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EMBEDDINGS AND THE (VIRTUAL) COHOMOLOGICAL DIMENSION OF THE BRAID AND MAPPING CLASS GROUPS OF SURFACES

DACIBERG LIMA GONÇALVES, JOHN GUASCHI, AND MIGUEL MALDONADO

Abstract. We use the relations between the braid and mapping class groups of a compact, connected, non-orientable surface N without boundary and those of its orientable double covering S to study embeddings of these groups and their (virtual) cohomological dimensions. We first generalise results of [4, 14] to show that the mapping class group $MCG(N;k)$ of N relative to a k-point subset embeds in the mapping class group $MCG(S;2k)$ of S relative to a $2k$-point subset. We then compute the cohomological dimension of the braid groups of all compact, connected aspherical surfaces without boundary, generalising results of [15]. Finally, if the genus of N is at least 2, we deduce upper bounds for the virtual cohomological dimension of $MCG(N;k)$ that are coherent with computations of Ivanov.

1. Introduction

Let S be a compact, connected surface, and let $X = \{x_1, \ldots, x_k\}$ be a finite (possibly empty) subset of S of cardinality $k \geqslant 0$. Let $\mathrm{Top}(S;X)$ denote the group of homeomorphisms of S for the operation of composition that leave X invariant. If S is orientable, let $\mathrm{Top}^+(S;X)$ denote the set of elements of $\mathrm{Top}(S;X)$ that preserve orientation. Note that $\mathrm{Top}^+(S;X)$ is a subgroup of $\mathrm{Top}(S;X)$ of index two. We define the *mapping class group $MCG(S;X)$ of S relative to X* by:

$$MCG(S;X) = \begin{cases} \pi_0 \mathrm{Top}^+(S;X) & \text{if } S \text{ is orientable} \\ \pi_0 \mathrm{Top}(S;X) & \text{if } S \text{ is non orientable.} \end{cases} \tag{1.1}$$

If S is orientable (resp. non orientable), the group $MCG(S;X)$ is thus the set of isotopy classes of $\mathrm{Top}^+(S;X)$ (resp. $\mathrm{Top}(S;X)$), the isotopies being relative to the set X. If $f \in \mathrm{Top}^+(S;X)$ (resp. $\mathrm{Top}(S;X)$) then we let $[f]$ denote its mapping class in $MCG(S;X)$. Up to isomorphism, $MCG(S;X)$ only depends on the cardinality k of the subset X, and we shall often denote this group by $MCG(S;k)$. If X is empty, then we omit it from the notation, and shall just write $\mathrm{Top}(S)$, $\mathrm{Top}^+(S)$, $MCG(S)$ etc. The mapping class group may also be defined in other categories (PL category, smooth category), the groups obtained being isomorphic [5]. The mapping class group has been widely studied from different points of view – see [1, 9, 10, 24, 30] for example.

If $k \in \mathbb{N}$, there is a surjective homomorphism $\sigma\colon \mathrm{Top}(S;X) \longrightarrow \Sigma_k$ defined by $f(x_i) = x_{(\sigma(f))(i)}$ for all $f \in \mathrm{Top}(S;X)$ and $i \in \{1, \ldots, k\}$, where Σ_k is the symmetric group on k letters. This homomorphism induces a surjective homomorphism from $MCG(S;X)$ to Σ_k that we also denote by σ, and its kernel is called the *pure mapping class group of S relative to X*, denoted by $PMCG(S;X)$.

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We have a short exact sequence:

$$1 \longrightarrow PMCG(S;X) \longrightarrow MCG(S;X) \xrightarrow{\sigma} \Sigma_k \longrightarrow 1, \tag{1.2}$$

and if $f \in \mathrm{Top}(S;X)$ then $[f] \in PMCG(S;X)$ if and only if f fixes X pointwise.

Mapping class groups are closely related to surface braid groups. If $k \in \mathbb{N}$, the k^{th} *ordered configuration space* $F_k(S)$ of S is the set of all ordered k-tuples of distinct points of S that we may consider as a subspace of the k-fold product of S with itself. The group Σ_k acts freely on $F_k(S)$ by permuting coordinates, and the associated quotient is the k^{th} *unordered configuration space* of S, denoted by $D_k(S)$. The fundamental group $\pi_1 D_k(S)$ (resp. $\pi_1 F_k(S)$), denoted by $B_k(S)$ (resp. $P_k(S)$) is the *braid group* (resp. *pure braid group*) of S on k strings [8, 11], and $B_k(S)$ and $P_k(S)$ are related by a short exact sequence similar to that of (1.2). It is well known that $B_k(\mathbb{R}^2)$ (resp. $P_k(\mathbb{R}^2)$) is isomorphic to *Artin's braid group* (resp. *Artin's pure braid group*) on k strings. If $\pi : \widetilde{S} \longrightarrow S$ is a d-fold covering map, where $d \in \mathbb{N}$, then for all $k \geqslant 1$, there is a continuous map:

$$\varphi_k : D_k(S) \longrightarrow D_{kd}(\widetilde{S}) \tag{1.3}$$

defined by $\varphi_k(A) = \pi^{-1}(A)$ for all $A \in D_k(S)$, and the induced homomorphism $\varphi_{k\#} : B_k(S) \longrightarrow B_{kd}(\widetilde{S})$ on the level of fundamental groups is injective [14]. Taking $\pi : \mathbb{S}^2 \longrightarrow \mathbb{R}P^2$ to be the standard 2-fold (universal) covering, where $\mathbb{S}^2$ is the 2-sphere and $\mathbb{R}P^2$ is the real projective plane, this result was then applied in [14] to classify the isomorphism classes of the finite subgroups of $B_k(\mathbb{R}P^2)$, and to show that $B_k(\mathbb{R}P^2)$ and $MCG(\mathbb{R}P^2;k)$ are linear groups for all $k \in \mathbb{N}$.

If $g \geqslant 0$ (resp. $g \geqslant 1$), let $S = S_g$ (resp. $N = N_g$) be a compact, connected orientable (resp. non-orientable) surface of genus g without boundary. In the non-orientable case, g is the number of projective planes in a connected sum decomposition. If $g \geqslant 1$ and $k \geqslant 0$, the orientable double covering $\pi : S_{g-1} \longrightarrow N_g$ induces a homomorphism $\phi_k : MCG(N_g;k) \longrightarrow MCG(S_{g-1};2k)$, where the $2k$-point subset of marked points in S is equal to the inverse image by π of a k-point subset of N. One of the main aims of this paper is to generalise the injectivity result of [14] to this homomorphism. If $k = 0$ and $g \geqslant 3$, it was shown in [4, 22] that there exists an injective homomorphism $\phi : MCG(N_g) \longrightarrow MCG(S_{g-1})$, and that via ϕ, $MCG(N_g)$ may be identified with the subgroup of $MCG(S_{g-1})$ that consists of isotopy classes of symmetric homeomorphisms. In Section 3, we show that a similar result holds for all $k \geqslant 0$, namely that the homomorphism ϕ_k induced by π is injective, and that via ϕ_k, $MCG(N_g;k)$ may be identified with the subgroup of $MCG(S_{g-1};2k)$ that consists of isotopy classes of symmetric homeomorphisms.

