CONFLUENTES MATHEMATICI

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Tome 11, nº 1 (2019), p. 59-78.

<http://cml.cedram.org/item?id=CML_2019__11_1_59_0>

Confluentes Math.
11, 1 (2019) 59-78

STRICHARTZ ESTIMATES WITH LOSS OF DERIVATIVES UNDER A WEAK DISPERSION PROPERTY FOR THE WAVE OPERATOR

VALENTIN SAMOYEAU

Abstract. This paper can be considered as a sequel of [4] by Bernicot and Samoyeau, where the authors have proposed a general way of deriving Strichartz estimates for the Schrödinger equation from a dispersive property of the wave propagator. It goes through a reduction of $H^1 - \mathrm{BMO}$ dispersive estimates for the Schrödinger propagator to $L^2 - L^2$ microlocalized (in space and in frequency) dispersion inequalities for the wave operator. This paper aims to contribute in enlightening our comprehension of how dispersion for waves implies dispersion for the Schrödinger equation. More precisely, the hypothesis of our main theorem encodes dispersion for the wave equation in an uniform way, with respect to the light cone. In many situations the phenomena that arise near the boundary of the light cone are the more complicated ones. The method we present allows to forget those phenomena we do not understand very well yet. The second main step shows the Strichartz estimates with loss of derivatives we can obtain under those assumptions. The setting we work with is general enough to recover a large variety of frameworks (infinite metric spaces, Riemannian manifolds with rough metric, some groups, ...) where the lack of knowledge of the wave propagator is an obstacle to our understanding of the dispersion phenomena.

Contents

Introduction

The family of so-called Strichartz estimates is a powerful tool to study nonlinear Schrödinger equations. Those estimates give a control of the size of the solution to a linear problem in term of the size of the initial data. The size notion is usually given by a suitable functional space $L^p_t L^q_x$. Such inequalities were first introduced by Strichartz in [25] for Schrödinger waves on the Euclidean space. They were then

2010 *Mathematics Subject Classification:* 35B30, 42B37, 47D03, 47D06.
Keywords: dispersive inequalities; Strichartz inequalities; heat semigroup; Schrödinger group; wave operator.

extended by Ginibre and Velo in [14] (and the endpoint is due to Keel and Tao in [18]) for the propagator operator associated with the linear Schrödinger equation in $\mathbb{R}^d$. So for an initial data u_0, we are interested in controlling $u(t,.) = e^{it\Delta}u_0$ which is the solution of the linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

It is well-known that the unitary group $e^{it\Delta}$ satisfies the following inequality:

$$\|e^{it\Delta}u_0\|_{L^pL^q([-T,T]\times\mathbb{R}^d)} \leqslant C_T\|u_0\|_{L^2(\mathbb{R}^d)}$$

for every pair (p,q) of admissible exponents which means : $2 \leqslant p,q \leqslant \infty$, $(p,q,d) \neq (2,\infty,2)$, and

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{0.1}$$

The Strichartz estimates can be deduced via a TT^* argument from the dispersive estimates

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}}\|u_0\|_{L^1(\mathbb{R}^d)}. \tag{0.2}$$

If $\sup_{T>0} C_T < +\infty$, we will say that a global-in-time Strichartz estimate holds. Such a global-in-time estimate has been proved by Strichartz for the flat Laplacian on $\mathbb{R}^d$ while the local-in-time estimate is known in several geometric situations where the manifold is non-trapping (asymptotically Euclidean, conic, or hyperbolic, Heisenberg group); see [7, 6, 16, 24, 1] or with variable coefficients [21, 27].

The situation for compact manifolds presents a new difficulty, since considering the constant initial data on the torus $u_0 = 1 \in L^2(\mathbb{T})$ yields a contradiction in (0.2) for large time.

Burq, Gérard, and Tzvetkov [9] and Staffilani and Tataru [24] proved that Strichartz estimates hold on a compact manifold $\mathcal{M}$ for finite time if one considers regular data $u_0 \in W^{1/p,2}(\mathcal{M})$. Those are called "with a loss of derivatives":

$$\|e^{it\Delta}u_0\|_{L^pL^q([-T,T]\times\mathcal{M})} \leqslant C_T\|u_0\|_{W^{1/p,2}(\mathcal{M})}.$$

An interesting problem is to determine for specific situations, which loss of derivatives is optimal (for example the work of Bourgain [8] on the flat torus and [26] of Takaoka and Tzvetkov). For instance, the loss of $\frac{1}{p}$ derivatives in [9] is shown to be optimal in the case of the sphere.

An important remark is that, by Sobolev embedding, the loss of $2/p$ derivatives is straightforward. Indeed, by Sobolev embedding, we have $W^{\frac{2}{p},2} \hookrightarrow L^q$ since $\frac{2}{p} - \frac{d}{2} = 0 - \frac{d}{q}$ so that

$$\|e^{it\Delta}u_0\|_{L^q} \lesssim \|e^{it\Delta}u_0\|_{W^{\frac{2}{p},2}} \leqslant \|u_0\|_{W^{\frac{2}{p},2}} \tag{0.3}$$

and taking the $L^p([-T,T])$ norm yields

$$\|e^{it\Delta}u_0\|_{L^pL^q} \leqslant C_T\|u_0\|_{W^{\frac{2}{p},2}}.$$

Therefore Strichartz estimates with loss of derivatives are interesting for a loss smaller than $2/p$.

Let us now set the general framework of our study. We consider (X,d,μ) a metric measured space equipped with a nonnegative σ-finite Borel measure μ. We assume moreover that μ is Alfhors regular, that is there exist a dimension d, and two absolute positive constants c and C such that for all $x \in X$ and $r > 0$

$$cr^d \leqslant \mu(B(x,r)) \leqslant Cr^d, \tag{0.4}$$

where $B(x,r)$ denote the open ball with center $x \in X$ and radius $0 < r < \operatorname{diam}(X)$. Thus we aim our results to apply in numerous cases of metric spaces such as open subsets of $\mathbb{R}^d$, smooth d-manifolds, some fractal sets, Lie groups, Heisenberg group, ...

Keeping in mind the canonical example of the Laplacian operator in $\mathbb{R}^d$: $\Delta = \sum_{1\leqslant j\leqslant d} \partial_j^2$, we will be more general in the following sense: we consider a nonnegative, self-adjoint operator H on $L^2 = L^2(X,\mu)$ densely defined, which means that its domain

$$\mathcal{D}(H) := \{f \in L^2,\ Hf \in L^2\}$$

is supposed to be dense in L^2.

