k} \pi_Y(\overline{Z}_{\sqcup\!\sqcup}). \tag{5.3}$$

In other words, we obtain the extended Abel like theorem [46, 47]

$$\lim_{z\to 1} e^{y_1\log(1-z)}\pi_Y(\overline{\mathrm{L}}(z)) = \lim_{n\to\infty} e^{\sum_{k\geqslant 1} \mathrm{H}_{y_k}(n)(-y_1)^k/k}\overline{\mathrm{H}}(n) = \pi_Y(\overline{Z}_{\sqcup\!\sqcup}).$$

By (4.1) and (5.2), one then deduces

Corollary 5.4. — L *is the unique solution of* (DE) *satisfying* $\mathrm{L}(z) \sim_0 e^{x_0\log(z)}$ *(i.e. for $e^C = 1_{X^*}$). It follows that $\Phi_{KZ} = Z_{\sqcup\!\sqcup}$ is unique.*

[30] This group contains the group $DM(A)$ introduced in [10, 55] (DM for *double mélange*).

[31] Note that since (see [38, 41]) $\langle Z_{\sqcup\!\sqcup} \mid x_0\rangle = \langle Z_{\sqcup\!\sqcup} \mid x_1\rangle = 0$, by identification of the coefficients one has $\langle \overline{Z}_{\sqcup\!\sqcup} \mid x_1\rangle = \langle e^C \mid x_1\rangle$ and $\langle \overline{Z}_{\sqcup\!\sqcup} \mid x_0\rangle = \langle e^C \mid x_0\rangle$ which are not 0.

Proposition 5.5 ([46, 47]). — *Let $\{\overline{\gamma}_w\}_{w\in Y^*}$ be the finite parts of the asymptotic expansions of $\{\overline{\mathrm{H}}_w\}_{w\in Y^*}$ in $\{n^{-a}\log^b(n)\}_{a,b\in\mathbb{N}}$, and let $\overline{Z}_\gamma$ be their noncommutative generating series. Then*

$$\overline{Z}_\gamma := \sum_{w\in Y^*} \overline{\gamma}_w w, \quad \Delta_{ш\!\!\!+}(\overline{Z}_\gamma) = \overline{Z}_\gamma \otimes \overline{Z}_\gamma, \quad \langle \overline{Z}_\gamma \mid 1_{Y^*}\rangle = 1.$$

In other words, the following map is a character

$$\overline{\gamma}_\bullet : (\mathbb{Q}\langle Y\rangle, ш\!\!\!+, 1_{Y^*}) \longrightarrow (\mathcal{Z}[\gamma], \times, 1), \quad w \longmapsto \overline{\gamma}_w.$$

Proposition 5.6 (Extended bridge equation, [46, 47]). — *Under the action of the group $\mathrm{Gal}_{\mathbb{C}}(DE)$, one gets*[32]

$$\begin{aligned}
\overline{Z}_{ш} &= \mathrm{F.P.}_{z\to 1}\overline{\mathrm{L}}(z), && \{(1-z)^a\log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\overline{Z}_{ш\!\!\!+} &= \mathrm{F.P.}_{n\to+\infty}\overline{\mathrm{H}}(n), && \{n^a \mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\
\overline{Z}_\gamma &= \mathrm{F.P.}_{n\to+\infty}\overline{\mathrm{H}}(n), && \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.
\end{aligned}$$

Moreover, by Proposition 5.5, the extension of MRS factorization and the extended Abel like theorem lead to $\overline{Z}_\gamma = e^{\gamma y_1}\overline{Z}_{ш\!\!\!+}$. Hence, for any $\overline{Z}_{ш} \in dm(A)$, by cancellation and with expressions of B, B' given in Proposition 4.3, one obtains

$$\overline{Z}_\gamma = B(y_1)\pi_Y(\overline{Z}_{ш}) \iff \overline{Z}_{ш\!\!\!+} = B'(y_1)\pi_Y(\overline{Z}_{ш}).$$

Elements of the group $dm(A)$ satisfying similar properties as Φ_{KZ} are called *associators*[33], as regularized solutions of (DE) [46, 47]. Moreover, by the identification of local coordinates in the second point of Proposition 5.6, one gets

Corollary 5.7 ([46, 47]). — *If $\gamma \notin A$ then γ is transcendent over the A-algebra generated by convergent zeta values.*

Remark 5.8. — As example of the action of the differential Galois group on the singular expansions, we are interested in the action of their monodromy group[34] [46] generated by $e^{2\mathrm{i}\pi\mathfrak{m}_0}$ and $e^{2\mathrm{i}\pi\mathfrak{m}_1}$, where [41, 38]

$$\mathfrak{m}_0 = x_0 \quad\text{and}\quad \mathfrak{m}_1 = Z_{ш} e^{-2\mathrm{i}\pi x_1} Z_{ш}^{-1} = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{-\zeta(S_l)\,\mathrm{ad}_{P_l}}(-x_1).$$

By Proposition 4.3 and (5.1), the actions of the monodromy group on the right of $Z_{ш}$ and Z_γ are the following

[32]Note that, once the scales of comparison are *fixed*, the coefficients $\{\langle\overline{Z}_{ш} \mid w\rangle\}_{w\in x_0X^*x_1}$, $\{\langle\overline{Z}_{ш\!\!\!+} \mid w\rangle\}_{w\in(Y^*-\{y_1\})Y^*}$ and $\{\langle\overline{Z}_\gamma \mid w\rangle\}_{w\in(Y^*-\{y_1\})Y^*}$, as finite parts of the asymptotic expansions of $\{\langle\overline{\mathrm{L}} \mid w\rangle\}_{w\in x_0X^*x_1^*}$ and $\{\langle\overline{\mathrm{H}} \mid w\rangle\}_{w\in(Y^*-\{y_1\})Y^*}$, are *determined*, by the extended Abel like theorem.

[33]In [34, 55], associators (or Drinfel'd series) are defined as group-like series in $\mathbb{R}\langle\langle X\rangle\rangle$ satisfying a system of algebraic relations (duality, pentagonal and hexagonal), but the authors do not produce any associator other than Φ_{KZ}, which was completely determined earlier in [40, 38] (without divergent zeta values as local coordinates).

[34]A proof of linear independence of multi-valued polylogarithms is obtained via this monodromy group. It can be also proved by use of the differential Galois group [12, 43, 44].

An other proof for mono-valued polylogarithm functions, as a special case of hyperlogarithms, can be also obtained over functions field [17, 19].

(1) If $e^C = e^{2\mathrm{i}\pi\mathfrak{m}_0}$ then $\overline{Z}_{⧢} = Z_{⧢} e^{2\mathrm{i}\pi x_0}$ and

$$\overline{Z}_\gamma = \exp\Big(\gamma y_1 - \sum_{k\geqslant 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_Y Z_{⧢} = Z_\gamma.$$

This means that the monodromy at 0 of L consists of the multiplication on the right of $Z_{⧢}$ by $e^{2\mathrm{i}\pi x_0}$ and does not modify Z_γ.

(2) If $e^C = e^{2\mathrm{i}\pi\mathfrak{m}_0}$ then $\overline{Z}_{⧢} = e^{-2\mathrm{i}\pi x_1} Z_{⧢}$ and

$$\overline{Z}_\gamma = \exp\Big((\gamma - 2\mathrm{i}\pi) y_1 - \sum_{k\geqslant 2}\zeta(k)\frac{(-y_1)^k}{k}\Big)\pi_Y Z_{⧢} = e^{-2\mathrm{i}\pi y_1} Z_\gamma.$$

This means that the monodromy at 1 of L consists of the multiplication on the left of $Z_{⧢}$ and Z_γ by, respectively, $e^{-2\mathrm{i}\pi x_1}$ and $e^{-2\mathrm{i}\pi y_1}$.

Finally, the actions of the monodromy group on L does not allow, in this case, neither to introduce $e^{\gamma x_1}$ on the left of $Z_{⧢}$ nor to eliminate the left factor $e^{\gamma y_1}$ of Z_γ [46, 47].

5.2. **Associator Φ_{KZ}.** Now, let us examine some properties of the noncommutative generating series $Z_{\uplus}$ and $Z_{⧢}$, *i.e.* Φ_{KZ} (see Corollary 5.4).