Theorem 1.1. — *Let $k, g \in \mathbb{N}$, let $N = N_g$ be a compact, connected, non-orientable surface of genus g without boundary, and S its orientable double covering. The homomorphism $\phi_k : MCG(N;k) \longrightarrow MCG(S;2k)$ induced by the covering $\pi : S \longrightarrow N$ is injective; if $g \geqslant 3$ we have the following commutative diagram:*

$$\begin{tikzcd}
1 \arrow[r] & B_k(N) \arrow[r] \arrow[d, hook, "\varphi_{k\#}"] & MCG(N;k) \arrow[r, "\tau_k"] \arrow[d, hook, "\phi_k"] & MCG(N) \arrow[r] \arrow[d, hook, "\phi"] & 1 \\
1 \arrow[r] & B_{2k}(S) \arrow[r] & MCG(S;2k) \arrow[r, "\widetilde{\tau}_k"] & MCG(S) \arrow[r] & 1,
\end{tikzcd} \tag{1.4}$$

where τ_k and $\widetilde{\tau}_k$ are the homomorphisms induced by forgetting the markings on the sets of marked points.

Note that in contrast with [4], Theorem 1.1 holds for all $g \geqslant 1$, not just for $g \geqslant 3$. As we recall in Remark 2.4, the result of [4] does not hold if $g = 2$. The proof of this exceptional case $g = 2$ in Theorem 1.1 will turn out to be the most difficult, in part due to the non injectivity of $\phi : MCG(N_2) \longrightarrow MCG(S_1)$.

In Section 4, we compute the cohomological dimension of the braid groups of all compact surfaces without boundary different from $\mathbb{S}^2$ and $\mathbb{R}P^2$, and we give an upper bound for the virtual cohomological dimension of the mapping class group $MCG(N_g; k)$ for all $g \geqslant 2$ and $k \geqslant 1$. Recall that the *virtual cohomological dimension* $\mathrm{vcd}(G)$ of a group G is a generalisation of the cohomological dimension $\mathrm{cd}(G)$ of G, and is defined to be the cohomological dimension of any torsion-free finite index subgroup of G [6]. In particular, if G is a braid or mapping class group of finite (virtual) cohomological dimension then the corresponding pure braid or mapping class group has the same (virtual) cohomological dimension. In [15, Theorem 5], it was shown that if $k \geqslant 4$ (resp. $k \geqslant 3$) then $\mathrm{vcd}(B_k(\mathbb{S}^2)) = k-3$ (resp. $\mathrm{vcd}(B_k(\mathbb{R}P^2)) = k-2$). These results are generalised in the following theorem, the proof being a little more straightforward due to the fact that the braid groups of S_g and N_{g+1} are torsion free if $g \geqslant 1$.

Theorem 1.2. — *Let $g, k \geqslant 1$. Then:*

$$cd(B_k(S_g)) = cd(P_k(S_g)) = cd(B_k(N_{g+1})) = cd(P_k(N_{g+1})) = k + 1.$$

We then turn our attention to the mapping class groups. If $2g + k > 2$, J. Harer proved that [21, Theorem 4.1]:

$$\mathrm{vcd}(MCG(S_g; k)) = \begin{cases} 4g-5 & \text{if } k = 0 \\ k-3 & \text{if } g = 0 \text{ and } k > 0 \\ 4g+k-4 & \text{if } g, k > 0, \end{cases}$$

and in the non-orientable case, N. Ivanov showed that [23, Theorem 6.9]:

$$\mathrm{vcd}(MCG(N_g; k)) = \begin{cases} 0 & \text{if } g = 1 \text{ and } k \leqslant 2 \\ k-2 & \text{if } g = 1 \text{ and } k \geqslant 3 \\ k & \text{if } g = 2 \\ 2g-3 & \text{if } g \geqslant 3 \text{ and } k = 0 \\ 2g+k-2 & \text{if } g \geqslant 3 \text{ and } k \geqslant 1. \end{cases} \tag{1.5}$$

We believe that there is a typographical error in the last case of [23, Theorem 6.9], and that '$k = 1$' should read '$k \geqslant 1$' as above. Using the embedding of $MCG(N_g)$ in $MCG(S_{g-1})$ given in [4], G. Hope and U. Tillmann showed that if $g \geqslant 3$ then $\mathrm{vcd}(MCG(N_g)) \leqslant 4g - 9$ [22, Corollary 2.2]. Using [15, Theorem 5], it was shown in [15, Corollary 6] that if $k \geqslant 4$ (resp. $k \geqslant 3$) then $\mathrm{vcd}(MCG(\mathbb{S}^2; k)) = k - 3$ (resp. $\mathrm{vcd}(MCG(\mathbb{R}P^2; k)) = k - 2$). In the case of $\mathbb{S}^2$, we thus recover the results of Harer. For non-orientable surfaces, our methods shed light on the cohomological structure of $MCG(N_g; k)$. In particular, we obtain the following result.

Corollary 1.3. — *Let $k > 0$. The mapping class groups $MCG(N_g; k)$ and $PMCG(N_g; k)$ have the same (finite) virtual cohomological dimension. Further:*

(a) *$vcd(MCG(N_2; k)) = k$.*
(b) *if $g \geqslant 3$ then $vcd(MCG(N_g; k)) \leqslant 4g + k - 8$.*

Part (a) reproves (1.5) in the case of the Klein bottle ($g = 2$), and part (b) is coherent with the result of (1.5) for $g \geqslant 3$.

This paper is organised as follows. In Section 2, we recall some definitions and results about orientation-true mappings and the orientable double covering of non-orientable surfaces, we describe how we lift an element of $\mathrm{Top}(N)$ to one of $\mathrm{Top}^+(S)$ in a continuous manner, and we show how this correspondence induces a homomorphism from the mapping class group of a non-orientable surface to that of its orientable double covering (Proposition 2.3). In Section 3, we prove Theorem 1.1 using long exact sequences of fibrations involving the groups that appear in equation (1.1) and the corresponding braid groups [3, 33]. In most cases, using [14, 20], we obtain commutative diagrams of short exact sequences, and the conclusion is obtained in a straightforward manner. The situation is however much more complicated in the case where N is the Klein bottle and S is the 2-torus T, due partly to the fact that the exact sequences that appear in the associated commutative diagrams are no longer short exact. This case requires a detailed analysis, notably in the case $k = 1$. In Section 4, we prove Theorem 1.2 and Corollary 1.3. Finally, in an Appendix, we provide presentations of $P_2(T)$ and $B_2(T)$ that we use in the proof of Theorem 1.1.

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2. Orientation-true mappings and the orientable double covering

Let M, N be manifolds with base points x_0, y_0 respectively. Following [29], a pointed map $f : (M, x_0) \longrightarrow (N, y_0)$ is called *orientation true* if the induced homomorphism

$$f_* : \pi_1(M, x_0) \longrightarrow \pi_1(N, y_0)$$

sends orientation-preserving (resp. orientation-reversing) elements to orientation-preserving (resp. orientation-reversing) elements. In other words, the map f is orientation-true if for all $\alpha \in \pi_1(M, x_0)$, either α and $f_*(\alpha)$ are both orientation preserving or they are are both orientation reversing. In the case of a branched covering $f : M \longrightarrow N$ it follows by [16, Proposition 1.4] that f is orientation-true. In what follows, we will be interested in the case of maps between surfaces. We will start with the case of non-orientable surfaces.

