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ON THE LIMITING ABSORPTION PRINCIPLE FOR A NEW CLASS OF SCHRÖDINGER HAMILTONIANS

ALEXANDRE MARTIN

Abstract. We prove the limiting absorption principle and discuss the continuity properties of the boundary values of the resolvent for a class of form bounded perturbations of the Euclidean Laplacian Δ that covers both short and long range potentials with an essentially optimal behaviour at infinity. For this, we give an extension of Nakamura's results (see [16]).

CONTENTS

1. Introduction	63
2. Notation and basic notions	69
2.1. Notation	69
2.2. Regularity	70
3. Nakamura's ideas in a more general setting: Boundary values of the resolvent	71
4. An extension of Nakamura's results	73
5. Concrete potentials	78
5.1. A non Laplacian-compact potential	79
5.2. A class of oscillating potential	84
5.3. An unbounded potential with high oscillations	86
5.4. A short range potential in a weak sense	89
6. Flow	93
References	93

1. INTRODUCTION

The purpose of this article is to prove a limiting absorption principle for a certain class of Schrödinger operator with real potential and to study their essential spectrum. Because this operators are self-adjoint, we already know that their spectrum is in the real axis. We also know that the non negative Laplacian operator Δ (Schrödinger operator with no potential) has for spectrum the real set $[0, +\infty)$ with purely absolutely continuous spectrum on this set. If we add to Δ a "small" potential (with compact properties with respect to Δ), the essential spectrum of this new operator is the same that Δ essential spectrum which is continuous. We are interested in the nature of the essential spectrum of the perturbed operator and in the behaviour of the resolvent operator near the essential spectrum.

We will say that a self-adjoint operator has *normal spectrum* in an open real set O if it has no singular continuous spectrum in O and its eigenvalues in O are of finite multiplicities and have no accumulation points inside O . Note that if we have a Limiting Absorption Principle for an operator H on O , H has normal spectrum on O .

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A general technique for proving this property is due to E. Mourre [14] and it involves a local version of the positive commutator method due to C.R. Putnam [17, 18]. For various extensions and applications of these techniques we refer to [1]. Roughly speaking, the idea is to search for a second self-adjoint operator A such that H is regular in a certain sense with respect to A and such that H satisfies the Mourre estimate on a set I in the following sense

$$E(I)[H, iA]E(I) \geq c_0 E(I) + K$$

where $E(I)$ is the spectral measure of H on I , $c_0 > 0$ and K a compact operator. Then one says that the operator A is *conjugate* to H on I .

When $H = \Delta + V$ is a Schrödinger operator, we usually apply the Mourre theorem with the generator of dilations

$$A_D = \frac{1}{2}(p \cdot q + q \cdot p),$$

where $p = -i\nabla$ and q is the vector of multiplication by x (see [1, Proposition 7.4.6] and [7, Section 4]). But in the commutator expressions, derivatives of V appears which can be a problem, if, for example, V has high oscillations at infinity.

In a recent paper S. Nakamura [16] pointed out the interesting fact that a different choice of conjugate operator for H can be used to have a limiting absorption principle. This allows us to avoid imposing conditions on the derivative of the long range part of the potential. More precisely, if the operator of multiplication by $V(q)$ is Δ -compact and two other multiplication operators, which include differences on V and not derivatives, are Δ -bounded, then H has normal spectrum in $(0, \pi^2/a^2)$ and the limiting absorption principle holds for H locally on this set, outside the eigenvalues. This fact is a consequence of the Mourre theorem with A_N (see (1.1)) as conjugate operator.

Our purpose in this article is to put the results of Nakamura in a more general abstract setting and get a generalisation of his result. Moreover, we will show that this generalisation can be applied to potentials for which the Mourre theorem with the generator of dilations as conjugate operator cannot apply (our potentials are not of long range type). Furthermore, Nakamura's result cannot apply to this type of potentials which are not Δ -bounded. Finally, as usual, we will derive from the limiting absorption principle an application of this theory to wave operators.

We denote $X = \mathbb{R}^\nu$ and $\mathcal{H} = L^2(X)$. Let \mathcal{H}^1 be the first order Sobolev space on X , denote \mathcal{H}^{-1} its adjoint space and similarly, we denote \mathcal{H}^2 the second order Sobolev space on X and \mathcal{H}^{-2} its adjoint space. All this spaces realised the following

$$\mathcal{H}^2 \subset \mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1} \subset \mathcal{H}^{-2}.$$

Set $\mathcal{B}_1 = B(\mathcal{H}^1, \mathcal{H}^{-1})$ and $\mathcal{B}_2 = B(\mathcal{H}^2, \mathcal{H}^{-2})$. If needed for clarity, if u is a measurable function on X we denote $u(q)$ the operator of multiplication by u whose domain and range should be obvious from the context. If $a \in X$ let T_a be the operator of translation by a , more precisely $(T_a f)(x) = f(x + a)$.

We say that $V \in \mathcal{B}$, $\mathcal{B} = \mathcal{B}_1$ or $\mathcal{B} = \mathcal{B}_2$, is a *multiplication operator* if $V\theta(q) = \theta(q)V$ for any $\theta \in C_c^\infty(X)$. Note that V is not necessarily the operator of multiplication by a function, it could be the operator of multiplication by a distribution of strictly positive order. For example, in the one dimensional case V could be equal to the derivative of a bounded measurable function. Anyway, if V is a multiplication operator then there is a uniquely defined temperate distribution v on X such that $Vf = vf$ for all $f \in C_c^\infty(X)$ and

then $T_a V T_a^* = v(\cdot + a)$. In general we simplify notations and do not distinguish between the operator V and the distribution v , so we write $V = V(q)$ and $T_a V T_a^* = V(q + a)$.

We will extend Nakamura's result in two directions. First, we will use the Mourre theory with a class of potential $V : \mathcal{H}^2 \rightarrow L^2$ or $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ (cf. [1]) and satisfying a weaker regularity. In particular, this includes potentials with Coulomb singularities, and also short range potentials (see Definition 1.3). Secondly, we will use the Mourre theory with a more general class of conjugate operators including A_N (see (1.5)).

Let $a > 0$ and let $\sin(ap) = (\sin(-ia\partial_{x_1}), \dots, \sin(-ia\partial_{x_\nu}))$. Let

$$A_N = \frac{1}{2}(\sin(ap) \cdot q + q \cdot \sin(ap)). \quad (1.1)$$

Fix a real function $\xi \in C^\infty(X)$ such that $\xi(x) = 0$ if $|x| < 1$ and $\xi(x) = 1$ if $|x| > 2$.

THEOREM 1.1. — *Let $a \in \mathbb{R}$ and $V : \mathcal{H}^2 \rightarrow L^2$ (respectively $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$) be a compact symmetric multiplication operator with, for all vector e of the canonical basis of \mathbb{R}^ν ,*

$$\int_1^\infty \|\xi(q/r)|q|(V(q+ae) - V(q))\|_{\mathcal{B}} \frac{dr}{r} < \infty \quad (1.2)$$

where $\mathcal{B} = \mathcal{B}_2$ (respectively $\mathcal{B} = \mathcal{B}_1$). Then the self-adjoint operator $H = \Delta + V$ on \mathcal{H} has normal spectrum in $(0, (\pi/a)^2)$ and, for some appropriate Besov space \mathcal{K} , the limits

$$(H - \lambda \pm i0)^{-1} := \text{w}^*\text{-}\lim_{\mu \downarrow 0} (H - \lambda \pm i\mu)^{-1} \quad (1.3)$$

exist in $\mathcal{B}(\mathcal{K}, \mathcal{K}^*)$, locally uniformly in $\lambda \in (0, (\pi/a)^2)$ outside the eigenvalues of H .

We make some comments in connection with the Theorem 1.1.

- (1) The Besov space is defined in Section 2 by (2.2).
- (2) Condition (1.2) is satisfied if $\|\langle q \rangle^\mu (V(q+a) - V(q))\|_{\mathcal{B}} < \infty$ for a fixed $\mu > 1$. To satisfy this conditions, it suffices that $\|\langle q \rangle^\mu V\|_{\mathcal{B}} < \infty$. In particular, in dimension $\nu \neq 2$, if V is a real function on \mathbb{R}^ν and if there is $\mu > 1$ such that $(\langle \cdot \rangle^\mu V(\cdot))^p$ is in the Kato class, with $p = 1$ if $\nu = 1$, and $p = \nu/2$ if $\nu \geq 3$, then condition (1.2) is satisfied (see Proposition 4.7). If this is the case for all $a \in \mathbb{R}$, then the limiting absorption principle is true on $(0, +\infty)$.
- (3) In the case where $V : \mathcal{H}^2 \rightarrow L^2$ is compact, in [16], V is assumed to satisfy $q(V(q+ae) - V(q))$ and $q^2(V(q+ae) + V(q-ae) - 2V(q))$ be Δ -bounded. This assumptions implies to the $C^2(A_N, \mathcal{H}^2, L^2)$ regularity. Observe that, since $q(V(q+ae) - V(q))$ appears in $[V, iA_N]$, (1.2) implies the $C^{1,1}(A_N, \mathcal{H}^2, \mathcal{H}^{-2})$ regularity which is implied by the $C^2(A_N, \mathcal{H}^2, L^2)$ regularity.
- (4) In one dimension, let V such that

$$\widehat{qV}(\xi) = \sum_{n=-\infty}^{+\infty} \lambda_n \chi(\xi - n), \quad (1.4)$$

where $\widehat{\cdot}$ is the Fourier transform, $\lambda_n \in \mathbb{R}$ and χ is compactly support, then V satisfies assumptions of Theorem 1.1 with $\mathcal{B} = \mathcal{B}_1$ but V is neither Δ -bounded nor of class $C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$. In particular, we can neither apply the Mourre Theorem with the generator of dilation (see [1, p.258]) nor Nakamura's Theorem (see lemma 5.3).

As in [16], the limiting absorption principle is limited to $(0, (\pi/a)^2)$. The bound $(\pi/a)^2$ is artificial and appears with the choice of vector field $\sin(ap)$. In fact, by a simple computation with the Laplacian Δ in $L^2(\mathbb{R}^\nu)$, we have

$$\begin{aligned} [\Delta, iA_N] &= [\Delta, i\frac{1}{2}(\sin(ap) \cdot q + q \cdot \sin(ap))] \\ &= \frac{1}{2}(\sin(ap) \cdot [\Delta, iq] + [\Delta, iq] \cdot \sin(ap)) \\ &= 2p \cdot \sin(ap) \end{aligned}$$

which implies a loss of positivity on $(\pi/a)^2$. This is a drawback except if we can apply (1.2) for all $a > 0$. We will use Nakamura's method in a broader framework that allows the removing of this drawback.

Let us denote $H_0 = \Delta = p^2$. Then H_0 is a self-adjoint operator in \mathcal{H} with domain \mathcal{H}^2 which extends to a linear symmetric operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ for which we keep the notation H_0 . Let $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ be a linear symmetric compact operator. Then $H = H_0 + V$ is a symmetric operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ which induces a self-adjoint operator in \mathcal{H} for which we keep the notation H . Let E_0 and E be the spectral measures of H_0 and H .

Note that for each non real z the resolvent $R(z) = (H - z)^{-1}$ of the self-adjoint operator H in \mathcal{H} extends to a continuous operator $R(z) : \mathcal{H}^{-1} \rightarrow \mathcal{H}^1$ which is in fact the inverse of the bijective operator $H - z : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$.

A_N and A_D belongs to a general class of conjugate operator, which appears in [1, Proposition 4.2.3]. This is the class of operator which can be written like

$$A_u = \frac{1}{2}(u(p) \cdot q + q \cdot u(p)) \quad (1.5)$$

where u is a C^∞ vector field with all the derivates bounded. We will see that this conjugate operator is self-adjoint on some domain (see Section 6). Conjugate operators of this form were already used in Mourre's paper [14, page 395].

Remark that the commutator of such conjugate operator with a function of p is quite explicit: denoting $h' = \nabla h$ then

$$[h(p), iA_u] = [h(p), iu(p)q] = u(p) \cdot h'(p) = (u \cdot h')(p). \quad (1.6)$$

In particular $[H_0, iA_u] = 2p \cdot u(p)$. We denote by the same notation $e^{i\tau A_u}$ the C_0 -group in \mathcal{H}^1 and in \mathcal{H}^{-1} .

One says that A is *strictly conjugate to H_0* on J if there is a real number $a > 0$ such that $E_0(J)[H_0, iA]E_0(J) \geq aE_0(J)$, which in our case means $2k \cdot u(k) \geq a$ for each $k \in X$ such that $|k|^2 \in J$. Taking A_u in this class, we have the following

THEOREM 1.2. — *Let $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ be a compact symmetric operator such that there is u a C^∞ bounded vector field with all derivatives bounded such that V is of class $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ in the following sense:*

$$\int_0^1 \|V_\tau + V_{-\tau} - 2V\|_{\mathcal{B}_1} \frac{d\tau}{\tau^2} < \infty, \quad \text{where } V_\tau = e^{i\tau A_u} V e^{-i\tau A_u}. \quad (1.7)$$

Let J be an open real set such that

$$\inf\{k \cdot u(k) \mid k \in X, |k|^2 \in J\} > 0. \quad (1.8)$$

Then H has normal spectrum in J and the limits

$$R(\lambda \pm i0) := \text{w}^*\text{-}\lim_{\mu \downarrow 0} R(\lambda \pm i\mu) \quad (1.9)$$

exist in $B(\mathcal{H}_{1/2,1}^{-1}, \mathcal{H}_{-1/2,\infty}^1)$, locally uniformly in $\lambda \in J$ outside the eigenvalues of H , where $\mathcal{H}_{1/2,1}^{-1}$ and $\mathcal{H}_{-1/2,\infty}^1$ are interpolation spaces which are defined on Section 2.

We make some remarks about this Theorem:

- (1) To check the $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ property, it is useful to have $e^{itA_u}\mathcal{H}^1 \subset \mathcal{H}^1$. For that, we will make a comment in the Section 6 on the flow generated by the vector field u associated to A_u .
- (2) If $k \cdot u(k)$ is positive for all $k \neq 0$, Theorem 1.2 applies with $J = (0, +\infty)$.
- (3) If V is the divergence of a short range potential (see Definition 1.3), then Theorem 1.2 applies. A certain class of this type of potential were already studied in [5] and [6].
- (4) Since $R(\lambda \pm i0)$ exists in $B(\mathcal{H}_{1/2,1}^{-1}, \mathcal{H}_{-1/2,\infty}^1)$, for D' the space of distributions this operator exists in $B(C_c^\infty(\mathbb{R}^\nu), D')$.
- (5) If V can be seen as a compact operator from \mathcal{H}^2 to L^2 , Theorem 1.2 is still valid if we replace the assumption " V is of class $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ " by the weak assumption " $V \in C^{1,1}(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$ " with the same proof.
- (6) Consider the Δ -compact operator $V(q)$ where

$$V(x) = (1 - \kappa(|x|)) \frac{\sin(|x|^\alpha)}{|x|^\beta},$$

with $\kappa \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\kappa(|x|) = 1$ if $|x| < 1$, $0 \leq \kappa \leq 1$, $\alpha > 0$ and $\beta > 0$. Note that this type of potential was already studied in [2, 8, 9, 10, 13, 21, 22]. In [13], they proved that if $|\alpha - 1| + \beta > 1$, then V has the good regularity with A_D but, if $|\alpha - 1| + \beta < 1$, $H \notin C^1(A_D)$. In the latter case, we cannot apply the Mourre theory with the generator of dilation. Here, we prove that, with a certain choice of u , $V \in C^{1,1}(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$ if $2\alpha + \beta > 3$ (see lemma 5.4). In that case, Theorem 1.2 applies. In particular, in the region $2\alpha + \beta > 3$ and $\alpha + \beta \leq 2$, we have the limiting absorption principle but H is not of class $C^1(A_D)$. In Section 5, we will see that Theorem 1.2 also applies if $\beta \leq 0$ under certain condition on α .