One of the motivations of our paper is to study the connection between the wave equation and the Schrödinger equation. We define the wave propagator $\cos(t\sqrt{H})$ as follows: for any $f \in L^2$, $u(t,.) := t \mapsto \cos(t\sqrt{H})f$ is the unique solution of the linear wave problem

$$\begin{cases} \partial_t^2 u + Hu = 0 \\ u_{|t=0} = f \\ \partial_t u_{|t=0} = 0. \end{cases} \tag{0.5}$$

One can find the explicit solutions of this problem in [13] for the Euclidean case and in [2] for the Riemannian manifold case through precise formula for the kernel of the wave propagator. Up to our knowledge, those explicit solutions are not available in our abstract setting. It would be of great interest to be able to compute exact expression of the solution of the wave equation in such a general case. The remarkable property of this operator comes from its finite speed propagation: for any two disjoint open subsets $U_1, U_2 \subset X$, and any functions $f_i \in L^2(U_i)$, $i = 1,2$, then

$$\langle \cos(t\sqrt{H})f_1, f_2\rangle = 0 \tag{0.6}$$

for all $0 < t < d(U_1,U_2)$. If $\cos(t\sqrt{H})$ is an integral operator with kernel K_t, then (0.6) simply means that K_t is supported in the "light cone" $\mathcal{D}_t := \{(x,y) \in X^2,\ d(x,y) \leqslant t\}$. We assume that H satisfies (0.6). In [11], Coulhon and Sikora proved that this property is equivalent to the Davies-Gaffney estimates

$$\|e^{-tH}\|_{L^2(E)\to L^2(F)} \lesssim e^{-\frac{d(E,F)^2}{4t}} \tag{0.7}$$

for any two subsets E and F of X, and $t > 0$.
It is known that $-H$ is the generator of a L^2-holomorphic semigroup $(e^{-tH})_{t\geqslant 0}$ (see [12]). We will also assume that the heat semigroup $(e^{-tH})_{t\geqslant 0}$ satisfies the typical upper estimates (for a second order operator): for every $t > 0$ the operator e^{-tH} admits a kernel p_t with

$$|p_t(x,x)| \lesssim \frac{1}{\mu(B(x,\sqrt{t}))}, \quad \forall\ t > 0, \text{ a.e. } x \in X. \tag{DUE}$$

It is well-known that such on-diagonal pointwise estimates self-improve into the full pointwise Gaussian estimates (see [15, Theorem 1.1] or [11, Section 4.2] e.g.)

$$|p_t(x,y)| \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,y)^2}{t}\Big), \quad \forall\, t>0, \text{ a.e. } x,y \in X. \qquad (UE)$$

One can find in [4] and the references therein some examples where the previous estimates hold. To sum up, as we assume (DUE) we then have (UE) and (0.7) and therefore the finite speed propagation property (0.6).

When dealing with Schrödinger equation on a manifold or a more general metric space, the $L^1 - L^\infty$ estimate (0.2) seems out of reach. In [4], the authors show how to replace it by a $H^1 - \mathrm{BMO}$ estimate, with the Hardy space H^1 and the Bounded Mean Oscillations space BMO both adapted to the semigroup. We do not recall the definition of those spaces here, but refer to [4] for more details.

For any integer $m \geqslant 0$ and $x \in \mathbb{R}_+$ we set $\psi_m(x) = x^m e^{-x}$. It forms a family of smooth functions that vanish at 0 (except when $m = 0$) and infinity, which allows us to consider a smooth partition of unity, using holomorphic functionnal calculus (and requiring $\mathbb{C}_0^\infty$-calculus).

The main assumption of our work is the following

Assumption 0.1. — *There exist $\kappa \in (0,\infty]$ and an integer m such that for every $s \in (0,\kappa)$ the wave propagator $\cos(s\sqrt{H})$ at time s satisfies the following dispersion property*

$$\|\cos(s\sqrt{H})\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \Big(\frac{r}{r+s}\Big)^{\frac{d-1}{2}}, \qquad (0.8)$$

for any two balls B, $\widetilde{B}$ of radius $r > 0$.

This estimate is microlocalized in the physical space due to the balls B and $\widetilde{B}$ at scale r and in frequency at scale $\frac{1}{r}$ through $\psi_m(r^2H)$, thus respecting the Heisenberg uncertainty principle. The parameter κ is linked to the geometry of the space X (its injectivity radius for instance).

In the Euclidean space $\mathbb{R}^d$, the $L^2(B) - L^2(\widetilde{B})$ dispersion phenomenon seems only to depend on the distance $d(B,\widetilde{B})$. Indeed, the intuition is that, in an isotropic medium a wave propagates the same way in all the directions. That is what leads us to think that Assumption 0.1 could be proved without using a pointwise explicit formula of its kernel, but with a more general approach, using functional tools only, that could be extend to other settings. To our knowledge the study of such behavior is not known and could be a good direction to investigate.

We mentioned that the finite speed propagation property (0.6) gives us the idea that after time s the solution to the wave problem (0.5) with initial data supported in a ball of radius r is supported in a ball of radius $r+s$. Given that $r \leqslant s$ (otherwise L^2 functional calculus yields Assumption 0.1) and the fact that waves propagate the same way in all directions in an isotropic medium, if we cover the sphere of radius $r+s$ by $N \simeq \big(\frac{r+s}{r}\big)^{d-1}$ balls of radius r and use Theorem 1.1, we can conjecture that the term $\big(\frac{r}{r+s}\big)^{\frac{d-1}{2}}$ is the natural dispersion one can hope for such waves, if we look for a uniform estimate (depending only on r, s).

Indeed we also emphasize that Assumption 0.1 is weaker than the one in [4], namely:

There exist $\kappa \in (0,\infty]$ and an integer m such that for every $s \in (0,\kappa)$ we have

$$\|\cos(s\sqrt{H})\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(\frac{r}{r+|s-L|}\right)^{\frac{d+1}{2}}, \tag{0.9}$$

where $L = d(B,\widetilde{B})$, which describes more precisely the dispersion inside the light cone.

However (0.9) can be difficult to prove in an abstract setting. That is why we are interested in proving what Assumption 0.1 could imply as far as Strichartz estimates are concerned. Therefore in all the situations where (0.9) is satisfied, we can assure that Assumption 0.1 is valid. When we have a good knowledge of the wave propagator, we can affirm that Assumption 0.1 holds. This is the case thanks to a parametrix in [2] in the following cases:

- The Euclidean spaces $\mathbb{R}^d$ with the usual Laplacian $H = -\Delta = -\sum_{j=1}^d \partial_j^2$;
- A compact Riemannian manifold of dimension d with the Laplace-Beltrami operator;
- A smooth non-compact Riemannian d-manifold with C_b^∞-geometry and Laplace-Beltrami operator;
- The Euclidean space $\mathbb{R}^d$ equipped with the measure $d\mu = \rho dx$ and $H = -\frac{1}{\rho}\mathrm{div}(A\nabla)$, where ρ is an uniformly non-degenerate function and A a matrix with bounded derivatives.

However in [17] the authors proved that for the Laplacian inside a convex domain of dimension $d \geqslant 2$ in $\mathbb{R}^d$, there was a loss of $s^{\frac{1}{4}}$ in the dispersion, namely

$$\|\cos(s\sqrt{H})\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}+\frac{1}{4}} \left(\frac{r}{r+|s-L|}\right)^{\frac{d+1}{2}}. \tag{0.10}$$

This loss indicates a difficulty when dealing with boundaries of a domain. The authors used oscillatory integrals techniques and a careful study of the reflections on the boundary of the domain.

A remark of J.-M. Bouclet to get around the use of a parametrix leads us to investigate Klainerman's commuting vectorfields method. It can be found in detail in [19] and [23]. Briefly, if one can find enough vectorfields commuting with the wave operator, using a version of Sobolev inequalities, also known as Klainerman-Sobolev inequalities, one can obtain dispersion estimations for the wave propagator. In our setting, we would obtain (see [23, Remark 1.4]) the following dispersion property:

$$\|\cos(s\sqrt{H})\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(\frac{r}{r+|s-L|}\right)^{\frac{1}{2}}. \tag{0.11}$$

It is very close to our Assumption 0.1, but it takes into account the dispersion inside the light cone. In that sense, it is intermediate between our Assumption 0.1 and estimate (0.9). A question we would like to pursue investigating is to find enough well-suited vectorfields to apply this method in generals settings. The framework in which one is interested in verifying (0.11) is when H is a given by divergence form, namely $H = -\mathrm{div}(A\nabla)$. When $A = \mathrm{I}_d$ the identity matrix of size d, H is the usual Laplacian. In this case and the one where A has $C^{1,1}$ coefficients, Klainerman obtained in [19] a dispersion property of the form (0.11). It is not new since it was already proven in [22]. But the novelty in [19] is to get around the use of a parametrix.