Lemma 2.1. — *Every homeomorphism $f : N \longrightarrow N$ of a non-orientable surface is orientation true. Consequently, the subgroup of all orientation-preserving loops is invariant with respect to f_*.*

Lemma 2.1 is an obvious consequence of [16, Proposition 1.4], but we shall give a more direct proof.

Proof of Lemma 2.1. — Let $x_0 \in N$ be a base point and consider $\alpha \in \pi_1(N, x_0)$, which we represent by a loop $h : S^1 \longrightarrow N$. We fix a local orientation O_1 at x_0, and we consider the orientation on N induced by f, denoted by O_2. Moving the orientation O_1 along the path h by a finite sequence of small open sets U_i, the images $f(U_i)$ may be used to transport the local orientation O_2 along the path $f \circ h$.

Consider the maps $h, f \circ h : S^1 \longrightarrow N$ and the tangent bundle TN of N. Pulling back by these maps, we obtain bundles over S^1 that are homeomorphic, and that are the trivial bundle if the loop is orientation-preserving or the twisted bundle if the loop is orientation-reversing. □

Let $g \geqslant 1$, and let N_g be as defined in the introduction. The unique orientable double covering of N_g may be constructed as follows (see [27] for example). Let the orientable surface S_{g-1} be embedded in $\mathbb{R}^3$ in such a way that it is invariant under the reflections with respect to the xy-, yz- and xz-planes. Consider the involution $J : S_{g-1} \longrightarrow S_{g-1}$ defined by $J(x, y, z) = (-x, -y, -z)$. The orbit space $S_{g-1}/\langle J\rangle$ is homeomorphic to the surface N_g, and the associated quotient map $\pi : S_{g-1} \longrightarrow N_g$ is a double covering.

To simplify the notation, from now on, we will drop the subscripts g and $g-1$ from N and S respectively unless there is risk of confusion, so S will be the orientable double covering of N. As indicated previously, the map π gives rise to a map on the level of configuration spaces that induces an injective homomorphism $B_k(N) \longrightarrow B_{2k}(S)$ of the corresponding braid groups, and this allows us to study the braid groups of a non-orientable surface in terms of those of its orientable double covering [14].

The following result is an immediate consequence of Lemma 2.1 and is basically contained in [22, Key-Lemma 2.1].

Lemma 2.2. — *Let $f : N \longrightarrow N$ be a homeomorphism of a non-orientable surface, and let $\pi : S \longrightarrow N$ be the orientable double covering. Then f admits a lift, and the number of lifts is exactly two.*

Proof. — Using Lemma 2.1, we know that $f_\#|_{\pi_1(S)}(\pi_1(S)) \subset \pi_1(S)$, where we identify $\pi_1(S)$ with a subgroup of $\pi_1(N)$. By basic properties of covering spaces, the map f lifts to a map $f' : S \longrightarrow S$, and since $S \longrightarrow N$ is a double covering, there are two lifts. □

The lifts of f are in one-to-one correspondence with the group $\langle J\rangle$ of deck transformations. There is thus a natural way to choose a lift in a continuous manner simply by choosing $\tilde{f}$ to be the lift of degree 1 (the other lift is of degree -1 since J is of degree -1). Let $\rho : \mathrm{Top}(N) \longrightarrow \mathrm{Top}^+(S)$ denote this choice of lift. We may use Lemma 2.2 to compare mapping class groups of orientable and non-orientable surfaces.

Proposition 2.3. — *Let N be a non-orientable surface, and let S be its orientable double covering. Then there is a homomorphism $\phi : MCG(N) \to MCG(S)$ such that the following diagram commutes:*

$$\begin{tikzcd} Top(N) \arrow[r, "\rho"] \arrow[d] & Top^{+}(S) \arrow[d] \\ MCG(N) \arrow[r, "\phi"] & MCG(S). \end{tikzcd} \tag{2.1}$$

Further, if the genus of N is greater than or equal to 3 then ϕ is injective.

Proof. — If $f, g \in \mathrm{Top}(N)$ are isotopic homeomorphisms then an isotopy between them lifts to an isotopy between the orientation-preserving homeomorphisms $\rho(f)$ and $\rho(g)$ of S. This proves the first part of the statement. For the second part, suppose that the genus of N is greater than or equal to 3. Let $MCG^{\pm}(S)$ denote the extended mapping class group of S consisting of the isotopy classes of all homeomorphisms of S, and let $C\langle J\rangle$ be the subgroup of $MCG^{\pm}(S)$ defined by:

$$C\langle J\rangle = \left\{ [f] \in MCG^{\pm}(S) \;\middle|\; \begin{array}{l} f \in \mathrm{Top}(S), \text{ and there exists } f' \simeq f \\ \text{such that } f' \circ J = J \circ f' \end{array} \right\},$$

where $f' \simeq f$ means that f' is isotopic to f. By [4], $MCG(N) \cong C\langle J\rangle/\langle [J]\rangle$. Let $\pi\colon C\langle J\rangle \longrightarrow C\langle J\rangle/\langle [J]\rangle$ denote the quotient map. Let $f \in \mathrm{Top}(N)$. Then $\phi([f]) = [\rho(f)]$ by diagram (2.1). By Lemma 2.2 and the comment that follows it, f admits exactly two lifts, $\rho(f)$ and $J \circ \rho(f)$, the first (resp. second) preserving (resp. reversing) orientation. Now $\rho(f) \circ J$ is an orientation-reversing lift of f, from which we conclude that $J \circ \rho(f) = \rho(f) \circ J$. Hence $[\rho(f)] \in C\langle J\rangle \cap MCG(S)$, and so $\mathrm{Im}(\phi) \subset C\langle J\rangle \cap MCG(S)$. On the other hand, $\pi \left|_{C\langle J\rangle \cap MCG(S)}\right. : C\langle J\rangle \cap MCG(S) \longrightarrow MCG(N)$ is an isomorphism using the proof of [22, Key-Lemma 2.1]. It then follows that $\pi \circ \phi = \mathrm{Id}_{MCG(N)}$, in particular ϕ is injective. □

In the case of surfaces with marked points, there is another continuous way to choose a lift. For example, given a finite subset $X = \{x_1, \ldots, x_k\}$ of N, denote its preimage under π by:

$$\widetilde{X} = \{x_1', x_1'', x_2', x_2'', \ldots, x_k', x_k''\},$$

where $\{x_i', x_i''\} = \pi^{-1}(x_i)$ for all $1 \leqslant i \leqslant k$. Pick a subset of $\widetilde{X}$ that contains exactly one element of $\{x_i', x_i''\}$, denoted by $\overline{x}_i$, for all $1 \leqslant i \leqslant k$. If $f \in \mathrm{Top}(N; X)$ then the restriction of f to X is a given permutation of X, and if $j \in \{1, \ldots, k\}$ is such that $f(x_1) = x_j$, we may define $\widetilde{f}$ to be the unique lift of f such that $\widetilde{f}(\overline{x}_1) = \overline{x}_j$. The two choices for $\overline{x}_j$ correspond to the two possible lifts of f.