- (7) Let $\kappa \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $\kappa = 1$ on $[-1, 1]$ and $0 \leq \kappa \leq 1$. Let

$$V(x) = (1 - \kappa(|x|)) \exp(3|x|/4) \sin(\exp(|x|)).$$

We can show that, for all u bounded, $V \in C^\infty(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ and Theorem 1.2 applies (see Lemma 5.6). Moreover, this implies good regularity properties on the boundary values of the resolvent. Since V is not Δ -bounded, we cannot use the $C^1(A_D, \mathcal{H}^2, \mathcal{H}^{-2})$ (see [1, Theorem 6.3.4]).

We can also prove that $V \notin C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$. In particular, Theorem 1.2 does not apply with $A_u = A_D$.

- (8) If $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact and if there is $\mu > 0$ such that $x \mapsto \langle x \rangle^{1+\mu} V(x)$ is in \mathcal{H}^{-1} (V is assumed to be short range in a quite weak sense), then we can apply Theorem 1.2 with an appropriate u (see lemma 5.8). We will provide in this class an concrete example which cannot be treated with the generator of dilations or Nakamura's result (see lemma 5.10).

Now we will see a third result concerning existence of wave operators which are useful in scattering theory (see [20]).

DEFINITION 1.3. — A linear operator $S \in \mathcal{B}_1$ is *short range* if it is compact, symmetric, and

$$\int_1^\infty \|\xi(q/r)S\|_{\mathcal{B}_1} dr < \infty. \quad (1.10)$$

Remark that (1.2) is satisfied if

$$\int_1^\infty \|\xi(q/r)|q|V\|_{\mathcal{B}} \frac{dr}{r} < \infty \quad (1.11)$$

which is a short range type condition.

Note that we do not require S to be local. Clearly this condition requires less decay than the condition (1.11). Then we have:

THEOREM 1.4. — *Let H be as in Theorem 1.1 and let S be a short range operator. Then the self-adjoint operator $K = H + S$ has normal spectrum in $(0, \infty)$ and the wave operators*

$$\Omega_\pm = \text{s-lim}_{t \rightarrow \pm\infty} e^{itK} e^{-itH} E_H^c \quad (1.12)$$

exist and are complete, where E_H^c is the projection onto the continuity subspace of H .

Proof. — We have $K = \Delta + V + S$ and from [1, 7.5.8] it follows that S , hence $V + S$, is of class $C^{1,1}(A_N, \mathcal{H}^1, \mathcal{H}^{-1})$ for all $a > 0$ so that we can use Theorem 1.2 to deduce that K has normal spectrum in $(0, \infty)$ and that the boundary values of its resolvent exist as in the case of H . For the existence and completeness of the wave operators we use [1, Proposition 7.5.6] with the following change of notations: H_0, H, V from the quoted proposition are our H, K, S respectively. It remains only to check that S satisfies the last condition required on V in that proposition: but this is a consequence of [12, Theorem 2.14]. \square

We will give on Section 4 more explicit conditions which ensure that the assumptions of Theorem 1.1 and 1.4 are satisfied in the case where V and S are real functions.

We make two final remarks. First, the assumption of compactness of V and S as operators $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is too strong for some applications, for example it is not satisfied if $X = \mathbb{R}^3$ and $V(x)$ has local singularities of order $|x|^{-2}$. But compactness can be replaced by a notion of smallness at infinity similar to that used in [12] which covers such singularities and the arguments there extend to the present setting. Second, let us mention that we treat only the case when H_0 is the Laplacian $\Delta = p^2$ but an extension to more general functions $h(p)$ is straightforward with the same class of conjugate operator A_u .

The paper is organized as follows. In Section 2, we will give some notations we will use below and we recall some basic fact about regularity with respect to a conjugate operator. In Section 3, we will prove Theorem 1.2 and extend Nakamura's results by getting properties about the boundary values of the resolvent. In Section 4, we will give an extension of Nakamura's theorem by using the Mourre theory with $C^{1,1}$ regularity with respect to the conjugate operator A_N . In Section 5, we will give some examples of potentials which satisfies Theorem 1.1 and Theorem 1.2 and which are not covered by Mourre Theorem with the generator of dilation and Nakamura's Theorem. In Section 6, we will study the flow associated to the unitary group generated by A_u .

2. NOTATION AND BASIC NOTIONS

2.1. Notation. Let $X = \mathbb{R}^\nu$ and for $s \in \mathbb{R}$ let \mathcal{H}^s be the usual Sobolev spaces on X with $\mathcal{H}^0 = \mathcal{H} = L^2(X)$ whose norm is denoted $\|\cdot\|$. We are mainly interested in the space \mathcal{H}^1 defined by the norm $\|f\|_1^2 = \int (|f(x)|^2 + |\nabla f(x)|^2) dx$ and its dual space \mathcal{H}^{-1} .

Recall that we set $\mathcal{B}_1 = B(\mathcal{H}^1, \mathcal{H}^{-1})$ and $\mathcal{B}_2 = B(\mathcal{H}^2, \mathcal{H}^{-2})$ which are Banach spaces with norm $\|\cdot\|_{\mathcal{B}}$, $\mathcal{B} = \mathcal{B}_1, \mathcal{B}_2$. These spaces satisfy $\mathcal{B}_1 \subset \mathcal{B}_2$.

We denote q_j the operator of multiplication by the coordinate x_j and $p_j = -i\partial_j$ considered as operators in \mathcal{H} . For $k \in X$ we denote $k \cdot q = k_1 q_1 + \dots + k_\nu q_\nu$. If u is a measurable function on X let $u(q)$ be the operator of multiplication by u in \mathcal{H} and $u(p) = \mathcal{F}^{-1}u(q)\mathcal{F}$, where \mathcal{F} is the Fourier transformation:

$$(\mathcal{F}f)(\xi) = (2\pi)^{-\frac{\nu}{2}} \int e^{-ix \cdot \xi} u(x) dx.$$

If there is no ambiguity we keep the same notation for these operators when considered as acting in other spaces.

Throughout this paper $\xi \in C^\infty(X)$ is a real function such that $\xi(x) = 0$ if $|x| < 1$ and $\xi(x) = 1$ if $|x| > 2$. Clearly the operator $\xi(q)$ acts continuously in all the spaces \mathcal{H}^s .

We are mainly interested in potentials V which are multiplication operators in the following more general sense.

DEFINITION 2.1. — A map $V \in \mathcal{B}$ is called a *multiplication operator* if $Ve^{ik \cdot q} = e^{ik \cdot q}V$ for all $k \in X$. Or, equivalently, if $V\theta(q) = \theta(q)V$ for all $\theta \in C_c^\infty(X)$.

For the proof of the equivalence, note first that from $Ve^{ik \cdot q} = e^{ik \cdot q}V, \forall k \in X$ we get $V\theta(q) = \theta(q)V$ for any Schwartz test function θ because $(2\pi)^{\frac{\nu}{2}}\theta(q) = \int e^{ikq}(\mathcal{F}\theta)(k)dk$ and second that if $\eta \in C^1(X)$ is bounded with bounded derivative then $\eta(q)$ is the strong limit in \mathcal{B} of a sequence of operators $\theta(q)$ with $\theta \in C_c^\infty(X)$.

As we mentioned in the introduction, such a V is necessarily the operator of multiplication by a distribution that we also denote V and we sometimes write the associated operator $V(q)$. For example, the distribution V could be the divergence $\operatorname{div} W$ of a measurable vector field $W : X \rightarrow X$ such that multiplication by the components of W sends \mathcal{H}^1 into \mathcal{H} . For example, W could be a bounded function and if this function tends to zero at infinity then V will be a compact operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$. We say that a multiplication operator V is Δ -compact if $V : \mathcal{H}^2 \rightarrow L^2$ is a compact operator.

As usual $\langle x \rangle = \sqrt{1 + |x|^2}$. Then $\langle q \rangle$ is the operator of multiplication by the function $x \mapsto \langle x \rangle$ and $\langle p \rangle = \mathcal{F}^{-1}\langle q \rangle\mathcal{F}$. For real s, t we denote \mathcal{H}_s^t the space defined by the norm

$$\|f\|_{\mathcal{H}_s^t} = \|\langle q \rangle^s f\|_{\mathcal{H}^t} = \|\langle p \rangle^t \langle q \rangle^s f\| = \|\langle q \rangle^s \langle p \rangle^t f\|. \quad (2.1)$$

Note that the adjoint space of \mathcal{H}_s^t may be identified with \mathcal{H}_{-s}^{-t} .

A finer Besov type version $\mathcal{H}_{1/2,1}^{-1}$ of $\mathcal{H}_{1/2}^{-1}$ appears naturally in the theory. To alleviate the writing we denote it \mathcal{K} . This space is defined by the norm

$$\|f\|_{\mathcal{K}} = \|\theta(q)f\|_{\mathcal{H}^{-1}} + \int_1^\infty \|\tau^{1/2}\psi(q/\tau)f\|_{\mathcal{H}^{-1}} \frac{d\tau}{\tau} \quad (2.2)$$

where $\theta, \psi \in C_c^\infty(X)$ with $\theta(x) = 1$ if $|x| < 1$, $\psi(x) = 0$ if $|x| < 1/2$, and $\psi(x) = 1$ if $1 < |x| < 2$. The adjoint space \mathcal{K}^* of \mathcal{K} is the Besov space $\mathcal{H}_{-1/2,\infty}^1$ (see [1, Chapter 4]).

We will see in Section 6 that if $u : X \rightarrow X$ is a C^∞ vector field all of whose derivatives are bounded then the operator

$$A_u = \frac{1}{2}(u(p) \cdot q + q \cdot u(p)) = \frac{1}{2} \sum_{j=1}^{\nu} (u_j(p)q_j + q_j u_j(p)) = u(p) \cdot q + \frac{i}{2}(\operatorname{div} u)(p) \quad (2.3)$$

with domain $C_c^\infty(X)$ is essentially self-adjoint in \mathcal{H} ; we keep the notation A_u for its closure. Remark that the unitary group $e^{i\tau A_u}$ generated by A_u leaves invariant all the spaces \mathcal{H}_t^s and \mathcal{K} (see Section 6).

Since we will use a lot the case of u bounded, let \mathcal{U} be the space of vector fields u bounded with all derivatives bounded such that $x \cdot u(x) > 0$ for all $x \neq 0$.

2.2. Regularity. Let F', F'' be two Banach space and $T : F' \rightarrow F''$ a bounded operator.

Let A a self-adjoint operator.

Let $k \in \mathbb{N}$. we say that $T \in C^k(A, F', F'')$ if, for all $f \in F'$, the map $\mathbb{R} \ni t \rightarrow e^{itA} T e^{-itA} f$ has the usual C^k regularity. The following characterisation is available:

PROPOSITION 2.2. — $T \in C^1(A, F', F'')$ if and only if $[T, A]$ has an extension in $\mathcal{B}(F', F'')$.

It follows that, for $k > 1$, $T \in C^k(A, F', F'')$ if and only if $[T, A] \in C^{k-1}(A, F', F'')$. We can define another class of regularity called the $C^{1,1}$ regularity:

PROPOSITION 2.3. — we say that $T \in C^{1,1}(A, F', F'')$ if and only if

$$\int_0^1 \|T_\tau + T_{-\tau} - 2T\|_{\mathcal{B}(F', F'')} \frac{d\tau}{\tau^2} < \infty,$$

where $T_\tau = e^{i\tau A_u} T e^{-i\tau A_u}$.

An easier result can be used:

PROPOSITION 2.4 (Proposition 7.5.7 from [1]). — Let $\xi \in C^\infty(X)$ such that $\xi(x) = 0$ if $|x| < 1$ and $\xi(x) = 1$ if $|x| > 2$. If T satisfies

$$\int_1^\infty \|\xi(q/r)[T, iA]\|_{\mathcal{B}(F', F'')} \frac{dr}{r} < \infty$$

then T is of class $C^{1,1}(A, F', F'')$.

If T is not bounded, we say that $T \in C^k(A, F', F'')$ if for $z \notin \sigma(T)$ we have $(T - z)^{-1} \in C^k(A, F', F'')$.

PROPOSITION 2.5. — For all $k > 1$, we have

$$C^k(A, F', F'') \subset C^{1,1}(A, F', F'') \subset C^1(A, F', F'').$$

If $F' = F'' = \mathcal{H}$ is an Hilbert space, we note $C^1(A) = C^1(A, \mathcal{H}, \mathcal{H}^*)$. If T is self-adjoint, we have the following:

THEOREM 2.6 (Theorem 6.3.4 from [1]). — Let A and T be self-adjoint operator in a Hilbert space \mathcal{H} . Assume that the unitary group $\{\exp(iA\tau)\}_{\tau \in \mathbb{R}}$ leaves the domain $D(T)$ of T invariant. Set $\mathcal{G} = D(T)$ endowed with its graph topology. Then

- (1) T is of class $C^1(A)$ if and only if $T \in C^1(A, \mathcal{G}, \mathcal{G}^*)$.
- (2) T is of class $C^{1,1}(A)$ if and only if $T \in C^{1,1}(A, \mathcal{G}, \mathcal{G}^*)$.

Remark that, if $T : \mathcal{H} \rightarrow \mathcal{H}$ is not bounded, since $T : \mathcal{G} \rightarrow \mathcal{G}^*$ is bounded, in general, it is easier to prove that $T \in C^1(A, \mathcal{G}, \mathcal{G}^*)$ than $T \in C^1(A)$.

If \mathcal{G} is the form domain of H , we have the following:

PROPOSITION 2.7 (see p. 258 of [1]). — *Let A and T be self-adjoint operators in a Hilbert space \mathcal{H} . Assume that the unitary group $\{\exp(iA\tau)\}_{\tau \in \mathbb{R}}$ leaves the form domain \mathcal{G} of T invariant. Then*

- (1) T is of class $C^k(A)$ if and only if $T \in C^k(A, \mathcal{G}, \mathcal{G}^*)$, for all $k \in \mathbb{N}$.
- (2) T is of class $C^{1,1}(A)$ if and only if $T \in C^{1,1}(A, \mathcal{G}, \mathcal{G}^*)$.