We envision inequality (0.8) should be easier to prove than (0.9) in concrete examples. Consequently the results we obtain will be weaker too. We recall Theorem 1.3 from [4] in order to compare it with our Theorem 0.3.

Theorem 0.2 ([4]). — *Suppose* (0.4) *with* $d > 1$, (*DUE*) *and Assumption* (0.9) *with* $\kappa \in (0, \infty]$. *For every solution* $u(t,.) = e^{itH}u_0$ *of the problem*

$$\begin{cases} i\partial_t u + Hu = 0 \\ u_{|t=0} = u_0, \end{cases}$$

two cases occur:

- *if* $\kappa = \infty$ *then we have global-in-time Strichartz estimates without loss of derivatives:*
$$\|u\|_{L^pL^q} \lesssim \|u_0\|_{L^2}; \tag{0.12}$$
- *if* $\kappa < \infty$ *then for every* $\varepsilon > 0$ *we have local-in-time Strichartz estimates with loss of* $\frac{1+\varepsilon}{p}$ *derivatives:*
$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p},2}}. \tag{0.13}$$

To prove this, the authors first reduced the $H^1 - \mathrm{BMO}$ estimation to a microlocalized $L^2 - L^2$ estimate, and then showed how dispersion for the wave propagator implies dispersion for the Schrödinger group. Theorem 2.1 is playing that role in the present paper.

Our main theorem follows the routine of [4] to deduce Strichartz inequalities from $L^2 - L^2$ estimates.

Theorem 0.3. — *Assume* (0.4) *with* $d > 2$, (*DUE*), *and Assumption 0.1. Then for every* $2 \leqslant p \leqslant +\infty$ *and* $2 \leqslant q < +\infty$ *satisfying*

$$\frac{2}{p} + \frac{d-2}{q} = \frac{d-2}{2},$$

and every solution $u(t,.) = e^{itH}u_0$ *of the problem*

$$\begin{cases} i\partial_t u + Hu = 0 \\ u_{|t=0} = u_0, \end{cases}$$

we have

- *if* $\kappa = \infty$, *then* u *satisfies local-in-time Strichartz estimates with loss of derivatives*
$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{2(\frac{1}{2}-\frac{1}{q}),2}}; \tag{0.14}$$
- *if* $\kappa < \infty$, *then for every* $\varepsilon > 0$, u *satisfies local-in-time Strichartz estimates with loss of derivatives*
$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q}),2}}. \tag{0.15}$$

We would like to point out that the straightforward loss of derivatives given by Sobolev embeddings when

$$\frac{2}{p} + \frac{d-2}{q} = \frac{d-2}{2}$$

is

$$\frac{2}{p} + 1 - \frac{2}{q}.$$

Thus the loss is nontrivial here. For more on the loss of derivatives, see Remarks 2.7 and 2.8. It is interesting to see how a weak dispersion property on the wave propagator implies dispersion for the Schrödinger operator.

The idea of the proof here is similar to the one in [4]. More particularly it is due to a precise tracking of the constants in some key estimations (from [18] for instance).

The aim of this paper is to give a better understanding of how dispersion for the wave propagator implies dispersion for the Schrödinger equation, and what Strichartz inequalities ensue in some contexts, where we do not have precise dispersive estimates on the wave propagators. In other words if one can compute, even inaccurate, information about the wave propagator in general settings, it would allow to have some knowledge of the Schrödinger equation in that framework.

The organization of the paper is as follow: In Section 1 we set the notations used throughout the paper and recall some preliminary facts concerning the semigroup, Hardy and BMO spaces. Then Section 2 is dedicated to the proofs of the Theorems.

1. Preliminaries

1.1. **Notations.** We denote $\operatorname{diam}(X) := \sup_{x,y\in X} d(x,y)$ the diameter of a metric space X. For $B(x,r)$ a ball ($x \in X$ and $r > 0$) and any parameter $\lambda > 0$, we denote $\lambda B(x,r) := B(x,\lambda r)$ the dilated and concentric ball. As a consequence of (0.4), a ball $B(x,\lambda r)$ can be covered by $C\lambda^d$ balls of radius r, uniformly in $x \in X$, > 0 and $\lambda > 1$ (C is a constant only depending on the ambient space).

If no confusion arises, we will note L^p instead of $L^p(X,\mu)$ for $p \in [1,+\infty]$. For $s > 0$ and $p \in [1,+\infty]$, we denote by $W^{s,p}$ the Sobolev space of order s based on L^p, equipped with the norm

$$\|f\|_{W^{s,p}} := \|(1+H)^{\frac{s}{2}} f\|_{L^p}.$$

We will use $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \leqslant Cv$ and $u \simeq v$ to say that both $u \lesssim v$ and $v \lesssim u$. We welle note $u \lesssim_\varepsilon v$ to emphasize that the constant C depends on the parameter ε.

If Ω is a set, $\mathbb{1}_\Omega$ is the characteristic function of Ω, defined by

$$\mathbb{1}_\Omega(x) = \begin{cases} 1 \text{ if } x \in \Omega \\ 0 \text{ if } x \notin \Omega. \end{cases}$$

Throughout the paper, unless something else is explicitly mentioned, we assume that $d > 2$ and that (0.4), (DUE), (0.7), and Assumption 0.1 are satisfied.

1.2. **The heat semigroup and associated functional calculus.** We consider a nonnegative, self-adjoint operator H on $L^2 = L^2(X,\mu)$ densely defined. We recall the bounded functional calculus theorem from [20]:

Theorem 1.1. — *H admits a L^∞-functional calculus on L^2: if $f \in L^\infty(\mathbb{R}_+)$, then we may consider the operator $f(H)$ as a L^2-bounded operator and*

$$\|f(H)\|_{L^2\to L^2} \leqslant \|f\|_{L^\infty}.$$

From the Gaussian estimates of the heat kernel (UE) and the analyticity of the semigroup (see [10]) it comes that for every integer $m \in \mathbb{N}$ and every $t > 0$, the operator $\psi_m(tH)$ has a kernel $p_{m,t}$ also satisfying upper Gaussian estimates:

$$|p_{m,t}(x,y)| \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \exp\Big(-c\frac{d(x,y)^2}{t}\Big), \quad \forall\, t > 0, \text{ a.e. } x, y \in X. \tag{1.1}$$

We now give some basic results about the heat semigroup thanks to our assumptions. The detailed proofs can be found in Section 2 of [4].