In a way similar to what was said of the character $\gamma_\bullet$ (see Proposition 4.2), Definition 3.9 and Proposition 3.10 lead to

Proposition 5.9 ([14, 16, 46, 47]). — *One has* $\langle Z_{\uplus} \mid 1_{Y^*}\rangle = \langle Z_{⧢} \mid 1_{X^*}\rangle = 1$ *and*

$$\Delta_{\uplus}(Z_{\uplus}) = Z_{\uplus}\otimes Z_{\uplus}, \quad \Delta_{\uplus}(\log(Z_{\uplus})) = \log(Z_{\uplus})\otimes 1_{Y^*} + 1_{Y^*}\otimes\log(Z_{\uplus}),$$
$$\Delta_{⧢}(Z_{⧢}) = Z_{⧢}\otimes Z_{⧢}, \quad \Delta_{⧢}(\log(Z_{⧢})) = \log(Z_{⧢})\otimes 1_{X^*} + 1_{X^*}\otimes\log(Z_{⧢}),$$

and

$$\log(Z_{\uplus}) = \sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\zeta_{\uplus}(u_1 \uplus \ldots \uplus u_k)u_1\ldots u_k,$$
$$\log(Z_{⧢}) = \sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^+}\zeta_{⧢}(u_1 ⧢ \ldots ⧢ u_k)u_1\ldots u_k.$$

Moreover, the polymorphism ζ can be extended as follows

$$\zeta_{⧢} : (\mathbb{Q}\langle X\rangle, ⧢, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1), \quad \zeta_{\uplus} : (\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1),$$

according to its products and satisfying, for any $l \in \mathcal{L}ynY - \{y_1\}$,

$$\zeta_{⧢}(\pi_X(l)) = \zeta_{\uplus}(l) = \gamma_l = \zeta(l).$$

and, for the generators of length (resp. weight) one, for X^* (resp. Y^*),

$$\zeta_{⧢}(x_0) = 0 = \mathrm{f.p.}_{z\to 1}\,\mathrm{Li}_{x_1}(z), \qquad \{(1-z)^a\log^b((1-z)^{-1})\}_{a\in\mathbb{Z}, b\in\mathbb{N}},$$
$$\zeta_{\uplus}(y_1) = 0 = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{y_1}(n), \qquad \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}.$$

By Lazard's elimination, the free Lie algebra $\mathcal{L}ie_A\langle X\rangle$, as an A-module, is the direct sum of Ax_0 and of a Lie ideal, denoted by $\mathcal{J}$ and freely generated by $\{\mathrm{ad}_{x_0}^l x_1\}_{l\in\mathbb{N}}$. Then, by the calculations in Appendix B and by the identities $(x_0 \cup x_1)^* = (x_0^* x_1)^* x_0^*$ and $e^{x_0}x_1e^{-x_0} = e^{\mathrm{ad}_{x_0}}x_1$, one has

Proposition 5.10 (Gradation of L and $Z_{ш}$, [43, 44]). — *Let the operation $\circ$ be defined, for any $l\in\mathbb{N}$ and $P\in\mathbb{C}\langle X\rangle$, by $x_1x_0^l\circ P=x_1(x_0^l ш P)$. Then*

$$\begin{aligned}\mathrm{L}(z)&=\sum_{k\geqslant 0}\sum_{w\in x_0^* ш x_1^k}\mathrm{Li}_w(z)w\\&=e^{x_0\log(z)}\Big(1_{X^*}+\sum_{k\geqslant 1}\sum_{l_1,\cdots,l_k\geqslant 0}\mathrm{Li}_{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}}(z)\prod_{i=1}^k\mathrm{ad}_{-x_0}^{l_i}x_1\Big)\\&=\sum_{k\geqslant 0}\int_0^z\omega_1(t_k)\cdots\int_0^{t_{k-1}}\omega_1(t_1)\kappa_k(z,t_1,\cdots,t_k),\end{aligned}$$

$$Z_{ш}=\sum_{k\geqslant 0}\sum_{l_1,\cdots,l_k\geqslant 0}\zeta_{ш}(x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k})\prod_{i=0}^k\mathrm{ad}_{-x_0}^{l_i}x_1,$$

where $\mathrm{supp}(x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k})=\{w\in x_1X^*\,||w|_{x_1}=k,|w|_{x_0}=l_1+\cdots+l_k\}$ *and* $\kappa_k(z,t_1,\cdots,t_k)$ *for any $k\geqslant 0$ is the formal power series given by*

$$\begin{aligned}\kappa_k(z,t_1,\cdots,t_k)&=e^{x_0[\log(z)-\log(t_1)]}x_1\cdots e^{x_0[\log(t_{k-1})-\log(t_k)]}x_1e^{x_0\log(t_k)}\\&=e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0\log(t_1)}}x_1\cdots e^{\mathrm{ad}_{-x_0\log(t_k)}}x_1\\&=e^{x_0\log(z)}\sum_{l_1,\cdots,l_k\geqslant 0}\prod_{i=1}^k\frac{\log^{l_i}(t_i)}{l_i!}\mathrm{ad}_{-x_0}^{l_i}x_1.\end{aligned}$$

On the one hand, by Theorem 3.1 the morphism $\mathrm{Li}_\bullet$ is injective and the two families $\{\mathrm{ad}_{-x_0}^{l_1}x_1\cdots\mathrm{ad}_{-x_0}^{l_k}x_1\}_{k\geqslant 0}^{l_1,\cdots,l_k\geqslant 0}$ and $\{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}\}_{k\geqslant 0}^{l_1,\cdots,l_k\geqslant 0}$ are dual bases of, respectively, $\mathcal{U}(\mathcal{J})$ and $\mathcal{U}(\mathcal{J})^\vee$.

On the other hand, by Proposition 5.9 it turns out that $\zeta_{ш}$ corresponds to the adjoin of the regularization proposed in [34, 52].

5.3. **Associators with rational coefficients.** Since for any $t\in\mathbb{C}$ with $|t|<1$ one has $\mathrm{Li}_{(tx_1)^*}(z)=(1-z)^{-t}$, and by [16]

$$\mathrm{H}_{\pi_Y(tx_1)^*}=\sum_{k\geqslant 0}\mathrm{H}_{y_1^k}t^k=\exp\Big(-\sum_{k\geqslant 1}\mathrm{H}_{y_k}\frac{(-t)^k}{k}\Big),\tag{5.4}$$

by Lemma 3.5 and Proposition 5.9 we can extend the characters $\zeta_{ш}$ and $\gamma_\bullet$, over $\mathbb{C}\langle X\rangle ш \mathbb{C}[x_1^*]$ and $\mathbb{C}\langle Y\rangle ⧢ \mathbb{C}[y_1^*]$, respectively, by using the Euler beta and gamma functions[35] and also the incomplete beta function, *i.e* for any $z,a,b\in\mathbb{C}$ such that $|z|<1$, $\Re a>0$ and $\Re b>0$,

$$\mathrm{B}(z;a,b):=\int_0^z dt\ t^{a-1}(1-t)^{b-1}$$

and

$$\mathrm{B}(1;a,b)=:\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

[35]Following [20], for any $z\in\mathbb{C}$ the function $\Gamma(z)$ is meromorphic, admitting simple poles in $-\mathbb{N}$ and satisfying $\Gamma(\overline{z})=\overline{\Gamma(z)}$. The function $\Gamma^{-1}(z)$ is entire and admits simple zeros in $-\mathbb{N}$.

It is immediate that[36]

$$\begin{aligned}\mathrm{B}(z;a,b) &= \mathrm{Li}_{x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]}(z)\\ &= \mathrm{Li}_{x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]}(z).\end{aligned}$$

Proposition 5.11. — *The characters $\zeta_{\sqcup\!\sqcup}$ and $\gamma_\bullet$ can be extended algebraically as follows for $t \in \mathbb{C}$ with $|t| < 1$:*

$$\begin{aligned}\zeta_{\sqcup\!\sqcup} : (\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}[x_1^*], \sqcup\!\sqcup, 1_{X^*}) &\longrightarrow (\mathbb{C}, \times, 1_{\mathbb{C}}),\\ (tx_1)^* &\longmapsto 1_{\mathbb{C}},\\ \gamma_\bullet : (\mathbb{C}\langle Y\rangle \uplus \mathbb{C}[y_1^*], \uplus, 1_{Y^*}) &\longrightarrow (\mathbb{C}, \times, 1_{\mathbb{C}}),\\ (ty_1)^* &\longmapsto \exp\Big(\gamma t - \sum_{n\geqslant 2}\zeta(n)\frac{(-t)^n}{n}\Big) = \frac{1}{\Gamma(1+t)}.\end{aligned}$$

It follows then that

$$\begin{aligned}\mathrm{B}(a,b) &= \zeta_{\sqcup\!\sqcup}(x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*])\\ &= \zeta_{\sqcup\!\sqcup}(x_1[((1-a)x_0)^* \sqcup\!\sqcup (-bx_1)^*]).\end{aligned}$$

Moreover, for any $u, v \in \mathbb{C}$ such that $|u| < 1$, $|v| < 1$ and $|u+v| < 1$, one has[37]

$$\begin{aligned}\exp\Big(\sum_{n\geqslant 2}\zeta(n)\frac{(u+v)^n-(u^n+v^n)}{n}\Big) &= \frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-u-v)}\\ &= \frac{\gamma_{(-(u+v)y_1)^*}}{\gamma_{(-uy_1)^*}\gamma_{(-vy_1)^*}}\\ &= \frac{\gamma_{(-(u+v)y_1)^*}}{\gamma_{(-uy_1)^* \uplus (-vy_1)^*}}\\ &= \zeta_{\sqcup\!\sqcup}(x_0[(-ux_0)^* \sqcup\!\sqcup (-(1+v)x_1)^*])\\ &= \zeta_{\sqcup\!\sqcup}(x_1[(-(1+u)x_0)^* \sqcup\!\sqcup (-vx_1)^*])\end{aligned}$$

and

$$\begin{aligned}\zeta_{\sqcup\!\sqcup}((-(u+v)x_1)^*) &= \zeta_{\sqcup\!\sqcup}((-ux_1)^* \sqcup\!\sqcup (-vx_1)^*)\\ &= \zeta_{\sqcup\!\sqcup}((-ux_1)^*)\zeta_{\sqcup\!\sqcup}((-vx_1)^*)\\ &= 1.\end{aligned}$$

With the notations in Corollary 4.10, the values $p(1)$ and $\hat{p}(1)$ obtained by (4.10) and (4.11), respectively, represent the following finite parts :

[36] see the form of rational series given in (F_2) and Lemma 2.4.