As previously, since $T : \mathcal{G} \rightarrow \mathcal{G}^*$ is always bounded, it is, in general, easier to prove that $T \in C^k(A, \mathcal{G}, \mathcal{G}^*)$ than $T \in C^k(A)$.

3. NAKAMURA'S IDEAS IN A MORE GENERAL SETTING: BOUNDARY VALUES OF THE RESOLVENT

In this section, we will prove the Theorem 1.2 and we will see how the regularity of the potential, in relation to A_u , can implies a good regularity for the boundary values of the resolvent.

Proof of Theorem 1.2. — By taking into account the equation (1.6) and the statement of Theorem 7.5.6 in [1] we only have to explain how the space \mathcal{K} introduced before (2.2) appears into the picture (this is not the space also denoted \mathcal{K} in [1]). In fact the quoted theorem gives a more precise result, namely instead of our \mathcal{K} one may take the real interpolation space $(D(A_u, \mathcal{H}^{-1}), \mathcal{H}^{-1})_{1/2,1}$, where $D(A_u, \mathcal{H}^{-1})$ is the domain of the closure of A_u in \mathcal{H}^{-1} . From (2.3) and since u is bounded with all its derivatives bounded it follows immediately that $D(A_u, \mathcal{H}^{-1})$ contains the domain of $\langle q \rangle$ in \mathcal{H}^{-1} , which is \mathcal{H}_1^{-1} . Hence $(D(A_u, \mathcal{H}^{-1}), \mathcal{H}^{-1})_{1/2,1}$ contains $(\mathcal{H}_1^{-1}, \mathcal{H}^{-1})_{1/2,1}$ which is $\mathcal{H}_{1/2,1}^{-1} = \mathcal{K}$. \square

We say that V is of class $C^k(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ for some integer $k \geq 1$ if the map $\tau \mapsto V_\tau \in B(\mathcal{H}^1, \mathcal{H}^{-1})$ is k times strongly differentiable. We clearly have $C^2(A_u, \mathcal{H}^1, \mathcal{H}^{-1}) \subset C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. In [14] the Limiting Absorption Principle is proved essentially for $V \in C^2(A)$ (see [11] for more details); notice that in [15] the limiting absorption principle is proved in a space better (i.e larger) than \mathcal{K} (see (2.1)), but not of Besov type.

If $s > 1/2$ then $\mathcal{H}_s^{-1} \subset \mathcal{K}$ with a continuous and dense embedding. Hence:

COROLLARY 3.1. — *For each $s > 1/2$ the limit $R(\lambda \pm i0) = \text{w}^*\text{-}\lim_{\mu \downarrow 0} R(\lambda \pm i\mu)$ exists in the spaces $B(\mathcal{H}_s^{-1}, \mathcal{H}_{-s}^1)$, locally uniformly in $\lambda \in J$ outside the eigenvalues of H .*

The $C^{1,1}(A_u)$ regularity condition (1.7) on V is not explicit enough for some applications. We now give a simpler condition which ensures that (1.7) is satisfied.

We recall some easily proven facts concerning the class $C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. First, V is of class $C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ if and only if the function $\tau \mapsto V_\tau \in B(\mathcal{H}^1, \mathcal{H}^{-1})$ is (norm or strongly) Lipschitz. Notice that we used $e^{i\tau A} \mathcal{H}^1 \subset \mathcal{H}^1$ to prove this. Second, note that for an arbitrary V the expression $[V, iA_u]$ is well-defined as symmetric sesquilinear form on $C_c^\infty(X)$ and V is of class $C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ if and only if this form is continuous for the topology induced by \mathcal{H}^1 . In this case we keep the notation $[V, iA_u]$ for its continuous extension to \mathcal{H}^1 and for the continuous symmetric operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ associated to it.

As a consequence of Proposition 7.5.7 from [1] with the choice $\Lambda = \langle q \rangle$ we get:

PROPOSITION 3.2. — Let $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ be a symmetric bounded operator of class $C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ such that

$$\int_1^\infty \|\xi(q/r)[V, iA_u]\|_{\mathcal{B}} \frac{dr}{r} < \infty \quad (3.1)$$

with $\mathcal{B} = \mathcal{B}_1$. Then V is of class $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

If the potential V is of a higher regularity class with respect to A_u then, by using results from [3], we also get an optimal result on the order of continuity of the boundary values of the resolvent $R(\lambda \pm i0)$ as functions of λ . From [15] and the improvements in [4] one may also get a precise description of the propagation properties of the dynamical group e^{itH} in this context, but we shall not give the details here.

To state this regularity result we recall the definition of the *Hölder-Zygmund continuity classes* of order $s \in (0, \infty)$. Let \mathcal{E} be a Banach space and $F : \mathbb{R} \rightarrow \mathcal{E}$ a continuous function. If $0 < s < 1$ then F is of class Λ^s if F is Hölder continuous of order s . If $s = 1$ then F is of class Λ^1 if it is of Zygmund class, i.e. $\|F(t + \varepsilon) + F(t - \varepsilon) - 2F(t)\| \leq C\varepsilon$ for all real t and $\varepsilon > 0$. If $s > 1$, let us write $s = k + \sigma$ with $k \geq 1$ integer and $0 < \sigma \leq 1$; then F is of class Λ^s if F is k times continuously differentiable and $F^{(k)}$ is of class Λ^σ . The corresponding local classes are defined as follows: if F is defined on an open real set U then F is locally of class Λ^s if θF is of class Λ^s for any $\theta \in C_c^\infty(U)$.

We say that V is of class $\Lambda^s(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ if the function $\tau \mapsto V_\tau \in \mathcal{B}_1$ is of class Λ^s . We mention that in a more general context this class is denoted by $C^{s,\infty}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$, but this does not matter here. In any case, one may easily check that

$$\Lambda^s(A_u, \mathcal{H}^1, \mathcal{H}^{-1}) \subset C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1}) \subset C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$$

if $s > 1$. If $s \geq 1$ is an integer then $C^s(A_u, \mathcal{H}^1, \mathcal{H}^{-1}) \subset \Lambda^s(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ strictly.

THEOREM 3.3. — Assume that u and J are as in Theorem 1.2 and let s be a real number such that $s > 1/2$. If $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is a compact symmetric operator of class $\Lambda^{s+1/2}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ then the functions

$$\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s^{-1}, \mathcal{H}_{-s}^1) \quad (3.2)$$

are locally of class $\Lambda^{s-1/2}$ on J outside the eigenvalues of H .

Proof. — We shall deduce this from the theorem on page 12 of [3]. First, note that H has a spectral gap because $H_0 \geq 0$ and $(H+i)^{-1} - (H_0+i)^{-1}$ is a compact operator hence H and H_0 have the same essential spectrum. Thus we may use the quoted theorem and we get the assertion of the present theorem but with $B(\mathcal{H}_s^{-1}, \mathcal{H}_{-s}^1)$ replaced by $B(\mathcal{H}_s, \mathcal{H}_{-s})$. Then it suffices to observe that if z belongs to the resolvent set of H then we have

$$R(z) = R(i) + (z - i)R(i)^2 + (z - i)^2R(i)R(z)R(i)$$

and to note that $R(i)$ sends \mathcal{H}_s^{-1} into \mathcal{H}_s^1 . \square

We state explicitly the particular case corresponding to the Mourre condition $V \in C^2(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. We mention that this is equivalent to the fact that the sesquilinear form $[[V, A_u], A_u]$, which is always well-defined on $C_c^\infty(X)$, extends to a continuous sesquilinear form on \mathcal{H}^1 .

COROLLARY 3.4. — Assume that we are in the conditions of the Theorem 1.2 but with the condition (1.7) replaced by the stronger one $V \in C^2(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. Then the map (3.2) is Hölder continuous of order $s - 1/2$ for all s such that $1/2 < s < 3/2$ and if $s = 3/2$ then the map (3.2) is of Zygmund class (but could be nowhere differentiable).

Previously, we saw that H_0 verified the Mourre estimate on I if and only if $k \cdot u(k) > 0$ for all k such that $|k|^2 \in I$. Moreover, we saw that if H_0 verified the Mourre estimate on I , because the potential V is H_0 -compact or compact on \mathcal{H}^1 , the form domain of H_0 , to \mathcal{H}^{-1} , H verified the Mourre estimate on the same interval I .

Because

$$u_N(x) = (\sin(ax_j))_{j=1, \dots, \nu}$$

we have the Mourre estimate only on $I_a = (0, (\pi/a)^2)$, i.e. where $u_N(x) \neq 0$; this function constructs some artificial thresholds. If we can choose a vector field u such that $x \cdot u(x) > 0$ if $x \neq 0$ which satisfied some good conditions of regularity for the potential V , we can extend the interval I_a to $I = (0, +\infty)$. For example, we can choose the vector field $u(x) = (\arctan(x_j))_{j=1, \dots, \nu}$, the function \arctan being non zero for $x \neq 0$, or $u(x) = x/\langle x \rangle$. In particular, with this type of vector field, if $\langle p \rangle^{-1} q V \langle p \rangle^{-1}$ is bounded, then $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ and we have the Mourre estimate on all compact subset of $(0, +\infty)$.

4. AN EXTENSION OF NAKAMURA'S RESULTS

In this section, we will prove Theorem 1.1 and we will give some conditions easy to verify which assure that assumptions of Theorem 1.1 and 1.4 are satisfied. Moreover, we will give a stronger version of Nakamura's Theorem with estimates on the boundary values of the resolvent.

In all this section, $\mathcal{B} = \mathcal{B}_1 = B(\mathcal{H}^1, \mathcal{H}^{-1})$.

We fix a real number $a > 0$ and denote $I_a = (0, \frac{\pi^2}{a^2})$. We apply the general results from Section 3 with the vector field u as in [16]:

$$u_N(p) = (\sin(ap_1), \dots, \sin(ap_\nu)). \quad (4.1)$$

Then $(\operatorname{div} u_N)(p) = \sum_j \partial_{p_j} \sin(ap_j) = \sum_j a \cos(ap_j)$ hence

$$2A_N = \sum_j (q_j \sin(ap_j) + \sin(ap_j) q_j) = \sum_j (2q_j \sin(ap_j) - ia \cos(ap_j)). \quad (4.2)$$

The operator A_N behaves well with respect to the tensor factorization $L^2(X) = L^2(\mathbb{R})^{\otimes \nu}$ and this simplifies the computations. Indeed, if we denote B the operator A acting in $L^2(\mathbb{R})$ we have $A_N = A_1 + \dots + A_\nu$ with $A_1 = B \otimes 1 \dots \otimes 1$, $A_2 = 1 \otimes B \otimes 1 \dots \otimes 1$, etc.

Let $T_j = e^{ia p_j}$ be the operator of translation by a in the j direction, i.e. $(T_j f)(x) = f(x + a e_j)$ where e_1, \dots, e_ν is the natural basis of $X = \mathbb{R}^\nu$. For any $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ set

$$\delta_j(V) = T_j V T_j^* - V \quad (4.3)$$

which is also an operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ hence we may consider $\delta_k \delta_j(V)$, etc. If $V = V(q)$ is a multiplication operator then

$$\delta_j(V) = V(q + a e_j) - V(q).$$

Remark that, when V is a multiplication operator, $\delta_j(V)$ appears in the first commutator $[V, iA_N]$. The operation δ_j can also be applied to various unbounded operators, for example we obviously have $\delta_j(q_k) = a \delta_{jk}$, where δ_{jk} is the Kronecker symbol, and $\delta_j(u(p)) = 0$.

If $S \in \mathcal{B}$ then $[q_j, S]$ and $q_j S$ are well-defined as sesquilinear forms on $C_c^\infty(X)$ and we say that one of these expressions is a bounded operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ if the corresponding form is continuous in the topology induced by \mathcal{H}^1 .

THEOREM 4.1. — *Let $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ be a compact symmetric operator such that for any j the forms $[q_j, V]$ and $q_j \delta_j(V)$ are bounded operators $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ and*

$$\int_1^\infty \left(\|\xi(q/r)[q_j, V]\|_{\mathcal{B}} + \|\xi(q/r)q_j \delta_j(V)\|_{\mathcal{B}} \right) \frac{dr}{r} < \infty. \quad (4.4)$$

Then H has normal spectrum in I_a and the limits $R(\lambda \pm i0) = \text{w}^\text{-}\lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon)$ exist in $B(\mathcal{K}, \mathcal{K}^*)$, locally uniformly in $\lambda \in I_a$ outside the set of eigenvalues of H .*

Remark 4.2. — This Theorem give a stronger result than Theorem 1.1, since, if V is a multiplication operator, $[q_j, V] = 0$ and (4.4) reduces to (1.2).

Proof. — This is a consequence of Theorem 1.2 and Proposition 3.2 once we have checked that V is of class $C^1(A_N)$ and the relation (3.1) is satisfied. In order to prove that $V \in C^1(A_N, \mathcal{H}^1, \mathcal{H}^{-1})$ it suffices to show that the sesquilinear form on $C_c^\infty(X)$ defined by

$$[2A_N, V] = \sum_j (2[q_j \sin(ap_j), V] - ia[\cos(ap_j), V]) \quad (4.5)$$

is continuous for the \mathcal{H}^1 topology. This is clear for the second term in the sum and for the first one we use

$$[q_j \sin(ap_j), V] = [q_j, V] \sin(ap_j) + q_j [\sin(ap_j), V]. \quad (4.6)$$

The first term on the right hand side defines a bounded operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ by one of the hypotheses of the theorem. For the second one we first note that

$$[T_j, V] = \delta_j(V)T_j \quad \text{and} \quad [T_j^*, V] = -T_j^* \delta_j(V) \quad (4.7)$$

from which we get

$$2i[\sin(ap_j), V] = [T_j - T_j^*, V] = \delta_j(V)T_j + T_j^* \delta_j(V). \quad (4.8)$$

It follows easily that $q_j [\sin(ap_j), V]$ is bounded. Thus V is of class $C^1(A_N, \mathcal{H}^1, \mathcal{H}^{-1})$.

It remains to show that (3.1) is satisfied. We use (4.5) again: the terms with $\sin(ap_j)$ are treated with the help of (4.6) and (4.8). The term with $\cos(ap_j)$ is treated similarly by using

$$2[\cos(ap_j), V] = [T_j + T_j^*, V] = \delta_j(V)T_j - T_j^* \delta_j(V). \quad (4.9)$$

Using (4.4), this proves (3.1). \square

Remark 4.3. — The “usual” version of the preceding theorem involves the derivatives $[p_j, V]$ of the potential, instead of the finite differences $\delta_j(V)$, cf. [1, Theorem 7.6.8]. Note that the quoted theorem is a consequence of Theorem 4.1 because $\|\delta_j(V)\|_{\mathcal{B}} \leq a\|[p_j, V]\|_{\mathcal{B}}$.