Proposition 1.2. — *Under* (0.4) *and* (DUE)*, the heat semigroup is uniformly bounded in every* L^p*-space for* $p \in [1,\infty]$*; more precisely for every* $f \in L^p$*, we have*

$$\sup_{t>0} \|e^{-tH} f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Moreover, for $m \in \mathbb{N}$ *and* $t > 0$*, since* $\psi_m(tH)$ *also satisfies* (DUE) *we have*

$$\sup_{t>0} \|\psi_m(tH)\|_{L^p \to L^p} \lesssim 1.$$

Let us now define some tools for the Littlewood-Paley theory we need in the sequel. For all $\lambda > 0$ we set

$$\varphi(\lambda) := \int_\lambda^{+\infty} \psi_m(u) \frac{du}{u},$$

$$\tilde{\varphi}(\lambda) := \int_0^\lambda \psi_m(v) \frac{dv}{v} = \int_0^1 \psi_m(\lambda u) \frac{du}{u}.$$

Remark 1.3. — Notice that φ is, by integration by parts, a finite linear combination of functions ψ_k for $k \in \{0,..,m\}$. Moreover for every $\lambda > 0$,

$$\tilde{\varphi}(\lambda) + \varphi(\lambda) = \int_0^{+\infty} u^{m-1} e^{-u} du = \Gamma(m) = \text{constant}.$$

The following theorem will be useful to estimate the L^p-norm through the heat semigroup:

Theorem 1.4. — *Assume* (0.4) *and* (DUE)*. For every integer* $m \geqslant 1$ *and all* $p \in (1,\infty)$*, we have*

$$\|f\|_{L^p} \simeq \|\varphi(H) f\|_{L^p} + \Big\| \Big(\int_0^1 |\psi_m(uH) f|^2 \frac{du}{u} \Big)^{\frac{1}{2}} \Big\|_{L^p}.$$

So if $q \geqslant 2$

$$\|f\|_{L^q} \lesssim \|\varphi(H) f\|_{L^q} + \Big(\int_0^1 \|\psi_m(uH) f\|_{L^q}^2 \frac{du}{u} \Big)^{\frac{1}{2}}.$$

Such a result can be seen as a semigroup version of the Littlewood-Paley characterization of Lebesgue spaces. A proof of this theorem can be found in [4] (look for Theorem 2.8 in [4]).

1.3. **Hardy and BMO spaces.** We now define atomic Hardy spaces adapted to our situation (dictated by a semigroup) using the construction introduced in [5]. Again we sum up the definitions and properties we need without proofs. A more detailed explanation with proofs is provided in [4].

Let M be a large enough integer.

Definition 1.5. — A function $a \in L^1_{loc}$ is an atom associated with the ball Q of radius r if there exists a function f_Q whose support is included in Q such that $a = (1 - e^{-r^2 H})^M (f_Q)$, with

$$\|f_Q\|_{L^2(Q)} \leqslant (\mu(Q))^{-\frac{1}{2}}.$$

That last condition allows us to normalize f_Q in L^1. Indeed by the Cauchy-Schwarz inequality

$$\|f_Q\|_{L^1} \leqslant \|f_Q\|_{L^2(Q)} \mu(Q)^{\frac{1}{2}} \leqslant 1.$$

Moreover, $(1 - e^{-r^2 H})^M$ is bounded on L^1 so every atom is in L^1 and they are also normalized in L^1:

$$\sup_a \|a\|_{L^1} \lesssim 1, \tag{1.2}$$

where we take the supremum over all the atoms.

We may now define the Hardy space by atomic decomposition

Definition 1.6. — A measurable function h belongs to the atomic Hardy space H^1_{ato}, which will be denoted H^1, if there exists a decomposition

$$h = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \mu - \text{a.e.}$$

where a_i are atoms and λ_i real numbers satisfying

$$\sum_{i \in \mathbb{N}} |\lambda_i| < +\infty.$$

We equip the space H^1 with the norm

$$\|h\|_{H^1} := \inf_{h = \sum_i \lambda_i a_i} \sum_{i \in \mathbb{N}} |\lambda_i|,$$

where we take the infimum over all the atomic decompositions.

For a more general definition and some properties about atomic spaces we refer to [3, 5], and the references therein. From (1.2), we deduce

Corollary 1.7. — *The Hardy space is continuously embedded into L^1:*

$$\|f\|_{L^1} \lesssim \|f\|_{H^1}.$$

From [5, Corollary 7.2], *the Hardy space H^1 is also a Banach space.*

We refer the reader to [5, Section 8], for details about the problem of identifying the dual space $(H^1)^*$ with a BMO space. For a L^∞-function, we may define the BMO norm

$$\|f\|_{\mathrm{BMO}} := \sup_Q \left(\fint_Q |(1 - e^{-r^2 H})^M (f)|^2 \, d\mu \right)^{1/2},$$

where the supremum is taken over all the balls Q of radius $r > 0$. If $f \in L^\infty$ then $(1 - e^{-r^2 H})^M (f)$ is also uniformly bounded (with respect to the ball Q), since the heat semigroup is uniformly bounded in L^∞ (see Proposition 1.2) and so $\|f\|_{\mathrm{BMO}}$ is finite.

Definition 1.8. — The functional space BMO is defined as the closure

$$\mathrm{BMO} := \overline{\{f \in L^\infty + L^2,\ \|f\|_{\mathrm{BMO}} < \infty\}}$$

for the BMO norm.

Following [5, Section 8], it comes that BMO is continuously embedded into the dual space $(H^1)^*$ and contains L^∞:

$$L^\infty \hookrightarrow \mathrm{BMO} \hookrightarrow (H^1)^*.$$

Hence

$$\|T\|_{H^1 \to (H^1)^*} \lesssim \|T\|_{H^1 \to \mathrm{BMO}}, \tag{1.3}$$

and we have the following interpolation result:

$$\forall \theta \in (0,1), \quad (L^2, \mathrm{BMO})_\theta \hookrightarrow (L^2, (H^1)^*)_\theta. \tag{1.4}$$

The following interpolation theorem between Hardy spaces and Lebesgue spaces is essential in our study.

Theorem 1.9. — *For all $\theta \in (0,1)$, consider the exponent $p \in (1,2)$ and $q = p' \in (2,\infty)$ given by*

$$\frac{1}{p} = \frac{1-\theta}{2} + \theta \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{2}.$$

Then (using the interpolation notations), we have

$$(L^2, H^1)_\theta = L^p \quad \text{and} \quad (L^2, (H^1)^*)_\theta \hookrightarrow L^q,$$

if the ambient space X is non-bounded and

$$L^p \hookrightarrow L^2 + (L^2, H^1)_\theta \quad \text{and} \quad L^2 \cap (L^2, (H^1)^*)_\theta \hookrightarrow L^q,$$

if the space X is bounded.

The same results hold replacing $(H^1)^*$ by BMO thanks to (1.4).

Remark 1.10. — We will not mention the case of a bounded space X in the proofs since interpolation is more delicate in that case. One can find the corresponding interpolation theorem (Theorem 2.17 in [4]) and check that the results apply in that case.

2. Proofs of the Theorems

This section is dedicated to the proofs of the announced result. It is divided into two main theorems. The first one shows which $L^2 - L^2$ dispersion property we can recover thanks to Assumption 0.1. In the second theorem we obtain Strichartz inequalities using such dispersive estimates. We recall that our goal is to investigate which properties (in terms of Strichartz inequalities) for the Schrödinger operator can be deduced from a weak assumption on the wave operator.

2.1. **Dispersive estimates for the Schrödinger operator.** The main theorem of this section is the following

Theorem 2.1. — *Assume $d > 2$, $m \geqslant \lceil \frac{d}{2} \rceil$, and that Assumption 0.1 is satisfied, then for all balls B, $\widetilde{B}$ of radius $r > 0$ and all $m' \geqslant 0$, we have:*

- *if $\kappa = +\infty$ then for all t such that $0 < |t| \leqslant 1$,*

$$\|e^{itH}\psi_{m'}(h^2 H)\psi_m(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \frac{r^d}{|t|^{\frac{d-2}{2}} h^2}; \tag{2.1}$$

- *if $\kappa < +\infty$ then for all for all $\varepsilon > 0$, $h \in (0;1]$ and t such that $h^2 \leqslant |t| \leqslant h^{1+\varepsilon}$,*

$$\|e^{itH}\psi_{m'}(h^2 H)\psi_m(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim_\varepsilon \frac{r^d}{|t|^{\frac{d-2}{2}} h^2}; \tag{2.2}$$

This shows how dispersion for the wave propagator implies dispersion for the Schrödinger group. The main tool to link those two operators is Hadamard's transmutation formula

$$\forall z \in \mathbb{C},\ \operatorname{Re}(z) > 0,\ e^{-zH} = \int_0^{+\infty} \cos(s\sqrt{H}) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{\pi z}}. \tag{2.3}$$

Proof. — Let B, $\widetilde{B}$ be balls with radius $r > 0$. We start our proof with some easy reductions.