Remark 2.4. — Suppose that the genus of N is 2, in which case N is the Klein bottle K. If T is the torus, the homomorphism $\phi\colon MCG(K) \longrightarrow MCG(T)$ is not injective, and it may be described as follows. Set $\pi_1(K) = \left\langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} \right\rangle$. The group $MCG(K)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ [26], and its elements are given by the mapping classes of the following automorphisms:

$$\begin{cases} \alpha \longmapsto \alpha \\ \beta \longmapsto \beta, \end{cases} \qquad \begin{cases} \alpha \longmapsto \alpha \\ \beta \longmapsto \alpha\beta, \end{cases} \qquad \begin{cases} \alpha \longmapsto \alpha \\ \beta \longmapsto \beta^{-1}, \end{cases} \qquad \begin{cases} \alpha \longmapsto \alpha \\ \beta \longmapsto \alpha\beta^{-1}. \end{cases} \tag{2.2}$$

Let $\pi : T \longrightarrow K$ be the orientable double covering of K and let $\pi_1(T) = \langle a, b \mid [a, b] \rangle$, so that $\pi_\#(a) = \alpha$ and $\pi_\#(b) = \beta^2$. Given a map $f : K \longrightarrow K$ such that the induced homomorphism on the fundamental group is given by $f_\#(\alpha) = \alpha^r$, $f_\#(\beta) = \alpha^u\beta^v$, if v is odd, then f lifts to a map from T to T, and there are exactly two lifts. The matrices of the induced homomorphisms of these lifts on $\pi_1(T)$ are $\left(\begin{smallmatrix} r & 0 \\ 0 & v \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -r & 0 \\ 0 & v \end{smallmatrix}\right)$. Observe that the determinant of one of these two matrices is positive. Identifying $MCG(T)$ with $SL(2, \mathbb{Z})$, we conclude that ϕ sends the first (resp. second) two automorphisms of equation (2.2) to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (resp. to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). In particular, the second part of Proposition 2.3 does not hold in this case.

3. Embeddings of mapping class groups

As in all of this paper, the surfaces N and S under consideration are compact and without boundary, and $\pi : S \longrightarrow N$ is the double covering defined in Section 2. If X is a finite k-point subset of N, $\widetilde{X} = \pi^{-1}(X)$ and $f \in \mathrm{Top}(N; X)$, then we define a map $\rho_k : \mathrm{Top}(N, X) \longrightarrow \mathrm{Top}^+(S, \widetilde{X})$ by $\rho_k(f) = \rho(f)$, where we consider f to be an element of $\mathrm{Top}(N)$ and ρ is as defined in Section 2. By Proposition 2.3, the map ρ_k induces a homomorphism $\phi_k : MCG(N; X) \longrightarrow MCG(S; \widetilde{X})$ defined by $\phi_k([f]) = [\rho_k(f)]$. On the other hand, the map $\psi_k : \mathrm{Top}(N) \longrightarrow D_k(N)$ (resp. $\widetilde{\psi}_{2k} : \mathrm{Top}^+(S) \longrightarrow D_{2k}(S)$) defined by $\psi_k(f) = f(X)$ for all $f \in \mathrm{Top}(N)$ (resp. $\widetilde{\psi}_{2k}(h) = h(\widetilde{X})$ for all $h \in \mathrm{Top}^+(S)$) is a locally-trivial fibration [3, 28] whose fibre is $\mathrm{Top}(N; k)$ (resp. $\mathrm{Top}^+(S; 2k)$). The long exact sequence in homotopy of these fibrations gives rise to the following exact sequences:

$$B_k(N) \longrightarrow MCG(N; X) \xrightarrow{\tau_k} MCG(N), \quad B_k(S) \longrightarrow MCG(S; \widetilde{X}) \xrightarrow{\widetilde{\tau}_k} MCG(S),$$

where τ_k, $\widetilde{\tau}_k$ are induced by suppressing the markings on X and $\widetilde{X}$ respectively [3]. In Theorem 1.1, we will see that the injectivity of ϕ given by Proposition 2.3 carries over to that of the homomorphism ϕ_k between mapping class groups of marked surfaces and holds for *all* $g \geqslant 1$, in contrast to the non injectivity in the case $g = 2$ described in Remark 2.4.

Let S be a compact, connected surface without boundary, and let $l \in \mathbb{N}$. We define the following subgroups of $\mathrm{Top}(S)$ and $\mathrm{Top}(S; X)$:

(a) if S is orientable and $X \subset S$ is a subset of cardinality l, let:

$$\begin{aligned}\mathrm{Top}^+_F(S; l) &= \{ f \in \mathrm{Top}^+(S) \mid f(x) = x \text{ for all } x \in X \} \\ &= \{ f \in \mathrm{Top}^+(S; X) \mid [f] \in PMCG(S; l) \}.\end{aligned}$$

(b) if S is orientable, $\pi : S \longrightarrow N$ is the orientable double covering, and $X \subset N$ is a subset of cardinality l, let:

$$\mathrm{Top}^+_{\widetilde{F}}(S; 2l) = \left\{ \widetilde{f} \in \mathrm{Top}^+(S) \;\middle|\; \widetilde{f}(\pi^{-1}(x)) = \pi^{-1}(x) \text{ for all } x \in X \right\}.$$

(c) if S is non orientable and $X \subset S$ is a subset of cardinality l, let:

$$\begin{aligned}\mathrm{Top}_F(S; l) &= \{ f \in \mathrm{Top}(S) \mid f(x) = x \text{ for all } x \in X \} \\ &= \{ f \in \mathrm{Top}(S; X) \mid [f] \in PMCG(S; l) \}.\end{aligned}$$

Observe that if S is orientable (resp. non orientable), $\mathrm{Top}^+_F(S;1) = \mathrm{Top}^+(S;1)$ (resp. $\mathrm{Top}_F(S;1) = \mathrm{Top}(S;1)$). Before giving the proof of Theorem 1.1, we state and prove the following two results.

Proposition 3.1. — *Let $k \geqslant 1$, and consider the homomorphism*

$$\widetilde{\psi}_{2k\#} \colon \pi_1(\mathit{Top}^+(T)) \longrightarrow B_{2k}(T)$$

induced by the map $\widetilde{\psi}_{2k}$ defined above. Then $\pi_1(\mathit{Top}^+(T)) \cong \mathbb{Z}^2$, and $\mathit{Im}(\widetilde{\psi}_{2k\#}) = Z(B_{2k}(T)) \subset P_{2k}(T)$.

Proof. — First, by [18, Theorem 2, p. 63]), $\pi_1(\mathrm{Top}^+(T)) \cong \mathbb{Z}^2$. Secondly, taking $\mathrm{Top}^+(T)$ to be equipped with Id_T as its basepoint, a representative loop of an element $\gamma \in \pi_1(\mathrm{Top}^+(T), \mathrm{Id}_T)$ is a path in $\mathrm{Top}^+(T)$ from Id_T to itself. It follows from the definition of $\widetilde{\psi}_{2k}$ that $\widetilde{\psi}_{2k\#}(\gamma) \in P_{2k}(T)$. It remains to show that $\mathrm{Im}(\widetilde{\psi}_{2k\#}) = Z(B_{2k}(T))$. Since $\mathrm{Im}(\widetilde{\psi}_{2k\#}) \subset P_{2k}(T)$, the homomorphism $\widetilde{\psi}_{2k\#}$ coincides with the homomorphism of [3, Theorem 1]. Using the exact sequence of that theorem and [3, Corollary 1.3], we see that $\mathrm{Im}(\widetilde{\psi}_{2k\#}) = \langle a_1, b_1 \rangle$, where a_1 and b_1 are the generators of $P_{2k}(T)$ defined in [2, Theorem 5]. But by [31, Proposition 4.2], these two elements generate the centre of $B_{2k}(T)$ as required. □

The first part of the following lemma generalises results of [19].