The condition (4.4) says that the operators $[q_j, V]$ and $q_j \delta_j(V)$ are not only bounded as maps $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ but also tend to zero at infinity in some weak sense. Then it is clear that the maps $\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{K}, \mathcal{K}^*)$ are strongly continuous outside the eigenvalues of H , but nothing else can be said in general. Stronger conditions on this decay improve the smoothness properties of the boundary values $R(\lambda \pm i0)$ as maps $\mathcal{H}_s^1 \rightarrow \mathcal{H}_{-s}^{-1}$. This question is solved in general by using Theorem 3.3 but here we consider only a particular case as an example. One may see in [12, Theorem 1.7] the type of assumptions V has to satisfy in order to improve the smoothness properties of the boundary values.

Remark that if V is a multiplication operator, we have

$$\delta_j \delta_k(V) = V(q + ae_j + ae_k) - V(q + ae_j) - V(q + ae_k) + V(q)$$

which appears in the second commutator $[[V, iA_N], iA_N]$.

The next result is an extension of [16, Theorem 1]: we make the regularity assumption $V \in C^2(A_N)$ but V is not necessarily an operator in \mathcal{H} and we give the precise Hölder continuity order of the boundary values.

THEOREM 4.4. — *Let $V = V(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ be a symmetric compact multiplication operator. Assume that there is a real number $a > 0$ such that for all j, k*

$$(H) \quad \begin{cases} q_j(V(q + ae_j) - V(q)) \text{ and} \\ q_j q_k (V(q + ae_j + ae_k) - V(q + ae_k) - V(q + ae_k) + V(q)) \end{cases}$$

are bounded operators $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$. Then H has normal spectrum in the interval I_a and the limits $R(\lambda \pm i0) = \text{w}^*\text{-}\lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon)$ exist in $B(\mathcal{K}, \mathcal{K}^*)$, locally uniformly in $\lambda \in I_a$ outside the eigenvalues of H . If $\frac{1}{2} < s < \frac{3}{2}$ then the operators $R(\lambda \pm i0) \in B(\mathcal{H}_s^{-1}, \mathcal{H}_{-s}^1)$ are locally Hölder continuous functions of order $s - \frac{1}{2}$ of the parameter $\lambda \in I_a$ outside the eigenvalues of H .

Proof. — We first show that (1.8) is satisfied for any open interval J whose closure is included in I_a , i.e. $\inf\{k \cdot u_N(k) \mid k \in X, |k|^2 \in J\} > 0$. Since $k \mapsto k \cdot u_N(k)$ is a continuous function and \bar{J} is a compact in I_a , it suffices to check that $ak \cdot u_N(k) = \sum ak_j \sin(ak_j) > 0$ for all k such that $|k|^2 \in I_a$. The last condition may be written $0 < |ak| < \pi$ and this implies $|ak_j| < \pi$ for all j and $|ak_j| > 0$ for at least one j . Clearly then we get $ak \cdot u_N(k) > 0$.

For the rest of the proof it suffices to check that V is of class $C^2(A_N, \mathcal{H}^1, \mathcal{H}^{-1})$. Indeed, then we may use Theorems 1.2, Corollary 3.1, and Theorem 3.3 (see also Corollary 3.4).

Thus we have to prove that the commutators $[A_N, V]$ and $[A_N, [A_N, V]]$, which are a priori defined as sesquilinear forms on $C_c^\infty(X)$, extend to continuous forms on \mathcal{H}^1 . Although the computations are very simple, we give the details for the convenience of the reader.

We have $[A_N, V] = \sum [A_j, V]$ and $[A_N, [A_N, V]] = \sum [A_j, [A_k, V]]$ and we recall the relations (4.7). Since $\sin(ap_j) = \frac{1}{2i}(T_j - T_j^*)$ and $\cos(ap_j) = \frac{1}{2}(T_j + T_j^*)$ we have

$$2iA_j = q_j 2i \sin(ap_j) + a \cos(ap_j) = q_j(T_j - T_j^*) + b(T_j + T_j^*) \quad (4.10)$$

where $b = a/2$. Thus by using the relations $[T_j, V] = \delta_j(V)T_j$ and $[T_j^*, V] = -T_j^* \delta_j(V)$ and since $T_j q_j T_j^* = q_j + a$, we get

$$\begin{aligned} [2iA_j, V] &= q_j [T_j - T_j^*, V] + b [T_j + T_j^*, V] \\ &= (b + q_j) \delta_j(V) T_j + T_j^* (-b + q_j) \delta_j(V) \end{aligned} \quad (4.11)$$

Because $T_j = e^{iap_j}$ and $T_j^* = e^{-iap_j}$ are bounded, by the assumption (H) the right hand side of this relation is a bounded operator from \mathcal{H}^1 to \mathcal{H}^{-1} , hence V is of class $C^1(A_N, \mathcal{H}^1, \mathcal{H}^{-1})$. It remains to treat the second order commutators.

Since $[iA_N, S^*] = [iA_N, S]^*$, we have

$$\begin{aligned} [iA_j, [iA_k, V]] &= [iA_j, (b + q_k) \delta_k(V) T_k] + [iA_j, T_k^* (-b + q_k) \delta_k(V)] \\ &= [iA_j, (b + q_k) \delta_k(V)] T_k + (b + q_k) \delta_k(V) [iA_j, T_k] \\ &\quad + [iA_j, T_k]^* (-b + q_k) \delta_k(V) + T_k^* [iA_j, (-b + q_k) \delta_k(V)]. \end{aligned} \quad (4.12)$$

Since we have $[iA_j, T_k] = b\delta_{jk}(1 - T_k^2)$, we get

$$\begin{aligned} [iA_j, (b + q_k)\delta_k(V)]T_k + (b + q_k)\delta_k(V)[iA_j, T_k] &= [iA_j, (b + q_k)\delta_k(V)]T_k \\ &\quad + b\delta_{jk}(b + q_k)\delta_k(V)(1 - T_k^2). \end{aligned} \quad (4.13)$$

The last term here is again a bounded operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ by assumption **(H)**, hence it remains to prove that the first term of the right hand side has the same property. For this we use (4.11) with $(b + q_k)\delta_k(V)$ instead of V and we get:

$$\begin{aligned} [iA_j, (b + q_k)\delta_k(V)] &= (b + q_j)\delta_j((b + q_k)\delta_k(V))T_j \\ &\quad + T_j^*(-b + q_j)\delta_j((b + q_k)\delta_k(V)). \end{aligned} \quad (4.14)$$

Since $\delta_j(MN) = \delta_j(M)T_jNT_j^* + M\delta_j(N)$ we have

$$\begin{aligned} \delta_j((b + q_k)\delta_k(V)) &= \delta_j(b + q_k)T_j\delta_k(V)T_j^* + (b + q_k)\delta_j\delta_k(V) \\ &= a\delta_{jk}T_j\delta_j(V)T_j^* + (b + q_k)\delta_j\delta_k(V). \end{aligned}$$

Since $(b + q_j)T_j = T_j(q_j - b)$ we then get

$$(b + q_j)\delta_j((b + q_k)\delta_k(V)) = a\delta_{jk}T_j(q_j - b)\delta_j(V)T_j^* + (b + q_j)(b + q_k)\delta_j\delta_k(V)$$

which is bounded as operator $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ by **(H)**, hence the right hand side of (4.14) has the same property.

Using the same argument, we can prove that

$$[iA_j, T_k]^*(-b + q_k)\delta_k(V) + T_k^*[iA_j, (-b + q_k)\delta_k(V)]$$

is bounded.

By (4.12), $[iA_j, [iA_k, V]]$ is bounded; we deduce that V is of class $C^2(A_N, \mathcal{H}^1, \mathcal{H}^{-1})$. \square

Remark 4.5. — This Theorem is a stronger version of Nakamura's result. In fact, in Nakamura's paper, V is a multiplication operator with the Δ -compact property (compact from \mathcal{H}^2 to L^2) which is a stronger assumption than compact from \mathcal{H}^1 to \mathcal{H}^{-1} . Moreover, we add in Theorem 4.4 a result concerning the regularity of the boundary values of the resolvent.

Now we assume that V and S are real functions and we give more explicit conditions which ensure that the assumptions of the Theorems 1.1 and 1.4 are satisfied. Let $p = 1$ if $\nu = 1$, any $p > 1$ if $\nu = 2$, and $p = \nu/2$ if $\nu \geq 3$ and denote

$$\llbracket f \rrbracket_p^x = \left(\int_{|y-x|<1} |f(y)|^p dy \right)^{1/p}.$$

PROPOSITION 4.6. — *Consider V and S multiplication operators such that:*

- $V \in L_{\text{loc}}^p(X)$ satisfies $\lim_{x \rightarrow \infty} \llbracket V \rrbracket_p^x = 0$ and for any $a \in X$ and any $r > 1$,

$$\int_1^\infty \varphi_a(r) \frac{dr}{r} < \infty$$

where

$$\varphi_a(r) = \sup_{|x|>r} \left\{ |x| \llbracket V(\cdot + a) - V(\cdot) \rrbracket_p^x \right\};$$

- $S \in L_{\text{loc}}^p(X)$ and $\int_1^\infty \sup_{|x|>r} \llbracket S \rrbracket_p^x dr < \infty$.

Then all the conditions of Theorems 1.1 and 1.4 are satisfied.

Proof. — If $U : \mathcal{H}^1 \rightarrow \mathcal{H}$ is a local operator then it is easy to see that there is $C' \in \mathbb{R}$ such that

$$\|U\|_{\mathcal{H}^1 \rightarrow \mathcal{H}} \leq C' \sup\{\|Uf\| \mid f \in \mathcal{H}^1 \text{ with } \|f\|_{\mathcal{H}^1} \leq 1 \text{ and } \text{diam supp } f \leq 1\}.$$

If V is a function and $\nu \geq 3$ then the Sobolev inequality gives a number C'' such that

$$|\langle f|Vf \rangle| \leq \| |V|^{1/2} f \|^2 \leq \|V\|_{L^{\frac{\nu}{2}}} \|f\|_{L^{\frac{2\nu}{\nu-2}}}^2 \leq C'' \|V\|_{L^{\frac{\nu}{2}}} \|f\|_{\mathcal{H}^1}^2.$$

If $\nu = 1, 2$ then the argument is simpler but $\nu/2$ has to be replaced by 1 or any $p > 1$ respectively. Thus, if we introduce the notation

$$\llbracket V \rrbracket_p = \sup_x \left(\int_{|y-x|<1} |V(y)|^p dy \right)^{1/p} \quad (4.15)$$

with $p = 1$ if $\nu = 1$, any $p > 1$ if $\nu = 2$, and $p = \nu/2$ if $\nu \geq 3$, we get the following estimate: there is a number $C = C(\nu, p)$ such that

$$\|V\|_{\mathcal{B}} \leq C \llbracket V \rrbracket_p. \quad (4.16)$$

Clearly that $C_c^\infty(X)$ is dense for the norm $\llbracket \cdot \rrbracket_p$ in the space of functions V with finite $\llbracket \cdot \rrbracket_p$ norm and such that $\int_{|y-x|<1} |V(y)|^p dy \rightarrow 0$ as $x \rightarrow \infty$. Thus for such functions the operator $V(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact.

Suppose that, $\varphi_a(r) = \sup_{|x|>r} \{ |x| \llbracket V(\cdot + a) - V(\cdot) \rrbracket_p^x \}$ verifies

$$\int_1^\infty \varphi_a(r) \frac{dr}{r} < \infty.$$

We will prove that V verifies (1.2).

Because $\xi(x) = 0$ if $\|x\| \leq 1$, and according to (4.16), we have

$$\|\xi(q/r)|q|(V(q+ae) - V(q))\|_{\mathcal{B}} \leq C \sup_{|x|>r} \llbracket \cdot | (V(\cdot + a) - V(\cdot)) \rrbracket_p^x. \quad (4.17)$$

By definition, we have

$$\llbracket \cdot | (V(\cdot + a) - V(\cdot)) \rrbracket_p^x = \left(\int_{|y-x|<1} |y|^p |V(y+a) - V(y)|^p dy \right)^{1/p}. \quad (4.18)$$

Because $p \geq 1$ we have $|y|^p \leq (|y-x| + |x|)^p$. Moreover, using the convexity of the function $x \mapsto x^p$ on \mathbb{R}^+ ,

$$\begin{aligned} (|y-x| + |x|)^p &= 2^p \left(\frac{|y-x| + |x|}{2} \right)^p \\ &\leq 2^{p-1} (|y-x|^p + |x|^p). \end{aligned}$$

So from (4.18), we have

$$\llbracket \cdot | (V(\cdot + a) - V(\cdot)) \rrbracket_p^x \leq C_2 (1 + |x|^p)^{1/p} \llbracket (V(\cdot + a) - V(\cdot)) \rrbracket_p^x \quad (4.19)$$

where C_i are constants independent of x

By hypothesis on V , we have the following

$$\|\xi(q/r)|q|(V(q+ae) - V(q))\|_{\mathcal{B}} \leq C_3 \sup_{|x|>r} \frac{(1 + |x|^p)^{1/p}}{|x|} \varphi_a(r) \leq C_4 \varphi_a(r) \quad (4.20)$$

and then

$$\int_1^\infty \|\xi(q/r)q|(V(q+ae) - V(q))\|_{\mathcal{B}} \frac{dr}{r} < C_4 \int_1^\infty \varphi_a(r) \frac{dr}{r} < \infty. \quad (4.21) \quad \square$$

A class of potentials that we may consider is the *Kato class* whose definition is as follows [7, Sec. 1.2]. A measurable function $V : \mathbb{R}^\nu \rightarrow \mathbb{R}$ is of class K_ν if

- $\lim_{\alpha \downarrow 0} \sup_x \int_{|y-x| < \alpha} |y-x|^{2-\nu} |V(y)| dy = 0$ in case $\nu > 2$,
- $\lim_{\alpha \downarrow 0} \sup_x \int_{|y-x| < \alpha} \ln |y-x|^{-1} |V(y)| dy = 0$ in case $\nu = 2$,
- $\lim_{\alpha \downarrow 0} \sup_x \int_{|y-x| < \alpha} |V(y)| dy = 0$ in case $\nu = 1$.

The K_ν norm of such a function is given by

$$\|V\|_{K_\nu} = \sup_x \int_{|y-x| < 1} L_\nu(y-x) |V(y)| dy$$

with the obvious definition of L_ν . Note that the operator $V(q)$ is form relatively bounded with respect to the Laplacian with relative bound zero if $V \in K_\nu$ [7, p. 8] hence $H = \Delta + V(q)$ is a well-defined self-adjoint and bounded from below operator.