[37] The first equality is already presented in [25]. Moreover, since $(-uy_1)^* \uplus (uy_1)^* = (-u^2y_2)^*$, letting $v = -u$ it follows that

$$\exp\Big(-\sum_{n\geqslant 1}\zeta(2n)\frac{u^{2n}}{n}\Big) = \Gamma(1-u)\Gamma(1+u) = \frac{1}{\gamma_{(-uy_1)^* \uplus (uy_1)^*}} = \frac{1}{\gamma_{(-u^2y_2)^*}}.$$

It is also a consequence obtained by expanding identities like (5.4), for any $y_r \in Y$, [14, 16]

$$y_r^k = \frac{(-1)^k}{k!}\sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}}\frac{(-y_r)^{\uplus s_1}}{1^{s_1}} \uplus \ldots \uplus \frac{(-y_{kr})^{\uplus s_k}}{k^{s_k}}.$$

Lemma 5.12. — (1) *Put* $\mathrm{P}_Q(z) := e^{-\log(1-z)}\,\mathrm{Li}_Q(z)$ *for any*

$$Q \in (\mathbb{Z}[x_0^*, (-x_0)^*, x_1^*], ш, 1_{X^*})/\{x_0^* ш x_1^* - x_1^* + 1\}.$$

Then $\mathrm{P}_Q = \mathrm{Li}_{x_1^* ш Q}$ *and* $\mathrm{Li}_Q, \mathrm{P}_Q \in \mathbb{Z}[z, z^{-1}, e^{-\log(1-z)}]$.

(2) *By Lemma 3.4 the converse holds. Moreover, by* (4.13) *and* (4.14) *one has*

$$\mathrm{f.p.}_{z\to 1}\mathrm{P}_Q(z) = \mathrm{f.p.}_{z\to 1}\,\mathrm{Li}_Q(z) \in \mathbb{Z}, \quad \{(1-z)^a \log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$

$$\mathrm{f.p.}_{n\to+\infty}\langle \mathrm{P}_Q \mid z^n\rangle = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y(Q)}(n) \in \mathbb{Q}, \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.$$

(3) *For any* $w \in Y^*$, *let* R_w *be explicitly determined as in Proposition 4.11. There exists a unique polynomial* $p \in \mathbb{Z}[t]$ *of valuation* 1 *and of degree* $(w)+|w|$ *such that* $R_w = \check{p}(x_1^*)$ *and*

$$\mathrm{f.p.}_{z\to 1}\,\mathrm{Li}_{R_w}(z) = p(1) \in \mathbb{Z}, \quad \{(1-z)^a \log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$

$$\mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y(R_w)}(n) = \hat{p}(1) \in \mathbb{Q}, \quad \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$

where $\hat{p} \in \mathbb{Q}[t]$ *is the exponential transform of* p.

As determined in Proposition 4.7, $B_\bullet^-$ and $C_\bullet^-$ do not realize characters for $(\mathbb{Q}\langle X\rangle, ш, 1_{X^*})$ and $(\mathbb{Q}\langle Y\rangle, ⧢, 1_{Y^*})$, respectively [21]. Hence, instead of regularizing the divergent sums $\zeta_ш(R_w)$ and $\zeta_\gamma(\pi_Y(R_w))$ by B_w^- and C_w^-, one can use, respectively, $p(1)$ and $\hat{p}(1)$ (depending on w) as shown in Theorem 5.15 below which is a consequence of Lemma 5.12, Propositions 4.11, 5.11 and Corollary 4.12 :

Definition 5.13. — Let Υ and Λ be the noncommutative generating series of, respectively, $\{\mathrm{H}_{\pi_Y(R_w)}\}_{w\in Y^*}$ and $\{\mathrm{Li}_{R_{\pi_Y(w)}}\}_{w\in X^*}$ (with $\langle\Lambda(z) \mid x_0\rangle = \log(z)$) :

$$\Upsilon := \sum_{w\in Y^*} \mathrm{H}_{\pi_Y(R_w)} w \in \mathbb{Q}[(n+\bullet)_n]\langle\langle Y\rangle\rangle,$$

$$\Lambda := \sum_{w\in X^*} \mathrm{Li}_{R_{\pi_Y(w)}}\, w \in \mathbb{Q}[e^{-\log(1-z)}][\log(z)]\langle\langle X\rangle\rangle.$$

Let Z_γ^- and $Z_ш^-$ be the noncommutative generating series of[38], respectively, $\{\gamma_{\pi_Y(R_w)}\}_{w\in Y^*}$ and $\{\zeta_ш(R_{\pi_Y(w)})\}_{w\in X^*}$:

$$Z_\gamma^- := \sum_{w\in Y^*} \gamma_{\pi_Y(R_w)} w \in \mathbb{Q}\langle\langle Y\rangle\rangle \quad \text{and} \quad Z_ш^- := \sum_{w\in X^*} \zeta_ш(R_{\pi_Y(w)}) w \in \mathbb{Z}\langle\langle X\rangle\rangle.$$

Via the diagonal series $\mathcal{D}_⧢, \mathcal{D}_⧢$ given in (2.2)-(2.8), one has

Lemma 5.14. — *The extension* $R_\bullet : (\mathbb{C}[x_0]\langle Y_0\rangle, \top, 1_{Y_0^*}) \to (\mathbb{C}[x_0][x_1^*], ш, 1_{X^*})$ *is bijective. Hence :*

(1) *Let* $\hat{\pi}_Y$ *be the morphism of algebras defined, over an algebraic basis, by* $\hat{\pi}_Y S_l = \pi_Y S_l$ *for any* $l \in \mathcal{L}ynX - \{x_0\}$, *and* $\hat{\pi}_Y(x_0) = x_0$ *(such that* $\mathrm{Li}_{R_{\hat{\pi}_Y x_0}}(z) = \log(z)$, *whence* $\zeta(R_{\hat{\pi}_Y x_0}) = 0$*). Then*

$$\Upsilon = ((\mathrm{H}_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y \quad \text{and} \quad \Lambda = ((\mathrm{Li}_\bullet \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X,$$

$$Z_\gamma^- = ((\gamma_\bullet \circ \pi_Y \circ R_\bullet) \otimes \mathrm{Id})\mathcal{D}_Y \quad \text{and} \quad Z_ш^- = ((\zeta_ш \circ R_\bullet \circ \hat{\pi}_Y) \otimes \mathrm{Id})\mathcal{D}_X.$$

[38]Note that, on the one hand, by Proposition 5.9 one has $\langle Z_ш^- \mid x_0\rangle = \zeta_ш(x_0) = 0$. On the other hand, since $R_{y_1} = (2x_1)^* - x_1^*$, one has $\mathrm{Li}_{R_{y_1}}(z) = (1-z)^{-2} - (1-z)^{-1}$ and $\mathrm{H}_{\pi_Y(R_{y_1})}(n) = \binom{n}{2} - \binom{n}{1}$. Hence, $\langle Z_ш^- \mid x_1\rangle = \zeta_ш(R_{\pi_Y(y_1)}) = 0$, and $\langle Z_\gamma^- \mid x_1\rangle = \gamma_{\pi_Y(R_{y_1})} = -1/2$.

(2) *For any* $u \in X^*$ *and* $v \in Y^*$ *one has*

$$\mathrm{f.p.}_{z\to 1}\langle \Lambda(z) \mid u\rangle = \langle Z_{\sqcup\!\sqcup}^- \mid u\rangle, \qquad \{(1-z)^a \log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$
$$\mathrm{f.p.}_{n\to+\infty}\langle \Upsilon(n) \mid v\rangle = \langle Z_{\gamma}^- \mid v\rangle, \qquad \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$

which means that (see also (4.5), (4.6) *and* (4.7)*)*

$$Z_\gamma^- = \mathrm{F.P.}_{n\to+\infty}\Upsilon(n) \quad \text{and} \quad Z_{\sqcup\!\sqcup}^- = \mathrm{F.P.}_{z\to 1}\Lambda(z).$$

Hence, by Propositions 4.11 and 5.11, Lemmas 2.4–3.5 and 5.12, one derives

Theorem 5.15. — (1) *For any* $(s_1,\ldots,s_r) \in \mathbb{N}_+^r$ *associated with* $l \in \mathcal{L}ynY$ *there exists a unique* $p \in \mathbb{Z}[t]$ *of valuation* 1 *and of degree* $(l)+|l|$ *such that*

$$\begin{aligned}
\check{p}(x_1^*) &= R_l & &\in (\mathbb{Z}[x_1^*], \sqcup\!\sqcup, 1_{X^*}),\\
p(e^{-\log(1-z)}) &= \mathrm{Li}_{R_l}(z) & &\in (\mathbb{Z}[e^{-\log(1-z)}], \times, 1_B),\\
\hat{p}((n+\bullet)_n) &= \mathrm{H}_{\pi_Y(R_l)}(n) & &\in (\mathbb{Q}[(n+\bullet)_n], \times, 1),\\
\zeta(-s_1,\ldots,-s_r) = p(1) &= \zeta_{\sqcup\!\sqcup}(R_l) & &\in (\mathbb{Z}, \times, 1),\\
\gamma_{-s_1,\ldots,-s_r} = \hat{p}(1) &= \gamma_{\pi_Y(R_l)} & &\in (\mathbb{Q}, \times, 1),
\end{aligned}$$

where $\hat{p} \in \mathbb{Q}[t]$ *is the exponential transform of* p*, and* p *is obtained as the exponential transform of* $\check{p} \in \mathbb{Z}[t]$.