First remark that we can restrict ourselves to prove the theorem for $h \leqslant r$. Indeed if the theorem is true for $h \leqslant r$ then for all $h > r$

$$\begin{aligned}
&\|e^{itH}\psi_{m'}(h^2 H)\psi_m(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \\
&\quad = \frac{2^{m'} r^{2m}}{(\frac{h^2}{2} + r^2)^m} \|e^{itH}\psi_{m'}(\frac{h^2}{2} H)\psi_m((\frac{h^2}{2} + r^2)H)\|_{L^2(B)\to L^2(\widetilde{B})} \\
&\quad \lesssim \Big(\frac{r}{h}\Big)^{2m} \|e^{itH}\psi_{m'}(\frac{h^2}{2} H)\psi_m((\frac{h^2}{2} + r^2)H)\|_{L^2(B_\rho)\to L^2(\widetilde{B}_\rho)}
\end{aligned}$$

where $\rho = \frac{h^2}{2} + r^2 > r$, $B_\rho = \frac{\rho}{r} B$ and $\widetilde{B}_\rho = \frac{\rho}{r}\widetilde{B}$ are of radius ρ. Since $\frac{h^2}{2} + r^2 \geqslant \frac{h^2}{2}$ we then obtain

$$\|e^{itH}\psi_{m'}(h^2 H)\psi_m(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \Big(\frac{r}{h}\Big)^{2m} \frac{\rho^d}{|t|^{\frac{d-2}{2}} h^2}.$$

We conclude using $\rho \lesssim h$ and

$$\frac{r^{2m} h^d}{h^{2m}} = r^d \Big(\frac{r}{h}\Big)^{2m-d} \leqslant r^d.$$

Moreover we only need to prove the theorem for $m' = 0$. Indeed if we show

$$\|e^{itH}\psi_0(h^2 H)\psi_m(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \frac{r^d}{|t|^{\frac{d-2}{2}} h^2}$$

then for all $m' \geqslant 0$ we have

$$\begin{aligned}\|e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \\ &= \left(\frac{h}{r}\right)^{2m'}\|e^{itH}\psi_0(h^2H)\psi_{m+m'}(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \\ &\lesssim \frac{r^d}{|t|^{\frac{d-2}{2}}h^2}\end{aligned}$$

since $h \leqslant r$

Finally it is sufficient to consider $r^2 \leqslant t$ because if $r^2 > t$ then by bounded functional calculus we have

$$\|e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim 1 \leqslant \left(\frac{r^2}{|t|}\right)^{\frac{d-2}{2}} \leqslant \frac{r^d}{|t|^{\frac{d-2}{2}}h^2}.$$

In summary, we fix $h \leqslant r$, $m' = 0$, and $r^2 \leqslant t$.

In order to avoid nonzero bracket terms in the forthcoming integrations by parts, we introduce a technical function $\chi \in C^\infty(\mathbb{R}_+)$ such that

$$\begin{cases} 0 \leqslant \chi \leqslant 1 \\ \chi(x) = 1 \text{ if } x \in [0, \frac{|t|}{r}] \\ \chi(x) = 0 \text{ if } x \in [2\frac{|t|}{r}, +\infty] \end{cases}.$$

Moreover we have $\forall n \in \mathbb{N}$, $\forall x \in \mathbb{R}_+$, $|\chi^{(n)}(x)| \lesssim \left(\frac{r}{|t|}\right)^n$. Thus we split (2.3) into

$$e^{-zH} = \int_0^{+\infty} \chi(s)\cos(s\sqrt{H})e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{\pi z}} + \int_0^{+\infty}(1-\chi(s))\cos(s\sqrt{H})e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{\pi z}} \quad (2.4)$$

and use it with $z = h^2 - it$. Note that $|z| \simeq |t|$.

We treat the first term by integrations by parts. Making $2n$ integration by parts (with n to be determined later) we get

$$\int_0^\infty \cos(s\sqrt{H})\psi_m(r^2H)\chi(s)e^{-\frac{s^2}{4z}}\,ds =$$

$$\int_0^\infty \cos(s\sqrt{H})r^{2n}\psi_{m-n}(r^2H)\sum_{k=0}^{2n}\chi^{(2n-k)}(s)e^{-\frac{s^2}{4z}}\left(c_k\frac{s^k}{z^k} + \ldots + c_{n-2\lfloor\frac{n}{2}\rfloor}\frac{s^{k-2\lfloor\frac{k}{2}\rfloor}}{z^{k-\lfloor\frac{k}{2}\rfloor}}\right)ds,$$

with $(c_i)_i$ being numerical constants playing no significant role. Keeping the extremal terms (one when $k = 0$ and two when $k = 2n$) we have to estimate

$$\int_0^{2\frac{|t|}{r}} \|\cos(s\sqrt{H})\psi_{m-n}(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})}r^{2n}\left(\left(\frac{r}{|t|}\right)^{2n} + \frac{s^{2n}}{|t|^{2n}} + \frac{1}{|t|^n}\right)\frac{ds}{\sqrt{|t|}}.$$

By continuity of our operators

$$\|\cos(s\sqrt{H})\psi_{m-n}(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim 1,$$

we can estimate

$$\int_0^{2\frac{|t|}{r}} \left(\frac{r^2}{|t|}\right)^{2n}\frac{ds}{\sqrt{|t|}} \lesssim \left(\frac{r^2}{|t|}\right)^{2n-\frac{1}{2}} \quad \text{and} \quad \int_0^{2\frac{|t|}{r}} \frac{r^{2n}}{|t|^n}\frac{ds}{\sqrt{|t|}} \lesssim \left(\frac{r^2}{|t|}\right)^{n-\frac{1}{2}}.$$

Using (0.1) we have

$$\int_0^{2\frac{|t|}{r}} \Big(\frac{r}{r+s}\Big)^{\frac{d-1}{2}} \Big(\frac{rs}{|t|}\Big)^{2n} \frac{ds}{\sqrt{|t|}} \leqslant \int_0^{2\frac{|t|}{r}} \frac{r^{\frac{d-1}{2}+2n}}{|t|^{2n+\frac{1}{2}}} s^{2n-\frac{d-1}{2}} ds \simeq \Big(\frac{r^2}{|t|}\Big)^{\frac{d-2}{2}}.$$

Thus, the intermediate terms having the same behaviour, for large enough n

$$\Big\| \int_0^\infty \cos(s\sqrt{H})\psi_m(r^2H)\chi(s)e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{z}} \Big\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \Big(\frac{r^2}{|t|}\Big)^{\frac{d-2}{2}}.$$

Moreover, since $h \leqslant r$ we have

$$\Big(\frac{r^2}{|t|}\Big)^{\frac{d-2}{2}} \leqslant \frac{r^d}{|t|^{\frac{d-2}{2}} h^2}.$$