PROPOSITION 4.7. — *Let V be a real function on \mathbb{R}^ν , with $\nu \neq 2$, such that there is $\mu > 1$ and $(\langle \cdot \rangle^\mu V(\cdot))^p \in K_\nu$. Then condition (1.2) is satisfied.*

Proof. — According to (4.16), there is $C > 0$ such that

$$\begin{aligned} \|\langle q \rangle^\mu V\|_{\mathcal{B}} &\leq C \|(\langle \cdot \rangle^\mu V)_p\| \\ &\leq C \sup_x \left(\int_{|y-x| < 1} |\langle y \rangle^\mu V(y)|^p dy \right)^{1/p} \\ &\leq C \left(\sup_x \int_{|y-x| < 1} |\langle y \rangle^\mu V(y)|^p dy \right)^{1/p} \end{aligned} \quad (4.22)$$

Because, if $\nu \neq 2$ then $L_\nu(y-x) \geq 1$ for $|y-x| < 1$, we have the following

$$\int_{|y-x| < 1} |\langle y \rangle^\mu V(y)|^p dy \leq \int_{|y-x| < 1} L_\nu(y-x) |\langle y \rangle^\mu V(y)|^p dy \quad (4.23)$$

So

$$\sup_x \int_{|y-x| < 1} |\langle y \rangle^\mu V(y)|^p dy \leq \sup_x \int_{|y-x| < 1} L_\nu(y-x) |\langle y \rangle^\mu V(y)|^p dy \quad (4.24)$$

According to (4.22), we have

$$\|\langle q \rangle^\mu V\|_{\mathcal{B}} \leq C (\|(\langle \cdot \rangle^\mu V(\cdot))^p\|_{K_\nu})^{1/p}. \quad (4.25)$$

So if there is $\mu > 1$ such that $(\langle \cdot \rangle^\mu V(\cdot))^p \in K_\nu$, then $\|\langle q \rangle^\mu V\|_{\mathcal{B}} < \infty$ and V satisfies (1.2). \square

5. CONCRETE POTENTIALS

In this section, we will give examples of concrete potential which satisfy the assumptions of Theorem 1.1 and the assumptions of Theorem 1.2. For these examples, we will discuss the application of the Mourre Theorem with the generator of dilation and/or Nakamura's result.

Note that since $H_0 = \Delta : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded, if $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact, then $H : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded. Furthermore $H_0 \in C^\infty(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ and we can deduce that:

PROPOSITION 5.1. — *Let $k \in \mathbb{N}^*$. We suppose that $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact. The following properties are equivalent:*

- (1) $H = \Delta + V \in C^k(A_u)$;
- (2) $H = \Delta + V \in C^k(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$;
- (3) $V \in C^k(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

When $V : \mathcal{H}^2 \rightarrow L^2$ is compact, we have the following

PROPOSITION 5.2. — *Let $k \in \mathbb{N}^*$. We suppose that V is Δ -compact. The following properties are equivalent:*

- (1) $H = \Delta + V \in C^k(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$;
- (2) $V \in C^k(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$.

Remark that if V is Δ -compact and $k \neq 1$, then $H \in C^k(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$ is not equivalent to $H \in C^k(A_u)$ (see [1, Theorem 6.3.4]).

5.1. A non Laplacian-compact potential. In this part, we work in one dimension.

Let $\chi \in C^1(\mathbb{R}, \mathbb{R})$ be such that $\chi(x) = 0$ if $|x| > 1$, $\chi(x) > 0$ if $|x| < 1$ and $\chi(-x) = \chi(x)$.

LEMMA 5.3. — *Let V such that*

$$\widehat{qV}(\xi) = \sum_{n=-\infty}^{+\infty} \lambda_n \chi(\xi - n),$$

where $\lambda_{-n} = \lambda_n \geq 0$, $\lambda_0 = 0$, and $(\lambda_n)_{n \in \mathbb{Z}}$ is not bounded. Moreover, we suppose that there is $0 < \epsilon < 1/2$ such that

$$\sum_{n=-\infty}^{+\infty} \lambda_n \langle n \rangle^{-1/2+\epsilon} < \infty. \quad (5.1)$$

Then, for all $u \in \mathcal{U}$,

- (1) V is symmetric and $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact.
- (2) $V \in C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.
- (3) $V \notin C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$.
- (4) V is not Δ -bounded.

We will give few remarks about this lemma.

- (a) Note that, since χ is compactly support, the sum which defines V is locally finite and so V is well defined.
- (b) Lemma 5.3 applies with A_N replacing A_u but since V is not Δ -bounded, and so V is not Δ -compact, we can not apply [16, Theorem 1]. Furthermore, because of (3), we can not apply the Mourre Theorem with A_D as conjugate operator.
- (c) The requirements on $(\lambda_n)_{n \in \mathbb{Z}}$ are satisfied in the case

$$\lambda_n = \begin{cases} p & \text{if } |n| = 2^p \\ 0 & \text{else} \end{cases}.$$

Proof of Lemma 5.3. —

(1) Let

$$T_n(x) = \begin{cases} \int_0^x \chi(s-n)ds & \text{if } n > 0 \\ -\int_x^0 \chi(s-n)ds & \text{if } n < 0 \end{cases}.$$

Remark that $T'_n = \chi(\cdot - n)$. For $n > 0$ and $f, g \in \mathcal{H}^1$, we have

$$\begin{aligned} \left| (\hat{f}, T_n * \hat{g}) \right| &= \left| \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) \int_0^{\xi-\eta} \chi(s-n) ds d\xi d\eta \right| \\ &= \left| \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) \int_{-n}^{\xi-\eta-n} \chi(s) ds d\xi d\eta \right|. \end{aligned}$$

Since $\chi(s) = 0$ if $s \leq -1$, we have $\int_{-n}^{\xi-\eta-n} \chi(s) ds = 0$ if $\xi - \eta - n \leq -1$. If $\xi - \eta - n > -1$, then $\int_{-n}^{\xi-\eta-n} \chi(s) ds = \int_{-1}^{\xi-\eta-n} \chi(s) ds$. So

$$\begin{aligned} \left| (\hat{f}, T_n * \hat{g}) \right| &= \left| \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) \int_{-1}^{\xi-\eta-n} \chi(s) ds d\xi d\eta \right| \\ &\leq \int_{\mathbb{R}^2} \langle \xi \rangle |\hat{f}(\xi)| \langle \eta \rangle |\hat{g}(\eta)| \langle \xi \rangle^{-1} \langle \eta \rangle^{-1} \int_{-1}^{\xi-\eta-n} \chi(s) ds d\xi d\eta. \end{aligned}$$

Since $\epsilon < 1/2$, there is $C > 0$ such that $\langle \xi \rangle^{-1/2+\epsilon} \langle \eta \rangle^{-1/2+\epsilon} \leq C \langle \xi - \eta \rangle^{-1/2+\epsilon}$.

$$\begin{aligned} \left| (\hat{f}, T_n * \hat{g}) \right| &\leq C \int_{\mathbb{R}^2} \langle \xi \rangle |\hat{f}(\xi)| \langle \eta \rangle |\hat{g}(\eta)| \langle \xi \rangle^{-1/2-\epsilon} \langle \eta \rangle^{-1/2-\epsilon} \\ &\quad \langle \xi - \eta \rangle^{-1/2+\epsilon} \int_{-1}^{\xi-\eta-n} \chi(s) ds d\xi d\eta \\ &\leq \langle n-1 \rangle^{-1/2+\epsilon} \int_{-1}^1 \chi(s) ds \int_{\mathbb{R}^2} \langle \xi \rangle |\hat{f}(\xi)| \langle \eta \rangle |\hat{g}(\eta)| \\ &\quad \langle \xi \rangle^{-1/2-\epsilon} \langle \eta \rangle^{-1/2-\epsilon} d\xi d\eta. \end{aligned}$$

So, since $K : (\xi, \eta) \rightarrow \langle \xi \rangle^{-1/2-\epsilon} \langle \eta \rangle^{-1/2-\epsilon}$ is in $L^2(\mathbb{R}^2)$, the operator $L^2 \ni \psi \mapsto \int_{\mathbb{R}} K(\xi, \eta) \psi(\eta) d\eta$ is compact (see [19, Theorem VI.23]). So T_n is compact from \mathcal{H}^1 to \mathcal{H}^{-1} if $n > 0$ and we can remark that we have similar inequalities for $n < 0$.

So

$$\begin{aligned} \left| (\hat{f}, \hat{V} * \hat{g}) \right| &\leq \sum_{n=-\infty}^{+\infty} |\lambda_n| \left| (\hat{f}, T_n * \hat{g}) \right| \\ &\leq \int_{\mathbb{R}^2} \langle \xi \rangle |\hat{f}(\xi)| \langle \eta \rangle |\hat{g}(\eta)| K(\xi, \eta) d\xi d\eta \\ &\quad \int_{-1}^1 \chi(s) ds \sum_{n=-\infty}^{+\infty} |\lambda_n| \max(\langle n-1 \rangle^{-1/2+\epsilon}, \langle n+1 \rangle^{-1/2+\epsilon}). \end{aligned}$$

So, since $\sum_{n=-\infty}^{+\infty} |\lambda_n| \langle n \rangle^{-1/2+\epsilon} < \infty$, we have that $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact.

Moreover, since $\chi(-x) = \chi(x)$ and $\lambda_n = \lambda_{-n}$, we have

$$\begin{aligned}\widehat{qV}(-\xi) &= \sum_{n=-\infty}^{+\infty} \lambda_n \chi(-\xi - n) \\ &= \sum_{n=-\infty}^{+\infty} \lambda_n \chi(\xi + n) \\ &= \sum_{n=-\infty}^{+\infty} \lambda_{-n} \chi(\xi - n) \\ &= \widehat{qV}(\xi)\end{aligned}$$

So, since $\widehat{qV}(\xi) \in \mathbb{R}$, we have $xV(x) \in \mathbb{R}, \forall x \in \mathbb{R}$, and we conclude that V is a symmetric multiplication operator.

(2) By a simple computation, in all dimension, we have

$$\begin{aligned}(g, [V, A_u]f) &= (Vg, A_u f) - (A_u g, Vf) \\ &= (Vg, qu(p)f) - (qu(p)g, Vf) \\ &\quad - \frac{i}{2} ((Vg, u'(p)f) + (u'(p)g, Vf)) \\ &= (qVg, u(p)f) - (u(p)g, qVf) \\ &\quad - \frac{i}{2} ((Vg, u'(p)f) + (u'(p)g, Vf)).\end{aligned}\tag{5.2}$$

If u' is bounded, since $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is a compact operator, there is $C > 0$ such that

$$|(Vg, u'(p)f) + (u'(p)g, Vf)| \leq C \|f\|_{\mathcal{H}^1} \|g\|_{\mathcal{H}^1}.$$

Moreover, we can remark that $\langle \cdot \rangle^{-1} \widehat{qV} \in L^1(\mathbb{R})$. So, there is $C > 0$ such that

$$\begin{aligned}|(u(p)f, qVg)| &= |(u(q)\hat{f}, \widehat{qV} * \hat{g})| \\ &= \left| \int_{\mathbb{R}^2} u(\xi) \hat{f}(\xi) \widehat{qV}(\xi - \eta) \hat{g}(\eta) d\xi d\eta \right| \\ &\leq C \int_{\mathbb{R}^2} |u(\xi)| |\langle \xi \rangle| |\hat{f}(\xi)| |\langle \xi - \eta \rangle^{-1}| |\widehat{qV}(\xi - \eta)| \\ &\quad |\langle \eta \rangle| |\hat{g}(\eta)| d\xi d\eta\end{aligned}\tag{5.3}$$

Since $f, g \in \mathcal{H}^1$ and u is bounded, $u(q)\langle q \rangle \hat{f}$ and $\langle q \rangle \hat{g}$ are in L^2 . So by Young inequality, we conclude that

$$\begin{aligned}|(u(p)f, qVg)| &\leq C \|u(q)\langle q \rangle \hat{f}\|_2 \|(\langle q \rangle^{-1} \widehat{qV}) * (\langle q \rangle \hat{g})\|_2 \\ &\leq C \|\langle q \rangle^{-1} \widehat{qV}\|_1 \|\langle q \rangle \hat{f}\|_2 \|\langle q \rangle \hat{g}\|_2 \\ &\leq C \|\langle q \rangle^{-1} \widehat{qV}\|_1 \|f\|_{\mathcal{H}^1} \|g\|_{\mathcal{H}^1}.\end{aligned}$$

So $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. Similarly, we have

$$\begin{aligned}(g, [[V, A_u], A_u]f) &= ([V, A_u]g, A_u f) - (A_u g, [V, A_u]f) \\ &= ([V, A_u]g, qu(p)f) - (qu(p)g, [V, A_u]f) \\ &\quad - \frac{i}{2} \left((u'(p)g, [V, A_u]f) + ([V, A_u]g, u'(p)f) \right).\end{aligned}$$

As $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$, we have that $(u'(p)g, [V, A_u]f) + ([V, A_u]g, u'(p)f)$ is bounded. Using (5.2), we have

$$\begin{aligned} (qu(p)g, [V, A_u]f) &= (qVqu(p)g, u(p)f) - (u(p)qu(p)g, qVf) \\ &\quad - \frac{i}{2}((Vqu(p)g, u'(p)f) + (u'(p)qu(p)g, Vf)) \\ &= (q^2Vu(p)g, u(p)f) - (u(p)qu(p)g, qVf) \\ &\quad - \frac{i}{2}((u(p)g, qVu'(p)f) + (u'(p)qu(p)g, Vf)). \end{aligned}$$

By a simple computation, we have

$$\begin{aligned} (u(p)qu(p)g, qVf) &= (qu(p)u(p)g, qVf) + ([u(p), q]u(p)g, qVf) \\ &= (u(p)u(p)g, q^2Vf) + \frac{i}{2}(u(p)u'(p)g, qVf) \end{aligned}$$

and

$$\begin{aligned} (u'(p)qu(p)g, Vf) &= (qu'(p)u(p)g, Vf) + ([u'(p), q]u(p)g, Vf) \\ &= (u(p)u'(p)g, qVf) + \frac{i}{2}(u''(p)u(p)g, Vf). \end{aligned}$$

As previously, we can remark that $\langle q \rangle^{-1}q^2\widehat{V} \in L^1$. So, since u and all of whose derivatives are bounded, we deduce that $[[V, A_u], A_u]$ is a bounded operator on $\mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ and $V \in C^2(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. Remark that all previous inequalities are true for any bounded u such that $u \in C^\infty$ with all derivatives bounded. In particular, by taking $u(x) = \sin(ax)$, we deduce that

$$V \in C^2(A_N, \mathcal{H}^1, \mathcal{H}^{-1}) \subset C^{1,1}(A_N, \mathcal{H}^1, \mathcal{H}^{-1}).$$

(3) Now we will prove that V is not in $C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$. For $N \in \mathbb{N}^*$, let

$$\widehat{f}_N = \mathbb{1}_{[N, N+1]} \langle N+1 \rangle^{-1} \quad \text{and} \quad \widehat{g} = \mathbb{1}_{[0,1]}.$$

Note that $\|\langle \cdot \rangle \widehat{f}_N\|_{L^2}^2 \|\langle \cdot \rangle \widehat{g}\|_{L^2}^2 \leq 2$ which implies that $\|f_N\|_{\mathcal{H}^1} \|g\|_{\mathcal{H}^1} \leq 4\sqrt{2}$. We have

$$\begin{aligned} (\widehat{f}_N, \widehat{\nabla}(q\widehat{V}) * \widehat{g}) &= \sum_{n=-\infty}^{+\infty} \lambda_n \int_{\mathbb{R}^2} (\xi - \eta) \widehat{f}_N(\xi) \chi(\xi - \eta - n) \widehat{g}(\eta) d\xi d\eta \\ &\geq \frac{\lambda_N}{\langle N+1 \rangle} \int_N^{N+1} \left(\int_0^1 (\xi - \eta) \chi(\xi - \eta - N + 1) d\eta \right) d\xi \\ &\geq \frac{\lambda_N}{\langle N+1 \rangle} (N-1) \int_N^{N+1} \left(\int_{\xi-N}^{\xi-N+1} \chi(\sigma) d\sigma \right) d\xi \\ &\geq \frac{N-1}{\langle N+1 \rangle} \lambda_N \int_0^2 \chi(\sigma) \left(\int_{N+\sigma-1}^{N+\sigma} d\xi \right) d\sigma \\ &\geq \frac{N-1}{\langle N+1 \rangle} \lambda_N \int_0^2 \chi(\sigma) d\sigma. \end{aligned}$$

So, since $\int_0^2 \chi(\sigma) d\sigma > 0$, we have that $((f_N, [V, iA_D]g))_{N \in \mathbb{N}}$ is not bounded with $\|f_N\|_{\mathcal{H}^1} \|g\|_{\mathcal{H}^1} \leq 4$. So V does not belong to the class $C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$.