(2) *One has* $\langle Z_\gamma^- \mid 1_{Y^*}\rangle = \langle Z_{\sqcup\!\sqcup}^- \mid 1_{X^*}\rangle = 1$ *and*

$$\Delta_{\uplus}(Z_\gamma^-) = Z_\gamma^- \otimes Z_\gamma^- \qquad \text{and} \qquad \Delta_{\sqcup\!\sqcup}(Z_{\sqcup\!\sqcup}^-) = Z_{\sqcup\!\sqcup}^- \otimes Z_{\sqcup\!\sqcup}^-,$$
$$Z_\gamma^- = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\pi_Y(R_{\Sigma_l})}\Pi_l} \qquad \text{and} \qquad Z_{\sqcup\!\sqcup}^- = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\zeta_{\sqcup\!\sqcup}(\pi_Y(S_l))P_l}.$$

(3) *Similarly,* $\langle \Upsilon \mid 1_{Y^*}\rangle = \langle \Lambda \mid 1_{X^*}\rangle = 1$ *and*

$$\Delta_{\uplus}(\Upsilon) = \Upsilon \otimes \Upsilon \qquad \text{and} \qquad \Delta_{\sqcup\!\sqcup}(\Lambda) = \Lambda \otimes \Lambda,$$
$$\Upsilon = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\pi_Y(R_{\Sigma_l})}\Pi_l} \qquad \text{and} \qquad \Lambda = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\pi_Y(S_l)}} P_l}.$$

(4) *Under the action of* $\mathcal{G}$ [35]*, as for* L [38, 41]*, for any* $g \in \mathcal{G}$ *there exists a letter substitution* σ_g *and a primitive series* C *such that*

$$\Lambda(g(z)) = \sigma_g(\Lambda(z))e^C \quad \text{and} \quad \Lambda(z) \sim_0 e^{x_0\log(z)}.$$

Remark 5.16. — By Corollary 5.4, Λ does not satisfy (DE) while $Z_{\sqcup\!\sqcup}^-$ and Z_γ^-, regularizing Λ and Υ respectively, satisfy similar properties as $Z_{\sqcup\!\sqcup}$ and Z_γ.

The series $Z_{\sqcup\!\sqcup}^-$ (or Z_γ^-) is not unique because in Theorem 5.15 the elements of the family $\{\mathrm{Li}_{R_l}\}_{l\in\mathcal{L}ynY}$ are polylogarithms with negative multiindices which are polynomial in $e^{-\log(1-z)}$.

Indeed, for any $l \in \mathcal{L}ynY$ one has $R_l \in \mathbb{Z}[x_1^*]$. Then, letting ρ_l be a monomial in $\mathbb{Z}[x_0^*, (-x_0)^*]$ with $\rho_l \neq 0$ and using Lemma 5.12, one gets the same regularized values $\zeta_{\sqcup\!\sqcup}(R_l)$ and $\gamma_{\pi_Y(R_l)}$ for the series $R_l \sqcup\!\sqcup \rho_l \in \mathbb{Z}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \sqcup\!\sqcup \mathbb{Z}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle = \mathbb{Z}_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle X\rangle\rangle$, *i.e.* (see Appendix C)

$$\mathrm{f.p.}_{z\to 1}\,\mathrm{Li}_{R_l \sqcup\!\sqcup \rho_l}(z) = \zeta_{\sqcup\!\sqcup}(R_l), \qquad \{(1-z)^a \log^b((1-z)^{-1})\}_{a\in\mathbb{Z},b\in\mathbb{N}},$$
$$\mathrm{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y(R_l \sqcup\!\sqcup \rho_l)}(n) = \gamma_{\pi_Y(R_l)}, \qquad \{n^a \log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.$$

For example, one can take ρ_l, by substituting each letter x_1 by x_0 in R_l.

6. Conclusion

In this paper, we have surveyed our recent results concerning the resolution of KZ_3 via a noncommutative symbolic computation, and the algebraic combinatorial aspects of the polylogarithms $\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}^{r\geqslant 1}$, the harmonic sums $\{\mathrm{H}_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}^{r\geqslant 1}$, and the zeta functions $\{\zeta(s_1,\ldots,s_r)\}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}^{r\geqslant 1}$ with the help of their commutative and noncommutative generating series.

This review is mainly based on the combinatorics on the shuffle bialgebras and their diagonal series, *i.e.* $\mathcal{D}_{\sqcup\!\sqcup}, \mathcal{D}_{\uplus}$ and $\mathcal{D}_X$. In particular, it used

(1) The construction of pairs of bases (Lie algebra bases and transcendence bases) in duality (Theorem 2.1) to factorize the noncommutative rational power series (Theorem 2.3) and to obtain the algebraic structure of $\{\zeta(s_1,\ldots,s_r)\}_{(s_1,\ldots,s_r)\in\mathbb{N}_{\geqslant 1}^r}^{r\geqslant 1}$ (polynomial relations homogenous in weight, and independence over a commutative extension of $\mathbb{Q}$, denoted by A) by identification of local coordinates, in infinite dimension (Corollary 4.4).

(2) The algebraic structures (Theorems 3.1 and 3.2) and the analysis of singularities (Theorems 4.1 and 4.9) of the polylogarithms and the harmonic sums, for which the global renormalizations has been obtained via Abel like theorems for the pairs of generating series L, H and $\mathrm{L}^-, \mathrm{H}^-$. In particular, the series L corresponds to the actual solution of (1.1) satisfying the standard asymptotic behaviors as given in (1.4) (Corollary 5.4).

(3) The paper culminates with the action (Theorem 5.2) of the differential Galois group $\mathrm{Gal}_{\mathbb{C}}(DE)$ (containing the group of associators $dm(A)$) on the asymptotic expansions of solutions of the equation (1.1) (see (5.2)–(5.3)).

    The group $dm(A)$ contains the unique associator Φ_{KZ}, *i.e.* the series $Z_{\sqcup\!\sqcup}$ determined by asymptotic conditions (Corollary 5.4), which is also associated with series $Z_{\uplus}$ and Z_γ. All of them are, for the corresponding co-products, group-like series and their logarithms are also provided (Propositions 4.2, 4.3 and 5.9).

(4) Non trivial expressions for associators with rational coefficients, *i.e.* $Z_{\sqcup\!\sqcup}^-$ and Z_γ^-, are also explicitly provided thanks to various processes of regularization via the noncommutative generating series Λ and Υ, which are group-like, respectively, for $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\uplus}$ (Theorem 5.15).

(5) Via the local coordinates of the power series $Z_{\sqcup\!\sqcup}, Z_{\sqcup\!\sqcup}^-, Z_\gamma, Z_\gamma^-$ and $Z_{\uplus}$, regularization maps for divergent zeta are constructed (Propositions 5.9, 5.11) over algebraic bases matching with analytical meaning : on the one hand, the character $\zeta_{\sqcup\!\sqcup}$ corresponds to the regularization, obtained as the finite parts of the singular expansions of $\{\mathrm{Li}_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{Z}^r}^{r\geqslant 1}$; on the other hand, the characters $\zeta_{\uplus}$ and $\gamma_\bullet$ correspond to the regularizations obtained as the finite parts of the asymptotic expansions of $\{\mathrm{H}_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{Z}^r}^{r\geqslant 1}$, in different comparison scales.

    In particular, the character $\gamma_\bullet$ furnished a generalization of the Euler's γ constant, $\{\gamma_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in\mathbb{N}_{\geqslant 1}^r}^{r\geqslant 1}$ (Corollary 4.6), and moreover, if $\gamma\notin A$ then γ is transcendent over the A-algebra generated by the convergent zeta values $\{\zeta(s_1,\ldots,s_r)\}_{(s_1,\ldots,s_r)\in\mathbb{N}_{\geqslant 1}^r, s_1\geqslant 2}^{r\geqslant 1}$ (Corollary 5.7).

Appendix A

By Proposition 4.3, identification of local coordinates, one obtains homogenous polynomials relations among the local coordinates $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ and $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$ (see Example 6.1).