Lemma 3.2. —

(a) *Let S be a compact, connected orientable (resp. non-orientable) surface without boundary for which $\pi_1(\mathit{Top}^+(S;1))$, (resp. $\pi_1(\mathit{Top}(S;1))$) is trivial, and let $l \geqslant 1$. Then $\pi_1(\mathit{Top}^+_F(S;l))$, (resp. $\pi_1(\mathit{Top}_F(S;l))$) is trivial. In particular, $\pi_1(\mathit{Top}^+_F(T;l))$ (resp. $\pi_1(\mathit{Top}_F(K;l))$) is trivial for all $l \geqslant 1$.*

(b) *$\pi_1(\mathit{Top}^+_{\widetilde{F}}(T;2l))$ is trivial for all $l \geqslant 1$.*

Proof. —

(a) Assume first that S is orientable. We prove the result by induction on l. If $l = 1$ then the result holds by the hypothesis, using the fact that $\mathrm{Top}^+_F(S;1) = \mathrm{Top}^+(S;1)$. Suppose by induction that the result holds for some $l \geqslant 1$. The map $\mathrm{Top}^+_F(S;l) \longrightarrow S \setminus \{x_1, \ldots, x_l\}$ given by evaluating an element of $\mathrm{Top}^+_F(S;l)$ at the point x_{l+1} is a fibration with fibre $\mathrm{Top}^+_F(S;l{+}1)$. Taking the long exact sequence in homotopy of this fibration and using the fact that $\pi_2(S \setminus \{x_1, \ldots, x_l\})$ is trivial, it follows that the homomorphism $\pi_1(\mathrm{Top}^+_F(S;l+1)) \longrightarrow \pi_1(\mathrm{Top}^+_F(S;l))$ induced by the inclusion of the fibre in the total space is injective, and since $\pi_1(\mathrm{Top}^+_F(S;l))$ is trivial by the induction hypothesis, $\pi_1(\mathrm{Top}^+_F(S;l+1))$ is too. This proves the first part of the statement if S is orientable. If S is non orientable, it suffices to replace $\mathrm{Top}^+_F(S;l)$ by $\mathrm{Top}_F(S;l)$ in the proof of the orientable case. The second part of the statement is a consequence of the first part and [18, Corollary, p. 65] and [19, Theorem 4.1].

(b) Let $l \geqslant 0$. The map $\mathrm{Top}^+_{\widetilde{F}}(T;2l) \longrightarrow D_2(T \setminus \{x_1, \ldots, x_{2l})\}$ that to a homeomorphism $f \in \mathrm{Top}^+_{\widetilde{F}}(T;2l)$ associates the set $\{f(x_{2l+1}), f(x_{2l+2})\}$ is a fibration whose fibre is $\mathrm{Top}^+_{\widetilde{F}}(T;2l+2)$. If $l = 0$ then $\mathrm{Top}^+_{\widetilde{F}}(T;2l)$ is just $\mathrm{Top}^+(T)$. Taking the long exact sequence in homotopy, and using the fact that $D_2(T \setminus \{x_1, \ldots, x_{2l}\})$ is a $K(\pi, 1)$, we obtain the following exact sequence:

$$1 \to \pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T; 2l+2)) \xrightarrow{(\widetilde{q}_{2l+2})_{\#}} \pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T; 2l)) \to B_2(T \setminus \{x_1, \ldots, x_{2l}\}), \tag{3.1}$$

where the homomorphism $(\widetilde{q}_{2l+2})_{\#}$ is induced by the map

$$\widetilde{q}_{2l+2} \colon \mathrm{Top}^{+}_{\widetilde{F}}(T; 2l+2) \longrightarrow \mathrm{Top}^{+}_{\widetilde{F}}(T; 2l)$$

that forgets the marking on the last two points. If $\pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T; 2l))$ is trivial for some $l \geqslant 1$ then clearly $\pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T; 2l+2))$ is also trivial. So applying induction on l, it suffices to prove the result for $l = 1$. It follows from Proposition 3.1 that the map $\pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T)) \longrightarrow B_2(T)$ sends $\pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T))$ isomorphically onto the centre of $B_2(T)$. It follows from exactness of (3.1) that $\pi_1(\mathrm{Top}^{+}_{\widetilde{F}}(T; 2))$ is trivial, and this completes the proof of the lemma. □

Remark 3.3. — If S is a surface that satisfies the hypotheses of Lemma 3.2(a) then S is different from the 2-sphere [28, lines 2–3, p. 303] and the real projective plane [19, Theorem 3.1], so S is an Eilenberg Mac Lane space of type $K(\pi, 1)$.

Proof of Theorem 1.1. — Consider the double covering $\pi : S \longrightarrow N$, fix a k-point subset X of N, and let $\widetilde{X} = \pi^{-1}(X)$. By the comments preceding the statement of the theorem, we obtain the following commutative diagram of fibrations:

$$\begin{tikzcd}
\mathrm{Top}(N; X) \arrow[r] \arrow[d, "\rho_k"] & \mathrm{Top}(N) \arrow[r, "\psi_k"] \arrow[d, "\rho"] & D_k(N) \arrow[d, "\varphi_k"] \\
\mathrm{Top}^{+}(S; \widetilde{X}) \arrow[r] & \mathrm{Top}^{+}(S) \arrow[r, "\widetilde{\psi}_{2k}"] & D_{2k}(S),
\end{tikzcd} \tag{3.2}$$

where ρ is as defined in Section 2, and $\varphi_k : D_k(N) \longrightarrow D_{2k}(S)$ is given by equation (1.3). The left-hand square clearly commutes because ρ_k is the restriction of ρ to $\mathrm{Top}(N; k)$. We claim that the right-hand square also commutes. On the one hand, $\pi(\widetilde{\psi}_{2k} \circ \rho(f)) = \pi \circ \rho(f)(\widetilde{X}) = f \circ \pi(\widetilde{X}) = f(X)$, using the fact that $\rho(f)$ is a lift of f, so $\widetilde{\psi}_{2k} \circ \rho(f) = \rho(f)(\widetilde{X}) \subset \pi^{-1}(f(X)) = \varphi_k \circ \psi_k(f)$. Conversely, if $y \in \pi^{-1}(f(X))$, then there exists $x \in X$ such that $\pi(y) = f(x)$. If $\widetilde{x} \in \widetilde{X}$ is a lift of x then $y \in \{\rho(f)(\widetilde{x}), J \circ \rho(f)(\widetilde{x})\} = \{\rho(f)(\widetilde{x}), \rho(f)(J(\widetilde{x}))\} \subset \rho(f)(\widetilde{X})$, which proves the claim. Let $\Phi_1 \colon \pi_1 \mathrm{Top}(N) \longrightarrow \pi_1 \mathrm{Top}^{+}(S)$ denote the homomorphism induced by ρ on the level of fundamental groups. We now take the long exact sequence in homotopy of the commutative diagram (3.2). The form of the resulting commutative diagram depends on the genus g of N, hence we consider the following three cases.

(a) Suppose that $g \geqslant 3$. Then by [20], $\pi_1 \mathrm{Top}(N)$ and $\pi_1 \mathrm{Top}^{+}(S)$ are trivial, and we obtain the commutative diagram of (1.4). Since $\varphi_{k\#}$ (resp. ϕ) is injective by [14] (resp. by Proposition 2.3), the injectivity of ϕ_k follows from the 5-Lemma.