To estimate the second term in (2.4), we treat separately the cases $s < \kappa$ and $s > \kappa$.

$$\int_0^{+\infty} (1-\chi(s)) \cos(s\sqrt{H}) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{\pi z}} = \int_{\frac{|t|}{r}}^{\kappa} \cos(s\sqrt{H}) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{\pi z}} + I_\kappa,$$

where

$$I_\kappa = \begin{cases} 0 & \text{if} \quad \kappa = +\infty \\ \int_\kappa^{+\infty} \cos(s\sqrt{H}) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{\pi z}} & \text{if} \quad \kappa < +\infty \end{cases}.$$

We use the exponential decay for $s > \kappa$. Recall that $z = h^2 - i|t|$. Using the L^2-boundedness of the $\cos(s\sqrt{H})\psi_m(r^2H)$ operator:

$$\begin{aligned}
\Big\| \int_\kappa^{+\infty} \cos(s\sqrt{H})\psi_m(r^2H) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{z}} \Big\|_{L^2(B)\to L^2(\widetilde{B})} &\lesssim \int_\kappa^{+\infty} e^{-\frac{s^2}{8}\operatorname{Re}\frac{1}{z}} e^{-\frac{s^2}{8}\operatorname{Re}\frac{1}{z}} \frac{ds}{\sqrt{|z|}} \\
&\lesssim \int_{\frac{\kappa}{8}\sqrt{\operatorname{Re}\frac{1}{z}}}^{+\infty} e^{-u^2} \frac{du}{\sqrt{|z|\operatorname{Re}\frac{1}{z}}} e^{-\frac{\kappa^2 \operatorname{Re}\frac{1}{z}}{8}} \\
&\lesssim \Big(\int_0^{+\infty} e^{-u^2} du\Big) \frac{\sqrt{|t|}}{h} e^{-\frac{\kappa^2 h^2}{16t^2}} \\
&\lesssim \Big(\frac{h^2}{t^2}\Big)^{-N} \frac{\sqrt{|t|}}{h}
\end{aligned}$$

for all $N \geqslant 1$ as large as we want and where we used $|z| \simeq |t|$ and $\operatorname{Re}\frac{1}{z} \geqslant \frac{h^2}{2t^2}$. Moreover

$$\frac{|t|^{2N+\frac{1}{2}}}{h^{2N+1}} \leqslant \frac{h^d}{|t|^{\frac{d-2}{2}} h^2} \leqslant \frac{r^d}{|t|^{\frac{d-2}{2}} h^2}$$

as soon as $|t|^{2N+\frac{d-2}{2}+\frac{1}{2}} \leqslant h^{2N+d-1}$ that is $|t| \leqslant h^{1+\frac{\frac{d-1}{2}}{2N+\frac{d-1}{2}}}$. Which is true since

$$|t| \leqslant h^{1+\varepsilon} \leqslant h^{1+\frac{\frac{d-1}{2}}{2N+\frac{d-1}{2}}}$$

for N large enough.

Remark 2.2. — We point out that this is the only moment we use that $|t| \leqslant h^{1+\varepsilon}$. That is why we do not need it when $\kappa = +\infty$ since this term does not step in. Therefore the loss of derivatives in Theorem 0.3 is better when $\kappa = +\infty$.

We use Assumption 0.1 when $s<\kappa$. Indeed it yields

$$\left\|\int_{\frac{|t|}{r}}^{\kappa}\cos(s\sqrt{H})\psi_m(r^2H)e^{-\frac{s^2}{4z}}\frac{ds}{\sqrt{z}}\right\|_{L^2(B)\to L^2(\widetilde{B})}\lesssim\int_{\frac{|t|}{r}}^{\kappa}\left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}e^{-\frac{s^2}{4}\operatorname{Re}\frac{1}{z}}\frac{ds}{\sqrt{|t|}}.$$

When $\frac{d-1}{2}>1$ (i.e. $d>3$) we have

$$\begin{aligned}\int_{\frac{|t|}{r}}^{\kappa}\left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}\frac{ds}{\sqrt{|t|}}&\leqslant\frac{r^{\frac{d-1}{2}}}{\sqrt{|t|}}\int_{\frac{|t|}{r}}^{\infty}s^{-\frac{d-1}{2}}ds\lesssim\frac{r^{\frac{d-1}{2}}}{\sqrt{|t|}}\left(\frac{|t|}{r}\right)^{-\frac{d-1}{2}+1}\\&\leqslant\frac{r^{d-2}}{|t|^{\frac{d-2}{2}}}\leqslant\frac{r^d}{|t|^{\frac{d-2}{2}}h^2}h^2\leqslant\frac{r^d}{|t|^{\frac{d-2}{2}}h^2}\end{aligned}$$

since $h^2\leqslant 1$.
When $\frac{d-1}{2}<1$ (i.e. $d<3$), since $\operatorname{Re}\frac{1}{z}\gtrsim\frac{h^2}{t^2}$ we have

$$\begin{aligned}\int_{\frac{|t|}{r}}^{\kappa}\left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}e^{-c\frac{s^2h^2}{t^2}}\frac{ds}{\sqrt{|t|}}&\lesssim\frac{r^{\frac{d-1}{2}}}{\sqrt{|t|}}\int_{\frac{h}{r}}^{\infty}e^{-u^2}\left(\frac{|t|u}{h}\right)^{-\frac{d-1}{2}}\frac{|t|}{h}du\\&\lesssim\frac{r^{\frac{d-1}{2}}h^{\frac{d-3}{2}}}{|t|^{\frac{d-2}{2}}}\int_0^{\infty}u^{-\frac{d-1}{2}}e^{-u^2}du\\&\lesssim\frac{r^d}{|t|^{\frac{d-2}{2}}h^2}\frac{h^{\frac{d-3}{2}}h^2}{r^{\frac{d+1}{2}}}\leqslant\frac{r^d}{|t|^{\frac{d-2}{2}}h^2}\end{aligned}$$

since $h\leqslant r$.
When $\frac{d-1}{2}=1$ (i.e. $d=3$) we have

$$\begin{aligned}\int_{\frac{|t|}{r}}^{\kappa}\frac{r}{r+s}e^{-\frac{s^2}{4}\operatorname{Re}\frac{1}{z}}\frac{ds}{\sqrt{|t|}}&\lesssim\frac{r}{\sqrt{|t|}}\int_{\frac{h}{r}}^{\kappa\frac{h}{|t|}}\left(\frac{|t|u}{h}\right)^{-1}e^{-u^2}\frac{|t|}{h}du\\&\leqslant\frac{r}{\sqrt{|t|}}e^{-\frac{h^2}{2r^2}}\frac{r}{h}\int_0^{+\infty}e^{-\frac{u^2}{2}}du\lesssim\frac{r^2}{\sqrt{|t|}h}\left(\frac{h}{r}\right)^{-1}\\&=\frac{r^3}{\sqrt{|t|}h^2}=\frac{r^d}{|t|^{\frac{d-2}{2}}h^2}.\end{aligned}$$

In the end, summing all the parts up, we have

$$\|e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})}\lesssim\frac{r^d}{|t|^{\frac{d-2}{2}}h^2}.\qquad\square$$

2.2. **Strichartz inequalities.** To obtain Strichartz estimates we are going to use Theorem 1.1 of [4], which we recall here with a slight modification in assuming (0.4), namely

THEOREM 2.3. — *Assume* (0.4) *with* (*DUE*). *Consider an L^2-bounded operator T (with $\|T\|_{L^2\to L^2}\lesssim 1$), which commutes with H and satisfies*

$$\|T\psi_m(r^2H)\|_{L^2(B_r)\to L^2(\widetilde{B_r})}\lesssim A\mu(B_r)^{\frac{1}{2}}\mu(\widetilde{B_r})^{\frac{1}{2}}\qquad(H_m(A))$$

for some $m\geqslant\frac{d}{2}$. Then T is bounded from H^1 to BMO and from L^p to $L^{p'}$ for $p\in(1,2)$ with

$$\|T\|_{H^1\to BMO}\lesssim A\qquad\text{and}\qquad\|T\|_{L^p\to L^{p'}}\lesssim A^{\frac{1}{p}-\frac{1}{p'}}$$

if the ambient space X is unbounded and

$$\|T\|_{H^1\to BMO} \lesssim \max(A,1) \qquad \text{and} \qquad \|T\|_{L^p\to L^{p'}} \lesssim \max(A^{\frac{1}{p}-\frac{1}{p'}}, B)$$

if the ambient space X is bounded, and where, for the last inequality, we assumed that $\|T\|_{L^p\to L^2} \lesssim B$.