Example 6.1 (Homogenous polynomials relations among local coordinates[39]). —

	Relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1})=\frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2})=\zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4})=\frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1})=\frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1})=\frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2})=\frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2})=\frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3})=\frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2})=3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2})=-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1})=-\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1})=-\frac{3}{2}\zeta(S_{x_0^4x_1})+\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1})=\frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-\frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3})=-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2})=\frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2})=\frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3})=\frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4})=\zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6})=\frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1})=\frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2})=\zeta(\Sigma_{y_3})^2-\frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2})=\frac{6}{35}\zeta(S_{x_0x_1})^3-\frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1})=\frac{2}{7}\zeta(\Sigma_{y_2})^3-\frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1})=\frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2})=-\frac{17}{30}\zeta(\Sigma_{y_2})^3+\frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3})=\frac{23}{70}\zeta(S_{x_0x_1})^3-\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1})=3\zeta(\Sigma_{y_3})^2-\frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2})=\frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2})=\frac{3}{10}\zeta(\Sigma_{y_2})^3-\frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1})=-\frac{89}{210}\zeta(S_{x_0x_1})^3+\frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2})=\frac{11}{63}\zeta(\Sigma_{y_2})^3-\frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4})=\frac{6}{35}\zeta(S_{x_0x_1})^3-\frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3})=\frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3})=\frac{8}{21}\zeta(S_{x_0x_1})^3-\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4})=\frac{17}{50}\zeta(\Sigma_{y_2})^3+\frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5})=\frac{8}{35}\zeta(S_{x_0x_1})^3$

One obtains also two families of polynomials homogenous for the weight, describing the kernel of the polymorphism ζ (see Example 6.2, $\{Q_l\}_{l\in\mathcal{L}yn\mathcal{X}}$).

[39] These relations are sorted by weight and are ordered by Lyndon words.

Example 6.2 (Homogenous polynomials[40] generating $\ker(\zeta)$). —

	$\{Q_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$	$\{Q_l\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}-\frac{3}{2}\Sigma_{y_3})=0$	$\zeta(S_{x_0x_1^2}-S_{x_0^2x_1})=0$
4	$\zeta(\Sigma_{y_4}-\frac{2}{5}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_3y_1}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^2}-\frac{1}{10}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_2y_1^2}-\frac{2}{3}\Sigma_{y_2}^{\uplus 2})=0$	$\zeta(S_{x_0x_1^3}-\frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2})=0$
5	$\zeta(\Sigma_{y_3y_2}-3\Sigma_{y_3}\uplus\Sigma_{y_2}-5\Sigma_{y_5})=0$	$\zeta(S_{x_0^3x_1^2}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_4y_1}-\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{2}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1x_0x_1}-\frac{3}{2}S_{x_0^4x_1}+S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1})=0$
	$\zeta(\Sigma_{y_2^2y_1}-\frac{3}{2}\Sigma_{y_3}\uplus\Sigma_{y_2}-\frac{25}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0^2x_1^3}-S_{x_0^2x_1}\sqcup\!\sqcup S_{x_0x_1}+2S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_3y_1^2}-\frac{5}{12}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1x_0x_1^2}-\frac{1}{2}S_{x_0^4x_1})=0$
	$\zeta(\Sigma_{y_2y_1^3}-\frac{1}{4}\Sigma_{y_3}\uplus\Sigma_{y_2})+\frac{5}{4}\Sigma_{y_5})=0$	$\zeta(S_{x_0x_1^4}-S_{x_0^4x_1})=0$
6	$\zeta(\Sigma_{y_6}-\frac{8}{35}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^5x_1}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$
	$\zeta(\Sigma_{y_4y_2}-\Sigma_{y_3}^{\uplus 2}-\frac{4}{21}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^4x_1^2}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_5y_1}-\frac{2}{7}\Sigma_{y_2}^{\uplus 3}-\frac{1}{2}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1x_0x_1}-\frac{4}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$
	$\zeta(\Sigma_{y_3y_1y_2}-\frac{17}{30}\Sigma_{y_2}^{\uplus 3}+\frac{9}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^3x_1^3}-\frac{23}{70}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_3y_2y_1}-3\Sigma_{y_3}^{\uplus 2}-\frac{9}{10}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0^2x_1x_0x_1^2}-\frac{2}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$
	$\zeta(\Sigma_{y_4y_1^2}-\frac{3}{10}\Sigma_{y_2}^{\uplus 2}-\frac{3}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^2x_0x_1}-\frac{89}{210}S_{x_0x_1}^{\sqcup\!\sqcup 3}+\frac{3}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_2^2y_1^2}-\frac{11}{63}\Sigma_{y_2}^{\uplus 2}-\frac{1}{4}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0^2x_1^4}-\frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}-\frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_3y_1^3}-\frac{1}{21}\Sigma_{y_2}^{\uplus 3})=0$	$\zeta(S_{x_0x_1x_0x_1^3}-\frac{8}{21}S_{x_0x_1}^{\sqcup\!\sqcup 3}-S_{x_0^2x_1}^{\sqcup\!\sqcup 2})=0$
	$\zeta(\Sigma_{y_2y_1^4}-\frac{17}{50}\Sigma_{y_2}^{\uplus 3}+\frac{3}{16}\Sigma_{y_3}^{\uplus 2})=0$	$\zeta(S_{x_0x_1^5}-\frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3})=0$

By substituting "=" by "→" in the previous homogenous polynomial relations one obtains a Noetherian rewriting system without critical pairs among local coordinates $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ (resp. $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$) (see Example 6.3).

[40] These polynomials are sorted by weight and are ordered by Lyndon words.

Example 6.3 (Noetherian homogenous rewriting system among local coordinates[41]). —

	Rewriting on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Rewriting on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1})\to\frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2})\to\zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4})\to\frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1})\to\frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1})\to\frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2})\to\frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2})\to\frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3})\to\frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2})\to3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2})\to-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1})\to-\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1})\to-\frac{3}{2}\zeta(S_{x_0^4x_1})+\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1})\to\frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-\frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3})\to-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2})\to\frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2})\to\frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3})\to\frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4})\to\zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6})\to\frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1})\to\frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2})\to\zeta(\Sigma_{y_3})^2-\frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2})\to\frac{6}{35}\zeta(S_{x_0x_1})^3-\frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1})\to\frac{2}{7}\zeta(\Sigma_{y_2})^3-\frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1})\to\frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2})\to-\frac{17}{30}\zeta(\Sigma_{y_2})^3+\frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3})\to\frac{23}{70}\zeta(S_{x_0x_1})^3-\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1})\to3\zeta(\Sigma_{y_3})^2-\frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2})\to\frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2})\to\frac{3}{10}\zeta(\Sigma_{y_2})^3-\frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1})\to-\frac{89}{210}\zeta(S_{x_0x_1})^3+\frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2})\to\frac{11}{63}\zeta(\Sigma_{y_2})^3-\frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4})\to\frac{6}{35}\zeta(S_{x_0x_1})^3-\frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3})\to\frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3})\to\frac{8}{21}\zeta(S_{x_0x_1})^3-\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4})\to\frac{17}{50}\zeta(\Sigma_{y_2})^3+\frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5})\to\frac{8}{35}\zeta(S_{x_0x_1})^3$

This means that for any $l\in\mathcal{L}ynY-\{y_1\}$ (resp. $l\in\mathcal{L}ynX-X$), the element $\zeta(\Sigma_l)$ (resp. $\zeta(S_l)$) is rewritten in a unique way as polynomials (normal forms) with coeffients in $\mathbb{Q}$ in irreducible local coordinates $\mathcal{Z}_{irr}^{\infty}(Y)$ (resp. $\mathcal{Z}_{irr}^{\infty}(X)$) forming an algebraic generator system for $\mathcal{Z}$ (see Example 6.4).