(b) Suppose that $g = 1$. Using [14, 33], we obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\pi_1 \mathrm{Top}(\mathbb{R}P^2)} \arrow[r] \arrow[d, "\Phi_1"] & B_k(\mathbb{R}P^2) \arrow[r] \arrow[d, "\varphi_{k\#}"] & MCG(\mathbb{R}P^2; k) \arrow[r] \arrow[d, "\phi_k"] & 1 \\ 1 \arrow[r] & \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\pi_1 \mathrm{Top}^+(\mathbb{S}^2)} \arrow[r] & B_{2k}(\mathbb{S}^2) \arrow[r] & MCG(\mathbb{S}^2; 2k) \arrow[r] & 1, \end{tikzcd} \tag{3.3}$$

where in both cases, $\mathbb{Z}/2\mathbb{Z}$ is identified with the subgroup generated by the full-twist braid of the corresponding braid group. Since $\varphi_{k\#}$ is injective by [14], so is Φ_1, and a routine diagram-chasing argument shows that ϕ_k is injective.

(c) Finally suppose that $g = 2$, so that N is the Klein bottle K and S is the torus T. Let $k \geqslant 1$. We claim that $\mathrm{Ker}(\phi_k) \subset PMCG(K; k)$.

To prove this, let $X = \{x_1, \ldots, x_k\}$, and for $i = 1, \ldots, k$, let $y_i \in \pi^{-1}(x_i)$. Let $\sigma\colon MCG(K; k) \longrightarrow \Sigma_k$ and $\widetilde{\sigma}\colon MCG(T; 2k) \longrightarrow \Sigma_{2k}$ denote the usual homomorphisms onto the symmetric groups of X and $\widetilde{X}$ respectively as described in equation (1.2). From the geometric construction of ρ_k, if $1 \leqslant i, j \leqslant k$ and $\sigma([f])(x_i) = x_j$ then

$$\widetilde{\sigma}([\rho_k(f)])(y_i) \in \pi^{-1}(x_j).$$

If $[f] \in \mathrm{Ker}(\phi_k)$, where $f \in \mathrm{Top}(K, X)$, then

$$[\rho_k(f)] = \phi_k([f]) = [\mathrm{Id}_S].$$

In particular, $\widetilde{\sigma}([\rho_k(f)]) = \mathrm{Id}_{\Sigma_{2k}}$, so $\widetilde{\sigma}([\rho_k(f)])(y_i) = y_i$ for all $1 \leqslant i \leqslant k$, and it follows that $\sigma([f])(x_i) = x_i$, whence $[f] \in PMCG(K; k)$ as claimed.

It thus suffices to prove that the restriction $\phi_k \big|_{PMCG(K;k)}$ of ϕ_k to $PMCG(K; k)$ is injective. Now assume that $k \geqslant 2$. Let

$$p_k\colon F_k(K) \longrightarrow F_{k-1}(K) \quad (\text{resp. } q_k\colon \mathrm{Top}_F(K; k) \longrightarrow \mathrm{Top}_F(K; k-1))$$

be the map given by forgetting the last coordinate (resp. the marking on the last point). Let

$$D_{2k}^{(2)}(T) = F_{2k}(T)/\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{k \text{ times}},$$

let $B_{2k}^{(2)}(T) = \pi_1 D_{2k}^{(2)}(T)$. Let $\widetilde{p}_{2k}\colon D_{2k}^{(2)}(T) \longrightarrow D_{2(k-1)}^{(2)}(T)$ be the map given by forgetting the last two coordinates, and let

$$\widetilde{q}_{2k}\colon \mathrm{Top}_{\widetilde{F}}^{+}(T; 2k) \longrightarrow \mathrm{Top}_{\widetilde{F}}^{+}(T; 2(k-1))$$

be the map defined in the proof of Lemma 3.2(b). Then we have the following commutative diagram whose rows are fibrations:

$$
\begin{tikzcd}
\mathrm{Top}^{\pm}_{\widetilde{F}}(T;2(k-1)) \arrow[r] & \mathrm{Top}^{+}(T) \arrow[r, "\widetilde{\delta}_{2(k-1)}"] & D^{(2)}_{2(k-1)}(T) \\
\mathrm{Top}^{\pm}_{\widetilde{F}}(T;2k) \arrow[r] \arrow[u, "\widetilde{q}_{2k}"'] & \mathrm{Top}^{+}(T) \arrow[r, "\widetilde{\delta}_{2k}"] \arrow[u, equal] & D^{(2)}_{2k}(T) \arrow[u, "\widetilde{p}_{2k}"'] \\
\mathrm{Top}_{F}(K;k) \arrow[r] \arrow[u, "\rho_k\big|_{\mathrm{Top}_F(K;k)}"'] \arrow[d, "q_k"] & \mathrm{Top}(K) \arrow[r, "\delta_k"] \arrow[u, "\rho"'] \arrow[d, equal] & F_k(K) \arrow[u, "\widehat{\varphi}_k"'] \arrow[d, "p_k"] \\
\mathrm{Top}_{F}(K;k-1) \arrow[r] \arrow[d, "\rho_{k-1}\big|_{\mathrm{Top}_F(K;k-1)}"] & \mathrm{Top}(K) \arrow[r, "\delta_{k-1}"] \arrow[d, "\rho"] & F_{k-1}(K) \arrow[d, "\widehat{\varphi}_{k-1}"] \\
\mathrm{Top}^{\pm}_{\widetilde{F}}(T;2(k-1)) \arrow[r] & \mathrm{Top}^{+}(T) \arrow[r, "\widetilde{\delta}_{2(k-1)}"] & D^{(2)}_{2(k-1)}(T).
\end{tikzcd}
\tag{3.4}
$$

The map $\delta_k\colon \mathrm{Top}(K) \longrightarrow F_k(K)$ (resp. $\widetilde{\delta}_{2k}\colon \mathrm{Top}^+(T) \longrightarrow D^{(2)}_{2k}(T)$) is defined by:

$$
\delta_k(f) = (f(x_1), \ldots, f(x_k)) \text{ (resp. } \widetilde{\delta}_{2k}(\widetilde{f}) = \big(\widetilde{f}(\pi^{-1}(x_1)), \ldots, \widetilde{f}(\pi^{-1}(x_k))\big)) \tag{3.5}
$$

for all $f \in \mathrm{Top}(K)$ (resp. for all $\widetilde{f} \in \mathrm{Top}^+(T)$), and the map $\widehat{\varphi}_k\colon F_k(K) \longrightarrow D^{(2)}_{2k}(T)$ is defined by

$$
\widehat{\varphi}_k(v_1, \ldots, v_k) = (\pi^{-1}(v_1), \ldots, \pi^{-1}(v_k))
$$

for all $(v_1, \ldots, v_k) \in F_k(K)$. Note also that the diagram remains commutative if we identify the corresponding terms of the first and last rows. Taking the long exact sequence in homotopy of the diagram (3.4) and applying Lemma 3.2, we obtain the following commutative diagram whose rows are exact:

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1\mathrm{Top}^{+}(T) \arrow[r, "(\widetilde{\delta}_{2(k-1)})_{\#}"] & B^{(2)}_{2(k-1)}(T) \arrow[r, "\widetilde{\partial}_{2(k-1)}"] & \pi_0\mathrm{Top}^{\pm}_{\widetilde{F}}(T;2(k-1)) \arrow[r, "\widetilde{\tau}_{2(k-1)}"] & MCG(T) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1\mathrm{Top}^{+}(T) \arrow[r, "(\widetilde{\delta}_{2k})_{\#}"] \arrow[u, equal] & B^{(2)}_{2k}(T) \arrow[r, "\widetilde{\partial}_{2k}"] \arrow[u, "(\widetilde{p}_{2k})_{\#}"'] & \pi_0\mathrm{Top}^{\pm}_{\widetilde{F}}(T;2k) \arrow[r, "\widetilde{\tau}_{2k}"] \arrow[u, "(\widetilde{q}_{2k})_{\#}"'] & MCG(T) \arrow[r] \arrow[u, equal] & 1 \\
1 \arrow[r] & \pi_1\mathrm{Top}(K) \arrow[r, "(\delta_k)_{\#}"] \arrow[u, "\Phi_1"'] \arrow[d, equal] & P_k(K) \arrow[r, "\partial_k"] \arrow[u, "(\widehat{\varphi}_k)_{\#}"'] \arrow[d, "(p_k)_{\#}"] & PMCG(K;k) \arrow[r, "\tau_k"] \arrow[u, "\phi_k\big|_{PMCG(K;k)}"'] \arrow[d, "(q_k)_{\#}"] & MCG(K) \arrow[r] \arrow[u, "\phi"'] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1\mathrm{Top}(K) \arrow[r, "(\delta_{k-1})_{\#}"] \arrow[d, "\Phi_1"] & P_{k-1}(K) \arrow[r, "\partial_{k-1}"] \arrow[d, "(\widehat{\varphi}_{k-1})_{\#}"] & PMCG(K;k-1) \arrow[r, "\tau_{k-1}"] \arrow[d, "\phi_{k-1}\big|_{PMCG(K;k-1)}"] & MCG(K) \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & \pi_1\mathrm{Top}^{+}(T) \arrow[r, "(\widetilde{\delta}_{2(k-1)})_{\#}"] & B^{(2)}_{2(k-1)}(T) \arrow[r, "\widetilde{\partial}_{2(k-1)}"] & \pi_0\mathrm{Top}^{\pm}_{\widetilde{F}}(T;2(k-1)) \arrow[r, "\widetilde{\tau}_{2(k-1)}"] & MCG(T) \arrow[r] & 1,
\end{tikzcd}
\tag{3.6}
$$

where $\widetilde{\partial}_{2(k-1)}, \widetilde{\partial}_{2k}, \partial_k$ and ∂_{k-1} are the boundary homomorphisms associated with the corresponding fibrations of (3.4), $\widetilde{\tau}_{2(k-1)}$, $\widetilde{\tau}_{2k}$, τ_{k-1} and τ_k are the restrictions of these homomorphisms to the given groups, $(\widetilde{q}_{2k})_\#$ and $(q_k)_\#$ are the maps induced by $\widetilde{q}_{2k}$ and q_k respectively on the level of π_0, and $(\widetilde{\delta}_{2(k-1)})_\#$, $(\widetilde{\delta}_{2k})_\#$, $(\delta_k)_\#$ and $(\delta_{k-1})_\#$ are the homomorphisms induced by $\widetilde{\delta}_{2(k-1)}, \widetilde{\delta}_{2k}, \delta_k$ and δ_{k-1} on the level of π_1. The homomorphism ϕ is given by Proposition 2.3, and $(\widehat{\varphi}_k)_\#$ is the restriction of $\varphi_{k\#}$ to $P_k(K)$, so is injective by [14]. We remark that $\pi_1 \mathrm{Top}(K) \cong \mathbb{Z}$ [19, Theorem 4.1] (resp. $\pi_1 \mathrm{Top}^+(T) \cong \mathbb{Z}^2$ [18, Theorem 2]). Once more, the diagram (3.6) remains commutative if we identify the corresponding groups of the first and last rows. In particular, the second and third columns yield respectively:

$$(\widetilde{p}_{2k})_\# \circ (\widehat{\varphi}_k)_\# = (\widehat{\varphi}_{k-1})_\# \circ (p_k)_\# \tag{3.7}$$

$$(\widetilde{q}_{2k})_\# \circ \phi_k \left|_{PMCG(K;k)}\right. = \phi_{k-1} \left|_{PMCG(K;k-1)}\right. \circ (q_k)_\#. \tag{3.8}$$

We recall at this point that our aim is to show that the homomorphism $\phi_k \left|_{PMCG(K;k)}\right.$ is injective. Note that by diagram (3.6), $(\widehat{\varphi}_k)_\# \circ (\delta_k)_\# = (\widetilde{\delta}_{2k})_\# \circ \Phi_1$, and it follows from the exactness of (3.6) and the injectivity of $(\widehat{\varphi}_k)_\#$ that Φ_1 is also injective. We claim that the restriction

$$\widetilde{\partial}_{2k} \Big|_{\mathrm{Ker}((\widetilde{p}_{2k})_\#)} : \mathrm{Ker}((\widetilde{p}_{2k})_\#) \longrightarrow \mathrm{Ker}((\widetilde{q}_{2k})_\#) \tag{3.9}$$

is an isomorphism. Indeed:

(i) It is well defined: if $x \in \mathrm{Ker}((\widetilde{p}_{2k})_\#)$, the commutative diagram (3.6) gives $(\widetilde{q}_{2k})_\# \circ \widetilde{\partial}_{2k}(x) = \widetilde{\partial}_{2(k-1)} \circ (\widetilde{p}_{2k})_\#(x) = 1$, so $\widetilde{\partial}_{2k}(x) \in \mathrm{Ker}((\widetilde{q}_{2k})_\#)$.

(ii) Suppose that $x \in \mathrm{Ker}(\widetilde{\partial}_{2k} \Big|_{\mathrm{Ker}((\widetilde{p}_{2k})_\#)})$. By exactness of the second row of the diagram (3.6), there exists $z \in \pi_1 \mathrm{Top}^+(T)$ such that $(\widetilde{\delta}_{2k})_\#(z) = x$, and by commutativity of the same diagram,

$$(\widetilde{\delta}_{2(k-1)})_\#(z) = (\widetilde{p}_{2k})_\# \circ (\widetilde{\delta}_{2k})_\#(z) = (\widetilde{p}_{2k})_\#(x) = 1.$$

But $(\widetilde{\delta}_{2(k-1)})_\#$ is injective, so $z = 1$, whence $x = 1$. Thus $\widetilde{\partial}_{2k}|_{\mathrm{Ker}((\widetilde{p}_{2k})_\#)}$ is injective too.

(iii) Let $y \in \mathrm{Ker}((\widetilde{q}_{2k})_\#)$. Using commutativity and exactness of the second row of the diagram (3.6), we have $y \in \mathrm{Ker}(\widetilde{\tau}_{2k})$, so there exists $w \in B_{2k}^{(2)}(T)$ such that $\widetilde{\partial}_{2k}(w) = y$, and thus $(\widetilde{p}_{2k})_\#(w) \in \mathrm{Ker}(\widetilde{\partial}_{2(k-1)})$. Hence there exists $z \in \pi_1 \mathrm{Top}^+(T)$ such that

$$(\widetilde{p}_{2k})_\#(w) = (\widetilde{\delta}_{2(k-1)})_\#(z) = (\widetilde{p}_{2k})_\# \circ (\widetilde{\delta}_{2k})_\#(z).$$

It follows that $w(\widetilde{\delta}_{2k})_\#(z^{-1}) \in \mathrm{Ker}((\widetilde{p}_{2k})_\#)$, and that

$$\widetilde{\partial}_{2k}(w(\widetilde{\delta}_{2k})_\#(z^{-1})) = \widetilde{\partial}_{2k}(w) = y$$

by exactness of the second row of (3.6), which shows that $\widetilde{\partial}_{2k}|_{\mathrm{Ker}((\widetilde{p}_{2k})_\#)}$ is surjective.