As we mentioned previously, we do not use the part where X is bounded. We apply the Theorem with $T = e^{itH}\psi_{m'}(h^2H)$ and $A = |t|^{-\frac{d-2}{2}}h^{-2}$. In view of (0.4) we can reformulate $(H_m(A))$ (see [4]) as

$$\|e^{itH}\psi_{m'}(h^2H)\psi_m(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \frac{r^d}{|t|^{\frac{d-2}{2}}h^2} \tag{2.5}$$

which we just proved in the previous section under our assumption. Therefore we obtain

$$\|e^{itH}\psi_{m'}(h^2H)\|_{H^1\to BMO} \lesssim |t|^{-\frac{d-2}{2}}h^{-2},$$

and for all $p\in(1,2)$

$$\|e^{itH}\psi_{m'}(h^2H)\|_{L^p\to L^{p'}} \lesssim \left[h^{-2}|t|^{-\frac{d-2}{2}}\right]^{\frac{1}{p}-\frac{1}{p'}}.$$

We now recall a slightly modified version of a result of Keel-Tao in [18]:

Theorem 2.4. — *If $(U(t))_{t\in\mathbb{R}}$ satisfies*

$$\sup_{t\in\mathbb{R}}\|U(t)\|_{L^2\to L^2} \lesssim 1$$

and for some $\sigma>0$

$$\forall t\neq s,\ \|U(t)U(s)^*\|_{H^1\to \mathrm{BMO}} \leqslant C|t-s|^{-\sigma}.$$

Then for all $2\leqslant p\leqslant +\infty$ and $2\leqslant q<+\infty$ satisfying

$$\frac{1}{p}+\frac{\sigma}{q}=\frac{\sigma}{2}$$

we have

$$\|U(t)f\|_{L^p_tL^q_x} \lesssim C^{\frac{1}{2}-\frac{1}{q}}\|f\|_{L^2}.$$

Proof. — We just sum up the main steps of the proof in [14] and [18] to keep track of the constant in the last estimation.

- By symmetry and a T^*T argument, it suffices to show
$$\left|\int_{s<t}\langle U(s)^*F(s),U(t)^*G(t)\rangle ds dt\right| \lesssim C^2\|F\|_{L^{p'}_tL^{q'}_x}\|G\|_{L^{p'}_tL^{q'}_x}.$$
- By the interpolation Theorem 1.9 we have
$$\|U(t)U(s)^*\|_{L^{q'}\to L^q} \lesssim C^{1-\frac{2}{q}}|t-s|^{-\frac{2}{p}}.$$
- We conclude by Hölder and Hardy-Littlewood-Sobolev inequalities. □

We use this theorem with $C=\frac{1}{h^2}$ and $\sigma=\frac{d-2}{2}$ to obtain the following result.

Theorem 2.5. — *Under Assumption 0.1, if $2\leqslant p\leqslant+\infty$ and $2\leqslant q<+\infty$ satisfy*

$$\frac{2}{p}+\frac{d-2}{q}=\frac{d-2}{2},$$

and $f\in L^2$ and $0<h\leqslant 1$ we have

- *if $\kappa=+\infty$ then for all $m'\in\mathbb{N}$*
$$\|e^{itH}\psi_{m'}(h^2H)f\|_{L^p([-1,1],L^q)}\lesssim\frac{1}{h^{2(\frac12-\frac1q)}}\|\psi_{m'}(h^2H)f\|_{L^2};$$
- *if $\kappa<+\infty$ then for all $0<\varepsilon<1$ and $m'\in\mathbb{N}$*
$$\|e^{itH}\psi_{m'}(h^2H)f\|_{L^p([-1,1],L^q)}\lesssim\frac{1}{h^{\frac{1+\varepsilon}{p}}h^{2(\frac12-\frac1q)}}\|\psi_{m'}(h^2H)f\|_{L^2}.$$

Proof. — The following proof is a slight modification of the one of Theorem 4.2 and 4.3 of [4]. We rewrite it here for more readability. We only deal with the case $\kappa<+\infty$ since it is more technical. We leave the minor modifications to obtain the case $\kappa=+\infty$ to the readers.
Fix an interval $J\subset[-1,1]$ of length $|J|=h^{1+\varepsilon}$, $m'\in\mathbb{N}$, and consider
$$U(t)=\mathbb{1}_J(t)e^{itH}\psi_{m'}(h^2H).$$
We aim to apply Theorem 2.4 with $C=\frac{1}{h^2}$ and $\sigma=\frac{d-2}{2}$. By functional calculus we have
$$\sup_{t\in\mathbb{R}}\|U(t)\|_{L^2\to L^2}\lesssim 1.$$
The estimation (2.5) which we proved in Theorem 2.1 will lead to the second hypothesis of Theorem 2.4. First
$$\begin{aligned}U(t)U(s)^*&=\mathbb{1}_J(t)\mathbb{1}_J(s)e^{itH}\psi_{m'}(h^2H)(e^{isH}\psi_{m'}(h^2H))^*\\&=\mathbb{1}_J(t)\mathbb{1}_J(s)e^{i(t-s)H}2^{-2m'}\psi_{2m'}(2h^2H)\end{aligned}$$
because H is self-adjoint. Since J has length equal to $h^{1+\varepsilon}$ then $U(t)U(s)^*$ is vanishing or else $|t-s|\leqslant h^{1+\varepsilon}$. Hence, by Theorem 2.1 we deduce
$$\|U(t)U(s)^*\|_{H^1\to(H^1)^*}\lesssim\frac{1}{h^2}\frac{1}{|t-s|^{\frac{d-2}{2}}}.$$
Up to the change of $2m'$ into m', Theorem 2.4 (with $C=h^{-2}$ and $\sigma=(d-2)/2$) then leads to
$$\Big(\int_J\|e^{itH}\psi_{m'}(h^2H)f\|^p_{L^q}dt\Big)^{\frac1p}\lesssim\frac{1}{h^{2(\frac12-\frac1q)}}\|f\|_{L^2}.$$
We then split $[-1,1]$ into $N\simeq\frac{1}{h^{1+\varepsilon}}$ intervals J_k of length $h^{1+\varepsilon}$ to obtain
$$\int_{-1}^1\|e^{itH}\psi_{m'}(h^2H)f\|^p_{L^q}dt\leqslant\sum_{k=1}^N\int_{J_k}\|e^{itH}\psi_{m'}(h^2H)f\|^p_{L^q}dt\leqslant N\Big(\frac{1}{h^{2(\frac12-\frac1q)}}\|f\|_{L^2}\Big)^p.$$
Hence
$$\|e^{itH}\psi_{m'}(h^2H)f\|_{L^p([-1,1],L^q)}\lesssim\frac{1}{h^{\frac{1+\varepsilon}{p}}h^{2(\frac12-\frac1q)}}\|\psi_{m'}(h^2H)f\|_{L^2}.\qquad\square$$

We are now able to prove Strichartz estimates with loss of derivatives.