Example 6.4. — At weight 12 one has

$$\Sigma_{y_2}=y_2,\quad \Sigma_{y_3}=y_3,\quad \Sigma_{y_5}=y_5,\quad \Sigma_{y_7}=y_7,\quad \Sigma_{y_9}=y_9,\quad \Sigma_{y_{11}}=y_{11}$$

[41]These rules are sorted by weight and are ordered by Lyndon words.

and

$$S_{x_0x_1} = x_0x_1, \qquad S_{x_0^2x_1} = x_0^2x_1, \qquad S_{x_0^4x_1} = x_0^4x_1,$$
$$S_{x_0^6x_1} = x_0^6x_1, \qquad S_{x_0^8x_1} = x_0^8x_1, \qquad S_{x_0^{10}x_1} = x_0^{10}x_1.$$

The identification of local coordinates leads to the irreducible polyzetas (see [47] for a short discussion)

$$\begin{aligned}
\mathcal{Z}_{irr}^{\leqslant 12}(Y) &= \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}),\\
&\qquad \zeta(\Sigma_{y_3y_1^7}), \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})\}.\\
\mathcal{L}_{irr}^{\leqslant 12}(Y) &= \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3y_1^5}, \Sigma_{y_9}, \Sigma_{y_3y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2y_1^9}, \Sigma_{y_3y_1^9}, \Sigma_{y_2^2y_1^8}\}.\\
\mathcal{Z}_{irr}^{\leqslant 12}(X) &= \{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}),\\
&\qquad \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}.\\
\mathcal{L}_{irr}^{\leqslant 12}(X) &= \{S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^4x_1}, S_{x_0^6x_1}, S_{x_0x_1^2x_0x_1^4}, S_{x_0^8x_1}, S_{x_0x_1^2x_0x_1^6},\\
&\qquad S_{x_0^{10}x_1}, S_{x_0x_1^3x_0x_1^7}, S_{x_0x_1^2x_0x_1^8}, S_{x_0x_1^4x_0x_1^6}\}.
\end{aligned}$$

Appendix B

$$\sum_{w\in x_0^*} \mathrm{Li}_w(z)w = e^{x_0\log(z)},$$

$$\sum_{w\in x_0^* \sqcup\!\sqcup\, x_1} \mathrm{Li}_w(z)w = \int_0^z e^{x_0[\log(z)-\log(t)]}x_1\omega_1(t)e^{x_0\log(t)} = \int_0^z \omega_1(t)\kappa_1(z,t),$$

where

$$\kappa_1(z,t) = e^{x_0[\log(z)-\log(t)]}x_1e^{x_0\log(t)} = e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0\log(t)}}x_1.$$

$$\begin{aligned}
\sum_{w\in x_0^* \sqcup\!\sqcup\, x_1^2} \mathrm{Li}_w(z)w &= \int_0^z e^{x_0[\log(z)-\log(t_1)]}x_1\omega_1(t_1)\int_0^{t_1} e^{x_0[\log(t_1)-\log(t_2)]}x_1\omega_1(t_2)e^{x_0\log(t_2)}\\
&= \int_0^z \omega_1(t_1)\int_0^{t_1}\omega_1(t_2)\kappa_2(z,t_1,t_2),
\end{aligned}$$

where

$$\begin{aligned}
\kappa_2(z,t_1,t_2) &= e^{x_0[\log(z)-\log(t_1)]}x_1e^{x_0[\log(t_1)-\log(t_2)]}x_1e^{x_0\log(t_2)}\\
&= e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0\log(t_1)}}x_1e^{\mathrm{ad}_{-x_0\log(t_2)}}x_1,
\end{aligned}$$

$$\sum_{w\in x_0^* \sqcup\!\sqcup\, x_1^3} \mathrm{Li}_w(z)w = \int_0^z \omega_1(t_1)\int_0^{t_1}\omega_1(t_2)\int_0^{t_2}\omega_1(t_3)\kappa_3(z,t_1,t_2,t_3),$$

where

$$\begin{aligned}
\kappa_3(z,t_1,t_2,t_3) &= e^{x_0[\log(z)-\log(t_1)]}x_1e^{x_0[\log(t_1)-\log(t_2)]}x_1e^{x_0[\log(t_2)-\log(t_3)]}x_1e^{x_0\log(t_3)}\\
&= e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0\log(t_1)}}x_1e^{\mathrm{ad}_{-x_0\log(t_2)}}x_1e^{\mathrm{ad}_{-x_0\log(t_3)}}x_1,\\
&\vdots
\end{aligned}$$

$$\sum_{w\in x_0^* \sqcup\!\sqcup\, x_1^k} \mathrm{Li}_w(z)w = \int_0^z \omega_1(t_1)\cdots\int_0^{k-1}\omega_1(t_k)\kappa_k(z,t_1,\cdots,t_k),$$

where

$$\begin{aligned}\kappa_k(z,t_1,\cdots,t_k) &= e^{x_0[\log(z)-\log(t_1)]}x_1\cdots e^{x_0[\log(t_{k-1})-\log(t_k)]}x_1 e^{x_0\log(t_k)}\\ &= e^{x_0\log(z)}e^{\mathrm{ad}_{-x_0\log(t_1)}}x_1\cdots e^{\mathrm{ad}_{-x_0\log(t_k)}}x_1\\ &= e^{x_0\log(z)}\sum_{l_1,\cdots,l_k\geqslant 0}\prod_{i=1}^{k}\frac{\log^{l_i}(t_i)}{l_i!}\mathrm{ad}^{l_i}_{-x_0}x_1.\end{aligned}$$

Hence (see the notations of Proposition 5.10) [43, 44],

$$\begin{aligned}\sum_{w\in x_0^*\sqcup\!\sqcup\, x_1^k}\mathrm{Li}_w(z)w &= e^{x_0\log(z)}\sum_{l_1,\cdots,l_k\geqslant 0}\int_0^z\omega_1(t_1)\frac{\log^{l_1}(t_1)}{l_1!}\cdots\\ &\qquad\int_0^{t_{k-1}}\omega_1(t_k)\frac{\log^{l_k}(t_k)}{l_k!}\prod_{i=1}^{k}\mathrm{ad}^{l_i}_{-x_0}x_1\\ &= e^{x_0\log(z)}\sum_{l_1,\cdots,l_k\geqslant 0}\mathrm{Li}_{x_1x_0^{l_1}\circ\cdots\circ x_1x_0^{l_k}}(z)\prod_{i=1}^{k}\mathrm{ad}^{l_i}_{-x_0}x_1.\end{aligned}$$

See also Example 3.3 and Appendix C, for the commutative generating series of polylogarithms.

Appendix C

For $k\geqslant 0$ and $|\,t\,|<1$ let us define $V_k=(tx_0^*)\sqcup\!\sqcup\, x_1^k$ and $W_k=(tx_1^*)\sqcup\!\sqcup\, x_0^k$. By (3.4) one has [35, 36, 37]

$$\mathrm{Li}_{V_k}(z)=z^t\frac{(-\log(1-z))^k}{k!}\quad\text{and}\quad \mathrm{Li}_{W_k}(z)=(1-z)^{-t}\frac{\log^k(z)}{k!}.$$

Hence [35, 36, 37],

$$\begin{aligned}\mathrm{Li}_{(tx_0^*)\sqcup\!\sqcup\, x_1^*}(z) &= \sum_{k\geqslant 0}\mathrm{Li}_{V_k}(z) = \frac{z^t}{1-z},\\ \mathrm{Li}_{x_0^*\sqcup\!\sqcup\,(tx_1^*)}(z) &= \sum_{k\geqslant 0}\mathrm{Li}_{W_k}(z) = \frac{z}{(1-z)^t},\end{aligned}$$

and then (see Remark 5.16)

$$\zeta_{\sqcup\!\sqcup}((tx_1^*)\sqcup\!\sqcup\, x_0^*)=\sum_{k\geqslant 0}\zeta_{\sqcup\!\sqcup}(W_k)=1,\qquad \zeta_{\sqcup\!\sqcup}((tx_0^*)\sqcup\!\sqcup\, x_1^*)=\sum_{k\geqslant 0}\zeta_{\sqcup\!\sqcup}(V_k)=1$$

$$\text{and}\qquad \gamma_{\pi_Y((tx_1^*)\sqcup\!\sqcup\, x_0^*)}=\frac{1}{\Gamma(1+t)}.$$

By (3.4), for any $k\geqslant 1$ one also has [35, 36, 37]

$$\mathrm{Li}_{(tx_1)^{*i+1}-(tx_1)^{*i}}(z)=t(1-z)^t\log(1-z)\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{(-t\log(1-z))^k}{k!}.$$

More generally, as in Theorem 2.3, let S belong to $\mathbb{C}^{\mathrm{rat}}\langle\!\langle X\rangle\!\rangle$ and be of linear representation (β,μ,η) of dimension $n\geqslant 1$. Then the following matrix is nothing else than the "Dyson series" [36, 37]

$$R(z) = \sum_{w\in X^*} \mathrm{Li}_w(z)\mu(w) = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l}(z)\mu(P_l)}.$$

If S is exchangeable, *i.e.* $[\mu(x_0), \mu(x_1)] = 0$, then R reduces to (see Lemma 2.4) [36, 37]

$$R(z) = e^{\log(z)\mu(x_0)-\log(1-z)\mu(x_1)}.$$

The matrix R belongs to $\mathcal{M}_{n,n}(\mathbb{C}[\log(z), \log(1-z)][z^a, (1-z)^b]_{a,b\in\mathbb{C}})$ and if $\mu(x_0)$ and $\mu(x_1)$ are diagonal matrices, then $R \in \mathcal{M}_{n,n}(\mathbb{C}[z^a, (1-z)^b]_{a,b\in\mathbb{C}})$ [36, 37]. On the one hand, for $|t_0| < 1$ and $|t_1| < 1$, let us introduce the concatenation morphism τ_1, mapping x_0 to 1 and x_1 to t. Similarly, let τ_0 map x_1 to 1 and x_0 to t. It follows then (see Appendix B)

$$\tau_1(\mathrm{L}(z)) = \mathrm{Li}_{(tx_1^*)\sqcup\!\sqcup\, x_0^*}(z) = \frac{z}{(1-z)^t} \quad \text{and} \quad \tau_0(\mathrm{L}(z)) = \mathrm{Li}_{(tx_0^*)\sqcup\!\sqcup\, x_1^*}(z) = \frac{z^t}{1-z}.$$

On the other hand, let τ map x_1 to t_0 and x_0 to t_1. Then

$$\tau(\mathrm{L}(z)) = \mathrm{Li}_{(t_0x_0)^*\sqcup\!\sqcup\,(t_1x_1)^*}(z) = \frac{z^{t_0}}{(1-z)^{t_1}}.$$

Appendix D

The algebra $\mathcal{H}(\Omega)$ is endowed with the topology of *compact convergence* whose seminorms are indexed by compact subsets of Ω, and defined by

$$p_K(f) := ||f||_K = \sup_{s\in K} |f(s)|.$$

Of course, $p_{K_1\cup K_2} = \sup(p_{K_1}, p_{K_2})$, and therefore the same topology is defined by extracting a *fundamental subset of seminorms*, which here can be chosen denumerable. As $\mathcal{H}(\Omega)$ is complete in this topology, it is a Frechet space and even, as $p_K(fg) \leqslant p_K(f)p_K(g)$, it is a Frechet algebra (even more, as $p_K(1_\Omega) = 1$, a Frechet algebra with unit).