(4) Now, we will prove that V is not Δ -compact.

Let $N \in \mathbb{N}$, $N \geq 2$ and let

$$\widehat{f}_N = \mathbb{1}_{[N+1, N+2]} \quad \text{and} \quad \widehat{g} = \mathbb{1}_{[0,1]}.$$

Remark that $\|f_N\|_{L^2} \|g\|_{\mathcal{H}^2}$ is a bounded sequence.

$$\begin{aligned} |(f_N, Vg)| &= \left| (\widehat{f}_N, \widehat{V} * \widehat{g}) \right| \\ &= \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \sum_{n=-\infty}^{+\infty} \lambda_n \int_0^{\xi-\eta} \chi(s-n) ds \\ &= - \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \sum_{n=-\infty}^{-1} \lambda_n \int_{\xi-\eta}^0 \chi(s-n) ds \\ &\quad + \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \sum_{n=1}^{N-1} \lambda_n \int_0^{\xi-\eta} \chi(s-n) ds \\ &\quad + \sum_{n=N}^{N+2} \lambda_n \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \int_0^{\xi-\eta} \chi(s-n) ds \\ &\quad + \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \sum_{n=N+3}^{+\infty} \lambda_n \int_0^{\xi-\eta} \chi(s-n) ds. \end{aligned} \quad (5.4)$$

Remark that, for $\xi \in [N+1, N+2]$ and $\eta \in [0, 1]$, we have $N+2 \geq \xi - \eta \geq N$. So, if $\xi \in [N+1, N+2]$ and $\eta \in [0, 1]$, we have:

- If $n \leq -1$, then $\chi(s-n) = 0$ for all $s \in [0, \xi - \eta]$. So $\int_{\xi-\eta}^0 \chi(s-n) ds = 0$.
- If $1 \leq n \leq N-1$, then $\xi - \eta \geq n+1$. So, since $\chi(s) = 0$ if $|s| \geq 1$,

$$\int_0^{\xi-\eta} \chi(s-n) ds = \int_{n-1}^{n+1} \chi(s-n) ds = \int_{-1}^1 \chi(s) ds > 0.$$

- If $n \geq N+3$, then $\xi - \eta \leq n-1$. So, since $\chi(s-n) = 0$ for all $s \in [0, \xi - \eta]$, we have $\int_0^{\xi-\eta} \chi(s-n) ds = 0$.

So, since $\lambda_n \geq 0$ for all n and $\chi(x) \geq 0$ for all x , from (5.4), we have:

$$\begin{aligned} |(f_N, Vg)| &= \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \sum_{n=1}^{N-1} \lambda_n \int_{-1}^1 \chi(s) ds \\ &\quad + \sum_{n=N}^{N+2} \lambda_n \int_{\xi \in [N+1, N+2]} \int_{\eta \in [0,1]} \int_0^{\xi-\eta} \chi(s-n) ds \\ &\geq \lambda_{N-1} \int_{-1}^1 \chi(s) ds. \end{aligned}$$

So, since $(\lambda_n)_{n \in \mathbb{N}}$ is not bounded, we can extract a subsequence $(\lambda_{\phi(n)})_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow +\infty} \lambda_{\phi(n)} = +\infty \quad \text{and we have} \quad \lim_{N \rightarrow +\infty} |(f_{\phi(N)+1}, Vg)| = +\infty.$$

So V is not Δ -bounded. □

5.2. A class of oscillating potential. Let $\alpha > 0$, $\beta \in \mathbb{R}$, $k \in \mathbb{R}^*$ and $\kappa \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $\kappa = 1$ on $[-1, 1]$ and $0 \leq \kappa \leq 1$. Let

$$W_{\alpha\beta}(x) = (1 - \kappa(|x|)) \frac{\sin(k|x|^\alpha)}{|x|^\beta}. \quad (5.5)$$

This potential can be seen as a Δ -compact potential (if $\beta > 0$) or as a potential on \mathcal{H}^1 to \mathcal{H}^{-1} for which we keep the same notation.

We will see that under certain condition on (α, β) , we can apply Theorems 1.1 and 1.2 with $W_{\alpha\beta}$ as potential. We will also compare our results (lemma 5.4) with results given in [13].

Recall that \mathcal{U} is the space of vector fields u bounded with all derivatives bounded such that $x \cdot u(x) > 0$ for all $x \neq 0$. We have the following:

LEMMA 5.4. — *Let $W_{\alpha\beta}$ be as in (5.5) and let $H = \Delta + W_{\alpha\beta}$. For all $u \in \mathcal{U}$, we have:*

- (1) *if $\alpha + \beta > 2$, then $W_{\alpha\beta} : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact and $W_{\alpha\beta} \in C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.*
- (2) *if $2\alpha + \beta > 3$ and $\beta > 0$, then $W_{\alpha\beta} \in C^{1,1}(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$.*

In particular, in this both cases, Theorems 1.1 and 1.2 apply.

Note that in (1), we do not require to have $\beta > 0$. In particular, if $\beta < 0$, then $W_{\alpha\beta}$ is an unbounded function.

In [13], if we suppose $\beta > 0$, we can see that the Limiting Absorption Principle can be proved with the generator of dilation A_D as conjugate operator for $H = \Delta + W_{\alpha\beta}$ if $|\alpha - 1| + \beta > 1$. If $|\alpha - 1| + \beta < 1$, they showed that $H \notin C^1(A_D)$. This implies that we cannot apply the Mourre Theorem with A_D as conjugate operator on this area. Moreover, they also proved a limiting absorption principle if $\alpha > 1$ and $\beta > 1/2$, in a certain energy window. If $|\alpha - 1| + \beta < 1$ and $2\alpha + \beta > 3$, Theorem 1.2 improves this result in two waves: first, there is no restriction of energy; second, we have some result on the boundary value of the resolvent. Furthermore, the region where $|\alpha - 1| + \beta < 1$, $2\alpha + \beta > 3$ and $\beta \leq \frac{1}{2}$ is not covered by [13] but Theorem 1.2 applies.

Proof of Lemma 5.4. — Let $f, g \in \mathcal{S}$ and let $0 < \mu < 1$. Let $u \in \mathcal{U}$. We will always suppose that μ is small enough. We have

$$\begin{aligned} & (f, \langle q \rangle^\mu [W_{\alpha\beta}, iA_u]g) \\ &= (\langle q \rangle^\mu W_{\alpha\beta} f, iA_u g) - (A_u \langle q \rangle^\mu f, iW_{\alpha\beta} g) \\ &= (\langle q \rangle^\mu W_{\alpha\beta} f, iq \cdot u(p)g) - (q \cdot u(p) \langle q \rangle^\mu f, iW_{\alpha\beta} g) \\ & \quad + (\langle q \rangle^\mu W_{\alpha\beta} f, \frac{1}{2} u'(p)g) + (\frac{i}{2} u'(p) \langle q \rangle^\mu f, iW_{\alpha\beta} g) \\ &= (\langle q \rangle^\mu W_{\alpha\beta} f, iq \cdot u(p)g) - (q \cdot u(p) f, i \langle q \rangle^\mu W_{\alpha\beta} g) \end{aligned} \quad (5.6)$$

$$+ (\langle q \rangle^\mu W_{\alpha\beta} f, \frac{1}{2} u'(p)g) + (\frac{i}{2} u'(p) f, i \langle q \rangle^\mu W_{\alpha\beta} g) \quad (5.7)$$

$$- (q \cdot [u(p), \langle q \rangle^\mu] f, iW_{\alpha\beta} g) + (\frac{i}{2} [u'(p), \langle q \rangle^\mu] f, iW_{\alpha\beta} g). \quad (5.8)$$

Remark that, since u and all its derivatives are bounded, for $\mu < 1$, we have that $[u(p), \langle q \rangle^\mu]$ and $[u'(p), \langle q \rangle^\mu]$ are bounded and $\|u(p)f\|_{\mathcal{H}^s}$ and $\|u'(p)f\|_{\mathcal{H}^s}$ are controlled by $\|f\|_{\mathcal{H}^s}$ for $s = 1, 2$. We will use this argument to treat terms in (5.7) and (5.8) and we note that they are bounded in the \mathcal{H}^1 norm when terms in (5.6) are bounded. For this reason, we focus

on this to terms which are quite similar. To control them, we will show that $qW_{\alpha\beta}(q)$ can be written with a different form.

Let $\tilde{\kappa} \in C_c^\infty(\mathbb{R}, \mathbb{R})$ be such that $\tilde{\kappa}(|x|) = 0$ if $|x| \geq 1$, $\tilde{\kappa} = 1$ on $[-1/2, 1/2]$ and $0 \leq \tilde{\kappa} \leq 1$. So, we can observe that $(1 - \tilde{\kappa}(|x|))(1 - \kappa(|x|)) = (1 - \kappa(|x|))$ for all $x \in \mathbb{R}^\nu$.

For $\gamma \in \mathbb{R}$, let

$$\tilde{W}_{\alpha\gamma}(x) = (1 - \tilde{\kappa}(|x|)) \frac{\cos(k|x|^\alpha)}{|x|^\gamma}.$$

By a simple computation, we have

$$(1 - \kappa(|x|))|x|\nabla\tilde{W}_{\alpha\gamma}(x) = -(1 - \kappa(|x|))\gamma\frac{x}{|x|}\tilde{W}_{\alpha\gamma}(x) - k\alpha xW_{\alpha\beta}(x) \quad (5.9)$$

with $\gamma = \alpha + \beta - 1$.

In a first time, remark that, since, in both cases, $\gamma > 0$,

$$\langle q \rangle^\mu \frac{\gamma}{k\alpha} \frac{q}{|q|} \tilde{W}_{\alpha\gamma}(q)$$

is bounded for all $0 < \mu \leq \gamma$. Thus, by using (5.9) in (5.6), it suffices to prove that

$$\langle q \rangle^\mu (1 - \kappa(|q|))|q|\nabla\tilde{W}_{\alpha\gamma}(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$$

is bounded to show that $W_{\alpha\beta} \in C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

To do this, remark that $\nabla F(q) = i[p, F(q)]$ for all functions F . So, we have for $\phi, \psi \in \mathcal{S}$,

$$\begin{aligned} & (u(p)\phi, (1 - \kappa(|q|))\langle q \rangle^\mu |q|\nabla\tilde{W}_{\alpha\gamma}(q)\psi) \\ = & (u(p)\phi, i[p, (1 - \kappa(|q|))\langle q \rangle^\mu |q|\tilde{W}_{\alpha\gamma}(q)]\psi) + (u(p)\phi, q\kappa'(|q|)\langle q \rangle^\mu \tilde{W}_{\alpha\gamma}(q)\psi) \\ & - \mu(u(p)\phi, q(1 - \kappa(|q|))\langle q \rangle^{\mu-1}|q|\tilde{W}_{\alpha\gamma}(q)\psi) \\ & - (u(p)\phi, (1 - \kappa(|q|))\langle q \rangle^\mu \frac{q}{|q|}\tilde{W}_{\alpha\gamma}(q)\psi). \end{aligned}$$

So, we have

$$\begin{aligned} & (u(p)\phi, (1 - \kappa(|q|))\langle q \rangle^\mu |q|\nabla\tilde{W}_{\alpha\gamma}(q)\psi) \\ = & (pu(p)\phi, i(1 - \kappa(|q|))\langle q \rangle^\mu |q|\tilde{W}_{\alpha\gamma}(q)\psi) \\ & - (u(p)\phi, i(1 - \kappa(|q|))\langle q \rangle^\mu |q|\tilde{W}_{\alpha\gamma}(q)p\psi) + (u(p)\phi, q\kappa'(|q|)\langle q \rangle^\mu \tilde{W}_{\alpha\gamma}(q)\psi) \\ & - \mu(u(p)\phi, q(1 - \kappa(|q|))\langle q \rangle^{\mu-1}|q|\tilde{W}_{\alpha\gamma}(q)\psi) \\ & - (u(p)\phi, (1 - \kappa(|q|))\langle q \rangle^\mu \frac{q}{|q|}\tilde{W}_{\alpha\gamma}(q)\psi). \end{aligned} \quad (5.10)$$

So, since u is bounded, by density, if $(1 - \kappa(|q|))\langle q \rangle^\mu |q|\tilde{W}_{\alpha\gamma}(q)$ is bounded, then $(1 - \kappa(|q|))\langle q \rangle^\mu |q|\nabla\tilde{W}_{\alpha\gamma}(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded.

- (1) Suppose that $\alpha + \beta > 2$. Since $\gamma > 1$, by (5.9), $\langle q \rangle^\mu qW_{\alpha\beta} : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded for $\mu > 0$ small enough. This implies that $W_{\alpha\beta}$ belongs to the class $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

Moreover, by (5.9), we have

$$W_{\alpha\beta}(x) = -(1 - \kappa(|x|))\frac{\gamma}{k\alpha}\frac{1}{|x|}\tilde{W}_{\alpha\gamma}(x) - (1 - \kappa(|x|))\frac{x}{k\alpha|x|}\nabla\tilde{W}_{\alpha\gamma}(x).$$

So, since $\gamma > 0$, we have that $(1 - \kappa(|q|)) \frac{\gamma}{k\alpha} \frac{1}{|q|} \tilde{W}_{\alpha\gamma}(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact. As in (5.10), we can prove that $(1 - \kappa(|q|)) \frac{q}{k\alpha|q|} \nabla \tilde{W}_{\alpha\gamma}(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact. Thus, by sum, $W_{\alpha\beta} : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact.