Theorem 2.6. — *If Assumption 0.1 is satisfied. Then for every $2\leqslant p\leqslant+\infty$ and $2\leqslant q<+\infty$ satisfying*
$$\frac2p+\frac{d-2}{q}=\frac{d-2}{2},$$

and every solution $u(t,.) = e^{itH}u_0$ *of the problem*

$$\begin{cases} i\partial_t u + Hu = 0 \\ u_{|t=0} = u_0, \end{cases}$$

we have

- *if* $\kappa = +\infty$, *then* u *satisfies local-in-time Strichartz estimates with loss of derivatives*
$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{2(\frac{1}{2}-\frac{1}{q}),2}}; \tag{2.6}$$
- *if* $\kappa < +\infty$, *then for all* $0 < \varepsilon < 1$ *and* $0 < h \leqslant 1$, u *satisfies local-in-time Strichartz estimates with loss of derivatives*
$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q}),2}}. \tag{2.7}$$

Remark 2.7. — The loss of derivatives in (2.7) is interesting when it is lower than the straightforward loss given by Sobolev embeddings. The relation $\frac{2}{p} + \frac{d-2}{q} = \frac{d-2}{2}$ yields

$$W^{\frac{2}{p}+1-\frac{2}{q},2} \hookrightarrow L^q.$$

Thus

$$\|e^{itH}u_0\|_{L^q} \lesssim \|e^{itH}u_0\|_{W^{\frac{2}{p}+1-\frac{2}{q},2}} \leqslant \|u_0\|_{W^{\frac{2}{p}+1-\frac{2}{q},2}}$$

and taking the $L^p([-1,1])$ norm shows

$$\|e^{itH}u_0\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{2}{p}+1-\frac{2}{q},2}}.$$

That is, the loss of derivatives is interesting when it is less than $\frac{2}{p} + 1 - \frac{2}{q}$. Hence, for all $\varepsilon \in (0,1)$, the loss we obtained is strictly better than the one directly given by Sobolev embeddings. The loss in (2.6) is also nontrivial by the same argument.

Remark 2.8. — One could work out our estimate with

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

In order to do so we remark that in (2.5) we could write

$$\frac{r^d}{t^{\frac{d-2}{2}}h^2} = \left(\frac{r^2}{t}\right)^{\frac{d}{2}} \frac{t}{h^2} \leqslant \left(\frac{r^2}{t}\right)^{\frac{d}{2}} \frac{1}{h},$$

because $t \leqslant h$. Then the loss of derivatives obtained in (2.7) is $\frac{1+\varepsilon}{p} + 1(\frac{1}{2} - \frac{1}{q})$ that need to be compared to the trivial loss $\frac{2}{p}$. Since $\frac{1}{2} - \frac{1}{q} = \frac{2}{dp}$, the loss is less than $\frac{2}{p}$ if an only if

$$d \geqslant \frac{2}{1-\varepsilon}.$$

That is, as soon as $d > 2$, one can find $\varepsilon \in (0,1)$ such that the loss is nontrivial. We chose to present the previous Theorem in that form because it allows a wider range of exponent q. Indeed, on the one hand

$$p \geqslant 2 \Rightarrow \frac{d-2}{q} = \frac{d-2}{2} - \frac{2}{p} \geqslant \frac{d-2}{2} - 1$$

that is

$$\frac{1}{q} \geqslant \frac{1}{2} - \frac{1}{d-2}.$$

On the other hand $p \geqslant 2$ and $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$ yields

$$\frac{1}{q} \geqslant \frac{1}{2} - \frac{1}{d},$$

and for all $d > 2$,

$$\frac{1}{2} - \frac{1}{d-2} \leqslant \frac{1}{2} - \frac{1}{d}.$$

That is why the relation

$$\frac{2}{p} + \frac{d-2}{q} = \frac{d-2}{2}$$

gives a wider range for exponent q.

Proof of Theorem 2.6. — Again we only deal with the more difficult case $\kappa < +\infty$.
We apply Theorem 1.4 to $u(t) = e^{itH} u_0$. It leads to

$$\|u(t)\|_{L^q} \lesssim \|\varphi(H)u(t)\|_{L^q} + \left\| \Big(\int_0^1 |\psi_{m'}(s^2 H)u(t)|^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \right\|_{L^q},$$

with $m' \geqslant 1$.
Taking the $L^p([-1,1])$ norm in time of that expression and using Minkowski inequality yields

$$\begin{aligned} &\|u(t)\|_{L^p([-1,1],L^q)} \\ &\qquad \lesssim \|\varphi(H)u(t)\|_{L^p([-1,1],L^q)} + \left\| \Big(\int_0^1 \|\psi_{m'}(s^2 H)u(t)\|_{L^q}^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \right\|_{L^p([-1,1])}. \end{aligned}$$

Thanks to the Gaussian pointwise estimate of $\varphi(H)$ the first term can be estimated as follow

$$\|\varphi(H)u(t)\|_{L^p([-1,1],L^q)} \lesssim \|e^{itH}u_0\|_{L^p([-1,1],L^2)} \lesssim \|u_0\|_{L^2} \lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q}),2}}.$$

Since $p \geqslant 2$, Theorem 2.5 and generalized Minkowski inequality allow to bound the second term

$$\begin{aligned} \left\| \Big(\int_0^1 \|\psi_{m'}(s^2 H)u(t)\|_{L^q}^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \right\|_{L^p([-1,1])} &\lesssim \Big(\int_0^1 \|\psi_{m'}(s^2 H)u\|_{L^p([-1,1],L^q)}^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \\ &\lesssim \Big(\int_0^1 s^{-\frac{1+\varepsilon}{p}-2(\frac{1}{2}-\frac{1}{q})} \|\psi_{m'}(s^2 H)u_0\|_{L^2}^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \\ &\lesssim \left\| \Big(\int_0^1 s^{-\frac{1+\varepsilon}{p}-2(\frac{1}{2}-\frac{1}{q})} |\psi_{m'}(s^2 H)u_0|^2 \frac{ds}{s} \Big)^{\frac{1}{2}} \right\|_{L^2} \\ &\lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q}),2}}, \end{aligned}$$

where we used $m' \geqslant \frac{1}{4}[\frac{1+\varepsilon}{p} + 2(1-\frac{2}{q})]$ since $m' \geqslant 1$ and $\frac{1+\varepsilon}{p} + 2(1-\frac{2}{q}) < 2$ and the fact that

$$s^{-\frac{1+\varepsilon}{p}-2(\frac{1}{2}-\frac{1}{q})} |\psi_{m'}(s^2 H)|^2 = \psi_{m'-\frac{1}{4}[\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q})]}(s^2 H) H^{\frac{1}{2}[\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q})]}.$$

Finally, we get

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p}+2(\frac{1}{2}-\frac{1}{q}),2}}. \qquad \square$$

Acknowledgements

The author is very grateful to Frédéric Bernicot for his useful advices and comments about this article.

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Manuscript received June 16, 2018,
revised February 25, 2019,
accepted February 25, 2019.

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