With the standard topology above, an operator $\phi \in \mathrm{End}(\mathcal{H}(\Omega))$ is continuous if and only if, with K_i compacts of Ω,

$$(\forall K_2)(\exists K_1)(\exists M_{21} > 0)(\forall f \in \mathcal{H}(\Omega))(||\phi(f)||_{K_2} \leqslant M_{21}||f||_{K_1}),$$

the algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ (and $\mathcal{H}(\Omega)$) is closed under the operators θ_i for $i = 0, 1$. We will first build sections of them, then see that they are continuous and, propose (discontinuous) sections more adapted to renormalisation and the computation of associators.

For $z_0 \in \Omega$, let us define $\iota_i^{z_0} \in \mathrm{End}(\mathcal{H}(\Omega))$ by

$$\iota_0^{z_0}(f) = \int_{z_0}^z f(s)\omega_0(s) \quad \text{and} \quad \iota_1^{z_0}(f) = \int_{z_0}^z f(s)\omega_1(s).$$

It is easy to check that $\theta_i\iota_i^{z_0} = \mathrm{Id}_{\mathcal{H}(\Omega)}$ and that they are continuous on $\mathcal{H}(\Omega)$ (for the topology of compact convergence), because for all $K \subset_{\text{compact}} \Omega$ we have

$$|p_K(\iota_i^{z_0}(f)| \leqslant p_K(f)[\sup_{z\in K} |\int_{z_0}^z \omega_i(s)|],$$

and this is sufficient to prove continuity. The operators $\iota_i^{z_0}$ are also well defined on $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$, and it is easy to check that

$$\iota_i^{z_0}(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}) \subset \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

Due to the decomposition of $\mathcal{H}(\Omega)$ into a direct sum of closed subspaces

$$\mathcal{H}(\Omega) = \mathcal{H}_{z_0\mapsto 0}(\Omega) \oplus \mathbb{C}1_\Omega,$$

it is not hard to see that the graphs of θ_i are closed. Thus the θ_i are also continuous. Much more interesting (and adapted to the explicit computation of associators), we have the operator ι_i (without superscripts), mentioned in the introduction and (more rigorously) defined by means of a $\mathbb{C}$-basis of

$$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} = \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$

As $\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \cong \mathbb{C}[\mathcal{L}yn(X)]$, one can partition the alphabet of this polynomial algebra in $(\mathcal{L}yn(X)\cap X^*x_1)\sqcup\{x_0\}$ and obtain the decomposition

$$\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*} \simeq \mathcal{C}\otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*x_1} \otimes_{\mathbb{C}} \mathbb{C}\{\mathrm{Li}_w\}_{w\in x_0^*}.$$

Due to the following identity [35],

$$ux_1x_0^n = ux_1 \sqcup\!\sqcup\, x_0^n - \sum_{k=1}^{n} (u \sqcup\!\sqcup\, x_0^k)x_1x_0^{n-k},$$

we have an algorithm to convert $\mathrm{Li}_{ux_1x_0^n}$ as

$$\mathrm{Li}_{ux_1x_0^n}(z) = \sum_{m\leqslant n} P_m(z)\log^m(z) = \sum_{m\leqslant n, w\in X^*x_1} \langle P_m(z) \mid w\rangle \,\mathrm{Li}_w(z)\log^m(z).$$

This means that

$$\begin{aligned}\mathcal{B} := {} & (z^k\,\mathrm{Li}_{ux_1}(z)\,\mathrm{Li}_{x_0^n}(z))_{(k,n,u)\in\mathbb{Z}\times\mathbb{N}\times X^*} \sqcup (z^k\,\mathrm{Li}_{x_0^n}(z))_{(k,n)\in\mathbb{Z}\times\mathbb{N}}\\ & \sqcup ((1-z)^{-l}\,\mathrm{Li}_{ux_1}(z)\,\mathrm{Li}_{x_0^n}(z))_{(l,n,u)\in\mathbb{N}^+\times\mathbb{N}\times X^*} \sqcup ((1-z)^{-l}\,\mathrm{Li}_{x_0^n}(z))_{(l,n)\in\mathbb{N}^+\times\mathbb{N}},\end{aligned}$$

is a $\mathbb{C}$-basis of $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$. With this basis, we can define ι_0 as follows.

Definition 6.5 ([22]). — Define the index map $\mathrm{ind} : \mathcal{B} \to \mathbb{Z}$ by

$$\mathrm{ind}(z^k(1-z)^{-l}\,\mathrm{Li}_{x_0^n}(z)) = k \quad\text{and}\quad \mathrm{ind}(z^k(1-z)^{-l}\,\mathrm{Li}_{ux_1}(z)\log^n(z)) = k + |ux_1|.$$

Then ι_0 is computed as follows

$$\iota_0(b) = \begin{cases} \displaystyle\int_0^z b(s)\omega_0(s), & \text{if} \quad \mathrm{ind}(b) \geqslant 1,\\ \displaystyle\int_1^z b(s)\omega_0(s), & \text{if} \quad \mathrm{ind}(b) \leqslant 0.\end{cases}$$

To show discontinuity of ι_0, one of the possibilities consists in exhibiting two sequences $f_n, g_n \in \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ converging to the same limit but such that

$$\lim \iota_0(f_n) \neq \lim \iota_0(g_n).$$

Here, we choose the function z to be approximated in a twofold way, and if ι_0 were continuous, we would have equality of the limits of the image-sequences (which is not the case). We first remark that

$$z = \sum_{n\geqslant 0} \frac{\log^n(z)}{n!} = \sum_{n\geqslant 1} (-1)^{n+1}\frac{\log^m((1-z)^{-1}}{n!}$$

Set

$$f_n = \sum_{0 \leqslant m \leqslant n} \frac{\log^m(z)}{m!} \quad \text{and} \quad g_n = \sum_{1 \leqslant m \leqslant n} (-1)^{m+1} \frac{\log^m((1-z)^{-1}))}{m!}$$

(these two sequences are in $\mathbb{C}\{\mathrm{Li}_w\}_{w \in X^*}$). It is easily seen that $\iota_0(f_n) = f_{n+1} - 1$, and then $\lim_{n\to+\infty} \iota_0(f_n)(z) = z - 1$. Now, for any $s \in [0, z]$ with $z \in]0, 1[$ one has

$$|\, g(s) \,| = |\sum_{m=1}^{n} (-1)^{m+1} \frac{\log^n(1-s)}{m!} \,| \leqslant \frac{s}{1-s}.$$

In order to exchange limits, we apply *Lebesgue's dominated convergence theorem* to the measure space $(]0, z], \mathcal{B}, dz/z)$ ($\mathcal{B}$ is the usual Borel σ-algebra) and the function $p(x) = s(1-s)^{-1}$ which is — as are the functions g_n — integrable on $]0, z]$ for every $z \in]0, 1[$. Then

$$\lim(\iota_0(g_n)) = \lim_{n\to+\infty} \int_0^z g_n(s) \frac{ds}{s} = \int_0^z \lim_{n\to+\infty} g_n(s) \frac{ds}{s} = \int_0^z s \frac{ds}{s} = z.$$

Hence, for $z \in]0, 1[$ we obtain $\lim(\iota_0(f_n)) = z - 1 \neq z = \lim(\iota_0(g_n))$ which completes the proof.

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References

[1] J. Berstel, C. Reutenauer. *Rational series and their languages*, Springer Verlag, 1988.

[2] M. Bigotte. *Etude symbolique et algorithmique des fonctions polylogarithmes et des nombres d'Euler-Zagier colorés*, Ph. D., Lille, 2000.

[3] V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh. *Schützenberger's factorization on the (completed) Hopf algebra of $q-$stuffle product*, Journal of Algebra, Number Theory and Applications, pp. 191-215, 30, No. 2, 2013.

[4] V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, L. Kane, C. Tollu. *Dual bases for non commutative symmetric and quasi-symmetric functions via monoidal factorization*, dans Journal of Symbolic Computation (2015).