- (2) Suppose that $\beta > 0$ and $2\alpha + \beta > 3$. In this case, remark that $W_{\alpha\beta}(q)$ is Δ -compact. Let $u \in \mathcal{U}$. Let $\underline{\kappa} \in C_c^\infty(\mathbb{R}, \mathbb{R})$ be such that $\underline{\kappa}(x) = 0$ if $|x| \geq 1/2$, $\underline{\kappa} = 1$ on $[-1/4, 1/4]$ and $0 \leq \underline{\kappa} \leq 1$. For $\delta \in \mathbb{R}$, let

$$\underline{W}_{\alpha\delta}(x) = (1 - \underline{\kappa}(|x|)) \frac{\sin(k|x|^\alpha)}{|x|^\delta}.$$

By a simple computation, we can write:

$$\tilde{W}_{\alpha\gamma}(x) = (1 - \tilde{\kappa}(|x|)) \frac{\delta}{k\alpha} \frac{1}{|x|} \underline{W}_{\alpha\delta} + (1 - \tilde{\kappa}(|x|)) \frac{x}{k\alpha|x|} \nabla \underline{W}_{\alpha\delta}(x)$$

with $\delta = \gamma + \alpha - 1 = \beta + 2\alpha - 2 > 1$.

Since we want to prove that $[W_{\alpha\beta}, iA_u] : \mathcal{H}^2 \rightarrow \mathcal{H}^{-2}$ is bounded, we can make twice the argument of (5.10). This implies that $W_{\alpha\beta}$ is in $C^{1,1}(A_u, \mathcal{H}^2, \mathcal{H}^{-2})$. \square

If we want more regularity on the potentials, we have the following

LEMMA 5.5. — *Let $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ be a compact symmetric multiplication operator. If $|q|^n V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded for some $n \in \mathbb{N}^*$, then, for any $u \in \mathcal{U}$, we have $V \in C^n(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. In particular, if $\alpha + \beta - 1 \geq n \in \mathbb{N}^*$, then $W_{\alpha\beta} \in C^n(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. In this case, if $n > 1$, we deduce that $\lambda \mapsto R(\lambda \pm i0)$ is locally of class Λ^s on \mathbb{R}^{+*} outside the eigenvalues of H , where s is the integer part of $\alpha + \beta - 2$.*

Proof of Lemma 5.5. — Let $u \in \mathcal{U}$ and $n \in \mathbb{N}^*$. Let $ad_{A_u}^n(V)$ the iterated commutator of order n , with $ad_{A_u}^1(V) = [V, A_u]$. By induction, we can prove that there is $(B_k(p))_{k \in \{0, \dots, n\}}$ and $(B'_k(p))_{k \in \{0, \dots, n\}}$ two sequences of bounded operators such that

$$ad_{A_u}^n(V) = \sum_{k=0}^n B_k(p) q^k V B'_k(p). \quad (5.11)$$

Moreover, we can see that $B_k(p)$ and $B'_k(p)$ depends only of u and its derivatives of order less than n . So, if $|q|^n V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded, then $V \in C^n(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

Moreover, by (5.9), we can see that if $\gamma = \alpha + \beta - 1 \geq s + 1$, $s \in \mathbb{N}^*$, then $|q|^{s+1} V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded. So $W_{\alpha\beta} \in C^{s+1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1}) \subset \Lambda^{s+1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ and by Theorem 3.3, we can deduce that $\lambda \mapsto R(\lambda \pm i0)$ is locally of class Λ^s on \mathbb{R}^{+*} outside the eigenvalues of H . \square

5.3. An unbounded potential with high oscillations. Now, we will show an example of potential V of class $C^\infty(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ for any $u \in \mathcal{U}$ such that V is neither in $C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$ nor Δ -bounded. In particular we cannot have the Mourre estimate with A_D as conjugate operator but we can prove a limiting absorption principle with A_u as conjugate operator and have a good regularity for the boundary value of the resolvent.

LEMMA 5.6. — *Let $\kappa \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $\kappa = 1$ on $[-1, 1]$ and $0 \leq \kappa \leq 1$. Let*

$$V(x) = (1 - \kappa(|x|)) \exp(3|x|/4) \sin(\exp(|x|)). \quad (5.12)$$

Then:

- (1) $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact;
- (2) For any $u \in \mathcal{U}$, we have $V \in C^\infty(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$;

- (3) V is not Δ -bounded;
 (4) V is not of class $C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$.

In particular, we can use neither the Mourre Theorem with the generator of dilation as conjugate operator nor Nakamura's Theorem. By Theorem 3.3, we have the following

COROLLARY 5.7. — *Let V as in (5.12) and $H = \Delta + V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$. Then Theorem 1.2 applies and, for all $s > 0$, the functions*

$$\lambda \mapsto R(\lambda \pm i0) \in B(\mathcal{H}_s^{-1}, \mathcal{H}_{-s}^1) \quad (5.13)$$

are locally of class $\Lambda^{s-1/2}$ on $(0, +\infty)$ outside the eigenvalues of H .

In particular, if we see $R(\lambda \pm i0)$ as an operator from C_c^∞ to \mathcal{D}' the space of distributions, the functions

$$\lambda \mapsto R(\lambda \pm i0) \in B(C_c^\infty, \mathcal{D}')$$

are of class C^∞ on $(0, +\infty)$ outside the eigenvalues.

Proof of Lemma 5.6. — Let $\tilde{\kappa} \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $\tilde{\kappa}(|x|) = 0$ if $|x| \geq 1$, $\tilde{\kappa} = 1$ on $[-1/2, 1/2]$ and $0 \leq \tilde{\kappa} \leq 1$. So, we can observe that $(1 - \tilde{\kappa}(|x|))(1 - \kappa(|x|)) = (1 - \kappa(|x|))$ for all $x \in \mathbb{R}^\nu$.

If we denote

$$\tilde{V}(x) = (1 - \tilde{\kappa}(|x|)) \cos(\exp(|x|)),$$

we have:

$$(1 - \kappa(|x|)) \nabla \tilde{V}(x) = -(1 - \kappa(|x|)) \frac{x}{|x|} \exp(|x|) \sin(\exp(|x|)).$$

So,

$$xV(x) = -|x|(1 - \kappa(|x|)) \exp(-|x|/4) \nabla \tilde{V}(x).$$

- (1) By a simple calculus, we have

$$V(x) = -\frac{x}{|x|} (1 - \kappa(|x|)) \exp(-|x|/4) \nabla \tilde{V}(x).$$

So, since \tilde{V} is bounded, by writing $\nabla \tilde{V}(q) = i[p, \tilde{V}(q)]$, as in (5.10), we can prove that $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact.

- (2) Similarly, since \tilde{V} is bounded, writing $\nabla \tilde{V}(q) = i[p, \tilde{V}(q)]$ we have $q|q|^n V(q) : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded for all $n \in \mathbb{N}$. So, by Lemma 5.5, for any $u \in \mathcal{U}$ we have $V \in C^n(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$ for all $n \in \mathbb{N}^*$. So $V \in C^\infty(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.
- (3) Let $\chi \in C_c^0(\mathbb{R}, \mathbb{R})$ be such that $\text{supp}(\chi) \subset [\frac{\pi}{4}, \frac{3\pi}{4}]$, $\chi \geq 0$ and $\chi(\pi/2) = 1$. Let $N \in \mathbb{N}^*$,

$$f(x) = \langle x \rangle^{-(\nu+1)/2}$$

and

$$g_N(x) = \ln \left(\frac{3\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \exp(|x|/2) \chi(\exp(|x|) - 2N\pi).$$

We denote $C > 0$ constants independent of N . Remark that $f \in \mathcal{H}^2$ and

$$\begin{aligned} \int_{\mathbb{R}^\nu} g_N^2(x) dx &= \int_{\mathbb{R}^\nu} \ln \left(\frac{3\pi}{4} + 2N\pi \right)^{(1-\nu)} \exp(|x|) \chi^2(\exp(|x|) - 2N\pi) dx \\ &= C \ln \left(\frac{3\pi}{4} + 2N\pi \right)^{(1-\nu)} \int_{\mathbb{R}} \exp(r) \chi^2(\exp(r) - 2N\pi) r^{\nu-1} dr \\ &\leq C \int_{\mathbb{R}} \exp(r) \chi^2(\exp(r) - 2N\pi) dr. \end{aligned}$$

So, if $\sigma = e^r - 2N\pi$, we have

$$\int_{\mathbb{R}^\nu} g_N^2(x) dx \leq C \int_{\mathbb{R}} \chi^2(\sigma) d\sigma. \quad (5.14)$$

So $\|f\|_{\mathcal{H}^2} \|g_N\|_{L^2} \leq C$. Remark that, since $f(x)V(x)g_N(x) \geq 0$ for all $x \in \mathbb{R}^\nu$, we have for N large enough

$$\begin{aligned} \int_{\mathbb{R}^\nu} f(x)V(x)g(x) dx &= \int_{\mathbb{R}^\nu} \langle x \rangle^{-(\nu+1)/2} \ln \left(\frac{3\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \exp(5|x|/4) \\ &\quad \chi(\exp(|x|) - 2N\pi) (1 - \kappa(|x|)) \sin(\exp(|x|)) dx \\ &\geq C \ln \left(\frac{3\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \int_{\mathbb{R}^\nu} \langle x \rangle^{-(\nu+1)/2} \exp(5|x|/4) \\ &\quad \chi(\exp(|x|) - 2N\pi) dx \\ &\geq C \ln \left(\frac{3\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \\ &\quad \int_{\mathbb{R}} \langle r \rangle^{-(\nu+1)/2} \exp(5r/4) \chi(e^r - 2N\pi) r^{\nu-1} dr \\ &\geq C \int_{\mathbb{R}} \exp(r/8) e^r \chi(e^r - 2N\pi) dr \\ &\geq C \int_{\pi/4}^{3\pi/4} (\sigma + 2N\pi)^{1/8} \chi(\sigma) d\sigma \\ &\geq C \left(\frac{\pi}{4} + 2N\pi \right)^{1/8} \int_{\pi/4}^{3\pi/4} \chi(\sigma) d\sigma. \end{aligned}$$

So $\lim_{N \rightarrow +\infty} \int_{\mathbb{R}^\nu} f(x)V(x)g_N(x) dx = +\infty$. So V is not Δ -bounded.

(4) By a simple calculus, we have

$$\begin{aligned} x \cdot \nabla V(x) &= -\kappa'(|x|)|x| \exp(3|x|/4) \sin(\exp(|x|)) \\ &\quad + (1 - \kappa(|x|))|x| \exp(7|x|/4) \cos(\exp(|x|)) \\ &\quad + (1 - \kappa(|x|)) \frac{3}{4} |x| \exp(3|x|/4) \sin(\exp(|x|)). \end{aligned}$$

Let $\chi \in C_c^1(\mathbb{R}, \mathbb{R})$ be such that $\text{supp}(\chi) \subset [0, \frac{\pi}{4}]$, $\chi \geq 0$ and $\chi(\pi/8) = 1$. Let $N \in \mathbb{N}^*$ and

$$f(x) = \langle x \rangle^{-(\nu+1)/2}$$

and

$$g_N(x) = \ln \left(\frac{\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \exp(-|x|/2) \chi(\exp(|x|) - 2N\pi).$$

As in (5.14), we can show that $\|f\|_{\mathcal{H}^1} \|g_N\|_{\mathcal{H}^1} \leq C$. Remark that, $f(x)x \cdot \nabla V(x)g_N(x) \geq 0$ for all $x \in \mathbb{R}^\nu$. Since $\kappa \in C_c^\infty$, we have for $N \in \mathbb{N}^*$ large enough

$$\int_{\mathbb{R}^\nu} f(x) \kappa'(|x|) |x| \exp(3|x|/4) \sin(\exp(|x|)) g_N(x) dx = 0.$$

Moreover, we have

$$\int_{\mathbb{R}^\nu} f(x) (1 - \kappa(|x|)) \frac{3}{4} |x| \exp(3|x|/4) \sin(\exp(|x|)) g_N(x) dx \geq 0.$$

Thus, for N large enough,

$$\begin{aligned}
\int_{\mathbb{R}^\nu} f(x)x \cdot \nabla V(x)g(x)dx &\geq \int_{\mathbb{R}^\nu} |x| \ln \left(\frac{\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \chi(\exp(|x|) - 2N\pi) \\
&\quad (1 - \kappa(|x|)) \langle x \rangle^{-(\nu+1)/2} \exp(5|x|/4) \cos(\exp(|x|)) dx \\
&\geq C \ln \left(\frac{\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \int_{\mathbb{R}^\nu} \langle x \rangle^{-(\nu+1)/2} \\
&\quad \chi(\exp(|x|) - 2N\pi) |x| \exp(5|x|/4) dx \\
&\geq C \ln \left(\frac{\pi}{4} + 2N\pi \right)^{(1-\nu)/2} \int_{\mathbb{R}^\nu} \langle r \rangle^{-(\nu+1)/2} \\
&\quad \chi(e^r - 2N\pi) r^\nu \exp(5r/4) dx \\
&\geq C \int_{\mathbb{R}^\nu} \chi(e^r - 2N\pi) \exp(5r/4) dx \\
&\geq C \int_0^{\pi/4} (\sigma + 2N\pi)^{1/4} \chi(\sigma) d\sigma \\
&\geq C(2N\pi)^{1/4} \int_0^{\pi/4} \chi(\sigma) d\sigma.
\end{aligned}$$

Thus, $\lim_{N \rightarrow +\infty} \int_{\mathbb{R}^\nu} f(x)x \cdot \nabla V(x)g_N(x)dx = +\infty$.

Therefore, $V \notin C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$. \square

Remark that, if $f(x) = \langle x \rangle^{-(\nu+1)/2}$, then $f \in \mathcal{H}^k$ for all $k \in \mathbb{N}$. This yields that, by the same proof, we can show that $V : \mathcal{H}^k \rightarrow L^2$ is not bounded for all $k \in \mathbb{N}$.

5.4. A short range potential in a weak sense. Now, we will show an example of potential with no decay at infinity for which Theorem 1.2 applies.

We have the following:

LEMMA 5.8. — *Suppose that $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is a symmetric bounded operator. There exists $u \in \mathcal{U}$ such that:*

- (1) *if $x \mapsto |x|V(x)$ is in \mathcal{H}^{-1} then $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.*
- (2) *if $x \mapsto \langle x \rangle^{1+\mu}V(x)$ is in \mathcal{H}^{-1} for some $\mu > 0$, then $V \in C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.*

For this type of potential, we can take u of the form $u(x) = x \langle x \rangle^{-\nu-1}$. Note that this u is in $L^2(\mathbb{R}^\nu)$.

We have the following

COROLLARY 5.9. — *Let $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is a symmetric compact operator. Suppose that there is $\mu > 0$ such that $x \mapsto \langle x \rangle^{1+\mu}V(x)$ is in \mathcal{H}^{-1} . Then Theorem 1.2 applies on $(0, +\infty)$.*

We will give an example of a potential which satisfies assumption of the previous corollary and for which we cannot apply the Mourre Theorem with A_D as conjugate operator.