[5] V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh. *Structure of Polyzetas and Explicit Representation on Transcendence Bases of Shuffle and Stuffle Algebras*, Journal of Symbolic Computation, **83**, pp. 93-111 (2017).

[6] V.C. Bui, G.H.E. Duchamp, N. Hoang, V. Hoang Ngoc Minh, C. Tollu. *Combinatorics on the ϕ-deformed stuffle product*, arXiv:1302.5391.

[7] V.C. Bui, G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, C. Tollu. *(Pure) transcendence bases in φ-deformed shuffle bialgebras*, Journal Electronique du Séminaire Lotharingien de Combinatoire, **74** (2018).

[8] N. Bourbaki. *Topologie générale*, Springer (1974)

[9] P. Cartier. *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, 687 (1987), 31–52.

[10] P. Cartier. *Fonctions polylogarithmes, nombres polyzetas et groupes pro-unipotents.* Séminaire Bourbaki, 53$^{\text{ème}}$, $n°$885, 2000-2001.

[11] K.-T. Chen. *Iterated integrals and exponential homomorphisms*, Proc. Lond. Mathem. Soc. (3) **4** (1954) 502–512.

[12] C. Costermans. *Calcul non nommutatif : analyse des constantes d'arbre de fouille*, thèse, Lille, 2008.
[13] C. Costermans, J.Y. Enjalbert and V. Hoang Ngoc Minh. *Algorithmic and combinatoric aspects of multiple harmonic sums*, *Discrete Mathematics & Theoretical Computer Science Proceedings*, 2005.
[14] C. Costermans, V. Hoang Ngoc Minh. *Some Results à l'Abel Obtained by Use of Techniques à la Hopf*, "Workshop on Global Integrability of Field Theories and Applications", Daresbury (UK), 1-3, November 2006.
[15] C. Chevalley. *Fundamental Concepts of Algebra*, Academic Press, New York, 1956.
[16] C. Costermans, V. Hoang Ngoc Minh. *Noncommutative algebra, multiple harmonic sums and applications in discrete probability*, Journal of Symbolic Computation, (2009), 801-817.
[17] M. Deneufchâtel. *Intégrales itérées et physique combinatoire*, Ph. D., Paris 13, 2012.
[18] P. Deligne. *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math, 163, Springer-Verlag (1970).
[19] M. Deneufchâtel, G.H.E. Duchamp, V. Hoang Ngoc Minh, A.I. Solomon. *Independence of hyperlogarithms over function fields via algebraic combinatorics*, dans Lec. N. in Comp. Sc. (2011), Volume 6742/2011, 127-139.
[20] J. Dieudonné. *Calcul infinitésimal*, Hermann (1968).
[21] G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo. *Harmonic sums and polylogarithms at negative multi-indices*, Journal of Symbolic Computation, **83**, pp. 166-186 (2017).
[22] G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo. *Kleene stars of the plane, polylogarithms and symmetries*, soumitted to TCS.
[23] V. Drinfel'd. *Quantum groups*, Proc. Int. Cong. Math., Berkeley, 1986.
[24] V. Drinfel'd. *Quasi-Hopf Algebras*, *Len. Math. J.*, 1, 1419-1457, 1990.
[25] V. Drinfel'd. *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, *Leningrad Math. J.*, 4, 829-860, 1991.
[26] F.J. Dyson, *The radiation theories of Tomonaga, Schwinger and Feynman*, Physical Rev, vol 75, (1949), pp. 486-502.
[27] S. Eilenberg and S. Mac Lane. *On the groups $H(\Pi, n)$ I*, Ann. of Math. **58** (1) (1953) 55–106.
[28] E. Wardi. Mémoire de DEA, Lille, 1999.
[29] J.Y. Enjalbert, V. Hoang Ngoc Minh. *Analytic and combinatorial aspects of Hurwitz polyzêtas*, J. Th. des Nomb. de Bord., Vol. 19, (2007), 599-644.
[30] J.Y. Enjalbert, V. Hoang Ngoc Minh. *Propriétés combinatoires et prolongement analytique effectif de polyzêtas de Hurwitz et de leurs homologues*, J. Th. des Nomb. de Bord., 23 no. 2 (2011), 353-386.
[31] J.Y. Enjalbert, G.H.E. Duchamp, V. Hoang Ngoc Minh, C. Tollu. *The contrivances of shuffle products and their siblings*, Discrete Mathematics 340(9): 2286-2300 (2017).
[32] P. Flajolet, R. Sedgewick. *Analytic combinatorics*, Cambridge University Press, 2009.
[33] M. Fliess. *Fonctionnelles causales non linéaires et indéterminées non commutatives*, Bull. Soc. Math. France, N°109, 1981, pp. 3-40.
[34] J. Gonzalez-Lorca. *Série de Drinfel'd, monodromie et algèbres de Hecke*, Ph. D., Ecole Normale Supérieure, Paris, 1998.
[35] V Hoang Ngoc Minh. *Summations of Polylogarithms via Evaluation Transform*, dans Math. & Computers in Simulations, 1336, pp 707-728, 1996.
[36] V. Hoang Ngoc Minh. *Fonctions de Dirichlet d'ordre n et de paramètre t*, Disc. Math. 180, pp 221-242, 1998.
[37] V. Hoang Ngoc Minh & Jacob G.. *Symbolic Integration of meromorphic differential equation via Dirichlet functions*, Disc. Math. 210, pp. 87-116, 2000.
[38] V. Hoang Ngoc Minh, G. Jacob, N.E. Oussous, M. Petitot. *Aspects combinatoires des polylogarithmes et des sommes d'Euler-Zagier*, *Journal Electronique du Séminaire Lotharingien de Combinatoire*, B43e, (2000).
[39] V. Hoang Ngoc Minh, G. Jacob, N.E. Oussous, M. Petitot. *De l'algèbre des ζ de Riemann multivariées à l'algèbre des ζ de Hurwitz multivariées*, *Journal Electronique du Séminaire Lotharingien de Combinatoire*, 44, (2001).
[40] V. Hoang Ngoc Minh, M. Petitot. *Lyndon words, polylogarithmic functions and the Riemann ζ function*, Discrete Math., 217, 2000, pp. 273-292.

[41] V. Hoang Ngoc Minh, M. Petitot, J. Van der Hoeven. Polylogarithms and Shuffle Algebra, *Proceedings of FPSAC'98*, 1998.
[42] V. Hoang Ngoc Minh. Finite polyzêtas, Poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series, *Proceedings of* 4^{th} *International Conference on Words*, pp. 232-250, 2003.
[43] V. Hoang Ngoc Minh. Differential Galois groups and noncommutative generating series of polylogarithms, *Automata, Combinatorics & Geometry*, World Multi-conf. on Systemics, Cybernetics & Informatics, Florida, 2003.
[44] V. Hoang Ngoc Minh. Shuffle algebra and differential Galois group of colored polylogarithms, Nuclear Physics B, 135 (2004), pp. 220-224.
[45] V. Hoang Ngoc Minh. Algebraic Combinatoric Aspects of Asymptotic Analysis of Nonlinear Dynamical System with Singular Inputs, Acta Academiae Aboensis, Ser. B, Vol. 67, no. 2, (2007), pp. 117-126.
[46] V. Hoang Ngoc Minh. *On a conjecture by Pierre Cartier about a group of associators*, Acta Math. Vietnamica (2013), 38, Issue 3, 339-398.
[47] V. Hoang Ngoc Minh. *Structure of polyzetas and Lyndon words*, Vietnamese Math. J. (2013), 41, Issue 4, 409-450.
[48] M. Hoffman. *Quasi-shuffle products*, J. Alg. Combin. **11** (1) (2000) 49–68.
[49] D. Knutson. *λ-rings and the representation theory of the symmetric group*, Lecture Notes in Mathematics, vol. 308, Springer-Verlag, 1973.
[50] J.A. Lappo-Danilevsky. *Théorie des systèmes des équations différentielles linéaires*, Chelsea, New York, 1953.
[51] A. Lascoux. *Fonctions symétriques*, *journal électronique du Séminaire Lotharingien de Combinatoire*, B08e, (1983).
[52] T.Q.T. Lê, J. Murakami. *Kontsevich's integral for Kauffman polynomial*, Nagoya Math., pp 39-65, 1996.
[53] M. Lothaire. *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.
[54] W. Magnus. *On the exponential solution of differential equations for a linear operator.*, AC on Pure and App. Math., VII:649–673, 1954.
[55] G. Racinet. *Séries génératrices non-commutatives de polyzêtas et associateurs de Drinfel'd*, Ph. D., Amiens, 2000.
[56] R. Ree. *Lie elements and an algebra associated with shuffles Ann. Math* **68** 210–220, 1958.
[57] C. Reutenauer. *Free Lie Algebras*, London Math. Soc. Monographs (1993).
[58] G. Viennot. *Algèbres de Lie libres et monoïdes libres*, Lecture Notes in Mathematics, Springer-Verlag, 691, 1978.
[59] D. Zagier. *Values of zeta functions and their applications*, in "First European Congress of Mathematics", vol. 2, Birkhäuser, pp. 497-512, 1994.

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