LEMMA 5.10. — *Let $\nu \geq 3$ and let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi \in C^3$, $\chi(|x|) = 0$ if $|x| > 1$ and $\chi'(0) = \chi''(0) = 1$. Let*

$$V(x) = \sum_{n=2}^{+\infty} n^{(3\nu-1)/2} \chi'(n^{3\nu/2}(|x| - n)) \quad \text{with a finite sum for each } x.$$

Then

- (1) $V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is compact;
- (2) there is $u \in \mathcal{U}$ such that V is of class $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$;
- (3) V is not Δ -bounded;
- (4) V is not of class $C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$.

Proof of Lemma 5.8. — Let $u(x) = x\langle x \rangle^{-\nu-1}$.

- (1) Suppose that $x \mapsto |x|V(x)$ is in \mathcal{H}^{-1} . By (5.2), we can see that, if $[qV, u(p)] : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded, then $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

Let $f, g \in C_c^\infty$. By (5.3) and Young inequality, there is $C > 0$ such that

$$\begin{aligned}
|(u(p)f, qVg)| &\leq C \int_{\mathbb{R}^{2\nu}} |u(\xi)|\langle \xi \rangle |\hat{f}(\xi)|\langle \xi - \eta \rangle^{-1} |\widehat{qV}(\xi - \eta)|\langle \eta \rangle |\hat{g}(\eta)| d\xi d\eta \\
&\leq C \|u(q)\langle q \rangle \hat{f}\|_1 \|(\langle q \rangle^{-1} |\widehat{qV}|) * (\langle q \rangle |\hat{g}|)\|_\infty \\
&\leq C \|u\|_2 \|\langle q \rangle \hat{f}\|_2 \|\langle q \rangle^{-1} \widehat{qV}\|_2 \|\langle q \rangle \hat{g}\|_2 \\
&\leq C \|u\|_2 \|f\|_{\mathcal{H}^1} \|qV\|_{\mathcal{H}^{-1}} \|g\|_{\mathcal{H}^1}.
\end{aligned} \tag{5.15}$$

We have a similar inequality for $|(qVf, u(p)g)|$. By density, we have the same inequality for all $f, g \in \mathcal{H}^1$. Thus $[qV, u(p)] : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded which implies that $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

- (2) Suppose that there is $\mu > 0$ such that $x \mapsto \langle x \rangle^{1+\mu}V(x)$ is in \mathcal{H}^{-1} . In particular, $x \mapsto |x|V(x)$ is in \mathcal{H}^{-1} . Therefore $V \in C^1(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. As we saw previously, we can deduce that $[\langle q \rangle^\mu qV, u(p)] : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded.

By a simple calculus, we have:

$$\begin{aligned}
[qV, u(p)] &= [\langle q \rangle^{-\mu} \langle q \rangle^\mu qV, u(p)] \\
&= \langle q \rangle^{-\mu} [\langle q \rangle^\mu qV, u(p)] + [\langle q \rangle^{-\mu}, u(p)] \langle q \rangle^\mu qV
\end{aligned}$$

By the pseudo-differential calculus, we can prove that $\langle q \rangle^\mu [\langle q \rangle^{-\mu}, u(p)] \langle p \rangle^{1+\nu}$ is a bounded operator. From that, we deduce

$$[\langle q \rangle^{-\mu}, u(p)] \langle q \rangle^\mu qV = \langle q \rangle^{-\mu} \langle q \rangle^\mu [\langle q \rangle^{-\mu}, u(p)] \langle p \rangle^{1+\nu} \langle p \rangle^{-1-\nu} \langle q \rangle^\mu qV.$$

As in (5.15), since $x \mapsto \langle x \rangle^{-1-\nu}$ is in $L^2(\mathbb{R}^\nu)$, we have that $\langle p \rangle^{-1-\nu} \langle q \rangle^\mu qV : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded and $\langle q \rangle^\mu [\langle q \rangle^{-\mu}, u(p)] \langle p \rangle^{1+\nu} \langle p \rangle^{-1-\nu} \langle q \rangle^\mu qV : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded.

Thus we can write $[qV, u(p)] = \langle q \rangle^{-\mu} B$ where $B : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ is bounded. Thus, for all ξ a real function of class $C^\infty(\mathbb{R}^\nu)$ such that $\xi(x) = 0$ if $|x| < 1$ and $\xi(x) = 1$ if $|x| > 2$, we have

$$\begin{aligned}
\|\xi(q/r)[qV, u(p)]\|_{\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})} &= \|\xi(q/r)\langle q \rangle^{-\mu} B\|_{\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})} \\
&\leq \langle r \rangle^{-\mu} \|B\|_{\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})}.
\end{aligned}$$

In particular, by (5.15), V satisfies (3.1) and, by Proposition 3.2, V is of class $C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$. \square

Proof of Lemma 5.10. — Remark that we can write $V(x) = \frac{x}{|x|} \nabla W(x)$ where

$$W(x) = \sum_{n=2}^{+\infty} \sqrt{n}^{-1} \chi(\sqrt{n}^{3\nu}(|x| - n)).$$

- (1) As $\lim_{|x| \rightarrow +\infty} W(x) = 0$, writing $\nabla W(q) = i[p, W(q)]$, we have that

$$V : \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$$

is compact.

(2) Let $\mu > 0$.

$$\int_{\mathbb{R}^\nu} \left| |x|^{1+\mu} W(x) \right|^2 dx = \sum_{n=2}^{\infty} n^{-1} \int_{\mathbb{R}^\nu} |x|^{2+2\mu} \chi^2(n^{3\nu/2}(|x| - n)) dx.$$

Since $\chi(|x|) = 0$ if $|x| > 1$, there is $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^\nu} \left| |x|^{1+\mu} W(x) \right|^2 dx &\leq C \sum_{n=2}^{\infty} n^{-1} (n + n^{-3\nu/2})^{2+2\mu} \int_{n-n^{-3\nu/2}}^{n+n^{-3\nu/2}} r^{\nu-1} dr \\ &\leq 2C \sum_{n=2}^{\infty} (n + n^{-3\nu/2})^{1+2\mu+\nu} n^{-3\nu/2-1}. \end{aligned}$$

In particular, since $\nu \geq 3$, for $\mu > 0$ sufficiently small, this sum is finite, and we can conclude that $\langle q \rangle^{1+\mu} W \in L^2$. Therefore

$$\langle x \rangle^{1+\mu} V(x) = \langle x \rangle^{1+\mu} \frac{x}{|x|} \nabla W = \frac{x}{|x|} \nabla \{ \langle \cdot \rangle^{1+\mu} W \} (x) - (1 + \mu) |x| \langle x \rangle^{\mu-1} W(x).$$

Since $x \mapsto |x| \langle x \rangle^{\mu-1} W(x)$ is in L^2 , we have that $x \mapsto \langle x \rangle^{1+\mu} V(x)$ is in \mathcal{H}^{-1} . Thus, by Lemma 5.8, $V \in C^{1,1}(A_u, \mathcal{H}^1, \mathcal{H}^{-1})$.

(3) Let $N \in \mathbb{N}^*$. Then

$$f(x) = \langle x \rangle^{-(\nu+1)/2}$$

and

$$g_N(x) = N^{\nu/4+1/2} \chi'(N^{3\nu/2}(|x| - N)).$$

Remark that $f \in \mathcal{H}^2$ and

$$\begin{aligned} \int_{\mathbb{R}^\nu} g_N^2(x) dx &= N^{\nu/2+1} \int_{\mathbb{R}^\nu} \chi'(N^{3\nu/2}(|x| - N))^2 dx \\ &\leq CN^{\nu/2+1} \int_{\mathbb{R}^\nu} r^{\nu-1} \chi'(N^{3\nu/2}(r - N))^2 dx \\ &\leq CN^{\nu/2+1} (N+1)^{\nu-1} N^{-3\nu/2} \int_{-1}^1 \chi'(t)^2 dt \\ &\leq C. \end{aligned} \tag{5.16}$$

Thus $\|g_N\|_{L^2} \leq C$.

To simplify notation, let $I = [N - N^{-3\nu/2}, N + N^{-3\nu/2}]$. Then

$$\begin{aligned}
\int_{\mathbb{R}^\nu} f(x)V(x)g_N(x)dx &= N^{\nu/4+1/2}N^{(3\nu-1)/2} \\
&\int_{\mathbb{R}^\nu} \langle x \rangle^{-(\nu+1)/2} \chi'(N^{3\nu/2}(|x| - N))^2 dx \\
&\geq N^{\nu/4+1/2}N^{(3\nu-1)/2} \langle N + N^{-3\nu/2} \rangle^{-(\nu+1)/2} \\
&\int_{|x| \in I} \chi'(N^{3\nu/2}(|x| - N))^2 dx \\
&\geq CN^{\nu/4+1/2}N^{(3\nu-1)/2} \langle N + N^{-3\nu/2} \rangle^{-(\nu+1)/2} \\
&\int_{r \in I} r^{\nu-1} \chi'(N^{3\nu/2}(r - N))^2 dr \\
&\geq C \langle N + 1 \rangle^{-(\nu+1)/2} (N - 1)^{5\nu/4-1} \\
&\int_{r \in I} \chi'(N^{3\nu/2}(r - N))^2 dr \\
&\geq C \langle N + 1 \rangle^{-(\nu+1)/2} (N - 1)^{5\nu/4-1} \int_{r \in [-1,1]} \chi'(r)^2 dr
\end{aligned}$$

Therefore, since $\nu \geq 3$, we have that $\lim_{N \rightarrow +\infty} \int_{\mathbb{R}^\nu} f(x)V(x)g_N(x)dx = +\infty$.

Since $\|f\|_{\mathcal{H}^2} \|g_N\|_{L^2} \leq C$, we have that V is not Δ -bounded.

(4) By a simple calculus, we have

$$x \cdot \nabla V(x) = |x| \sum_{n=2}^{+\infty} n^{3\nu/2} n^{(3\nu-1)/2} \chi''(n^{3\nu/2}(|x| - n)).$$

Let $N \in \mathbb{N}^*$ and

$$f(x) = \langle x \rangle^{-(\nu+1)/2} \quad \text{and} \quad g_N(x) = N^{\nu/4+1/2} N^{-3\nu/2} \chi''(N^{3\nu/2}(|x| - N)).$$

Remark that $f \in \mathcal{H}^1$ and by (5.16) $\|g_N\|_{\mathcal{H}^1} \leq C$.

To simplify notation, let $I = [N - N^{-3\nu/2}, N + N^{-3\nu/2}]$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^\nu} f(x)x \cdot \nabla V(x)g_N(x)dx \\
&= N^{\nu/4+1/2}N^{(3\nu-1)/2} \int_{\mathbb{R}^\nu} |x| \langle x \rangle^{-(\nu+1)/2} \chi''(N^{3\nu/2}(|x| - N))^2 dx \\
&\geq N^{\nu/4+1/2}N^{(3\nu-1)/2} \langle N + N^{-3\nu/2} \rangle^{-(\nu+1)/2} \int_{|x| \in I} \chi''(N^{3\nu/2}(|x| - N))^2 dx \\
&\geq CN^{\nu/4+1/2}N^{(3\nu-1)/2} \langle N + N^{-3\nu/2} \rangle^{-(\nu+1)/2} \int_{r \in I} r^{\nu-1} \chi''(N^{3\nu/2}(r - N))^2 dr \\
&\geq C \langle N + 1 \rangle^{-(\nu+1)/2} (N - 1)^{5\nu/4-1} \int_{r \in I} \chi''(N^{3\nu/2}(r - N))^2 dr \\
&\geq C \langle N + 1 \rangle^{-(\nu+1)/2} (N - 1)^{5\nu/4-1} \int_{r \in [-1,1]} \chi''(r)^2 dr.
\end{aligned}$$

Thus, since $\nu \geq 3$, we have that $\lim_{N \rightarrow +\infty} \int_{\mathbb{R}^\nu} f(x)x \cdot \nabla V(x)g_N(x)dx = +\infty$ with $\|f\|_{\mathcal{H}^1} \|g_N\|_{\mathcal{H}^1} \leq C$. Thus $V \notin C^1(A_D, \mathcal{H}^1, \mathcal{H}^{-1})$. \square

6. FLOW

In this section we make a comment concerning the unitary group generated by the operator A_u which could be useful in checking the $C^{1,1}(A_u)$ property, subject that however we shall not pursue further in this note. For A_u as in (2.3) one may give an explicit description of $e^{i\tau A_u}$ in terms of the classical flow generated by the vector field u as follows (we refer to Subsection 4.2 in [1] for details). For each $x \in X$ denote $\phi_\tau(x)$ the solution of the system

$$\begin{cases} \frac{d}{d\tau} \phi_\tau(x) = u(\phi_\tau(x)) \\ \phi_0(x) = x \end{cases}, \quad (6.1)$$

which exists for all real τ and $\phi_\tau(x)$ is a C^∞ function of τ, x . Then $\phi_\tau : X \rightarrow X$ is a C^∞ diffeomorphism and we have $\phi_\sigma \circ \phi_\tau = \phi_{\sigma+\tau}$.

Remark that because $\mathcal{F}A_u\mathcal{F}^{-1} = \frac{1}{2}(p \cdot u(q) + u(q) \cdot p)$, we have that A_u is essentially self-adjoint (see [1, Proposition 4.2.3]). Moreover, because $\mathcal{F}^{-1}\mathcal{H}_t^s\mathcal{F} = \mathcal{H}_t^s$, for $u \in \mathcal{U}$, the C_0 groups $e^{i\tau A_u}$ leaves invariant the \mathcal{H}_t^s spaces (see [1, Proposition 4.2.4]).

Denote $\phi'_\tau(x)$ the derivative of ϕ_τ at the point x , so that $\phi'_\tau(x) : X \rightarrow X$ is a linear map with $J_\tau(x) = \det \phi'_\tau(x) > 0$. Then:

$$\mathcal{F}e^{i\tau A_u}\mathcal{F}^{-1}f = J_\tau^{1/2}f \circ \phi_\tau \quad (6.2)$$

where \mathcal{F} is the Fourier transformation.

For the operator A_N given by (4.2) it suffices to consider the one dimensional case, because of the factorization properties mentioned at the beginning of Section 4. Then $X = \mathbb{R}$ and $u(k) = \sin(ak)$. For simplicity, and without loss of generality, we take $a = 1$. Then the system (6.1) has an elementary solution: if $0 \leq x \leq \pi$ then

$$\phi_\tau(x) = \arccos\left(\frac{(1 - e^{2\tau}) + \cos(x)(1 + e^{2\tau})}{(1 + e^{2\tau}) + \cos(x)(1 - e^{2\tau})}\right) = 2 \arctan\left(e^\tau \tan\left(\frac{x}{2}\right)\right) \quad (6.3)$$

and similarly outside $[0, \pi]$. Note that if $x = k\pi$ with $k \in \mathbb{Z}$ then $\phi_\tau(x) = x$.

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