In a similar manner, one may show that the restriction

$$\partial_k \left|_{\mathrm{Ker}((p_k)_\#)}\right. : \mathrm{Ker}((p_k)_\#) \longrightarrow \mathrm{Ker}((q_k)_\#) \tag{3.10}$$

is also an isomorphism, and if $w \in \operatorname{Ker}((q_k)_\#)$ then

$$(\widetilde{q}_{2k})_\#(\phi_k(w)) = \phi_{k-1}((q_k)_\#(w)) = 1$$

by equation (3.8), and so $\phi_k(w) \in \operatorname{Ker}((\widetilde{q}_{2k})_\#)$. We thus obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Ker}((q_k)_\#) \arrow[r] \arrow[d, "\phi_k\left|_{\operatorname{Ker}((q_k)_\#)}\right."] & PMCG(K;k) \arrow[r, "(q_k)_\#"] \arrow[d, "\phi_k\left|_{PMCG(K;k)}\right."] & PMCG(K;k-1) \arrow[r] \arrow[d, "\phi_{k-1}\left|_{PMCG(K;k-1)}\right."] & 1 \\
1 \arrow[r] & \operatorname{Ker}((\widetilde{q}_{2k})_\#) \arrow[r] & \pi_0 \operatorname{Top}^+_{\widetilde{F}}(T;2k) \arrow[r, "(\widetilde{q}_{2k})_\#"'] & \pi_0 \operatorname{Top}^+_{\widetilde{F}}(T;2(k-1)) \arrow[r] & 1,
\end{tikzcd} \tag{3.11}$$

the second arrow in each row being inclusion. We claim that $\phi_k\left|_{\operatorname{Ker}((q_k)_\#)}\right.$ is injective. This being the case, if $\phi_{k-1}\left|_{PMCG(K;k-1)}\right.$ is injective then $\phi_k\left|_{PMCG(K;k)}\right.$ is also injective by the commutativity and exactness of the rows of (3.11). By induction on k, to complete the proof in the case $g = 2$, it will thus suffice to prove that the homomorphism $\phi_1 \colon MCG(K;1) \longrightarrow MCG(T;2)$ is injective, which we shall do shortly (note that $PMCG(K;1) = MCG(K;1)$, so we may remove the restriction symbol from ϕ_1). We first prove the claim. Let $y \in \operatorname{Ker}((q_k)_\#)$ be such that $\phi_k(y) = 1$. From the isomorphism (3.10), there exists a unique $x \in \operatorname{Ker}((p_k)_\#)$ such that $\partial_k(x) = y$. By (3.7), we have $(\widetilde{p}_{2k})_\# \circ (\widehat{\varphi}_k)_\#(x) = (\widehat{\varphi}_{k-1})_\# \circ (p_k)_\#(x) = 1$, so $(\widehat{\varphi}_k)_\#(x) \in \operatorname{Ker}((\widetilde{p}_{2k})_\#)$. On the other hand, $\widetilde{\partial}_{2k} \circ (\widehat{\varphi}_k)_\#(x) = \phi_k \circ \partial_k(x) = \phi_k(y) = 1$ by commutativity of the diagram (3.6), hence $(\widehat{\varphi}_k)_\#(x) \in \operatorname{Ker}(\widetilde{\partial}_{2k})$. So $(\widehat{\varphi}_k)_\#(x) = 1$ by the isomorphism (3.9). But $(\widehat{\varphi}_k)_\#$ is the restriction of $\varphi_{k\#}$ to $P_k(K)$, so is injective [14]. Thus $x = 1$, $y = 1$, and therefore $\phi_k\left|_{\operatorname{Ker}((q_k)_\#)}\right.$ is injective as claimed.

It thus remains to show that the homomorphism $\phi_1 \colon MCG(K;1) \longrightarrow MCG(T;2)$ is injective. Suppose that $w \in \operatorname{Ker}(\phi_1)$. Taking $k = 1$, the second and third rows of diagram (3.6) become:

$$\begin{tikzcd}
1 \arrow[r] & Z(P_1(K)) \arrow[r, "(\delta_1)_\#"] \arrow[d, "\Phi_1"] & P_1(K) \arrow[r, "\partial_1"] \arrow[d, "\varphi_{1\#}"] & MCG(K;1) \arrow[r, "\tau_1"] \arrow[d, "\phi_1"] & \underbrace{MCG(K)}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2} \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & Z(P_2(T)) \arrow[r, "(\widetilde{\delta}_2)_\#"] & B_2(T) \arrow[r, "\widetilde{\partial}_2"] & MCG(T;2) \arrow[r, "\widetilde{\tau}_2"] & \underbrace{MCG(T)}_{\mathrm{SL}(2,\mathbb{Z})} \arrow[r] & 1.
\end{tikzcd} \tag{3.12}$$

Since $\phi \circ \tau_1(w) = \widetilde{\tau}_2 \circ \phi_1(w) = 1$, we see that $\tau_1(w) \in \operatorname{Ker}(\phi)$, and so by Remark 2.4, $\tau_1(w)$ is either equal to the mapping class of the identity of $MCG(K)$, in which case we say that w is of *type one*, or is equal to the mapping class of the automorphism $\begin{cases} \alpha \longmapsto \alpha \\ \beta \longmapsto \alpha\beta, \end{cases}$ in which case we say that w is of *type two*. Recall that

$$P_1(K) = \left\langle \alpha, \beta \;\middle|\; \alpha\beta\alpha\beta^{-1} \right\rangle, \tag{3.13}$$

$Z(P_1(K)) = \mathbb{Z}(\beta^2)$, and every element of $P_1(K)$ may be written (uniquely) in the form $\alpha^r\beta^s$, where $r, s \in \mathbb{Z}$. By the Appendix, $B_2(T) = \langle x, y, a, b, \sigma\rangle$ and $Z(P_1(T)) = \mathbb{Z}(a) \oplus \mathbb{Z}(b)$, where the given generators of $B_2(T)$ are described geometrically, and are subject to the relations of Theorem A.2. The elements a and b are represented by pairs of parallel strings, and they generate the centre of $B_2(T)$, the elements x and y are braids whose second string is fixed, and σ is the standard Artin generator that exchanges the two basepoints. The group $B_2(T)$ contains

$$P_2(T) = \mathbb{F}_2(x, y) \oplus \mathbb{Z}(a) \oplus \mathbb{Z}(b) \tag{3.14}$$

as an index two subgroup, where $\mathbb{F}_2(x, y)$ is the free group on $\{x, y\}$. The homomorphism $\varphi_{1\#}$ is given by $\varphi_{1\#}(\alpha) = a^{-1}x^2$ and $\varphi_{1\#}(\beta) = y\sigma^{-1}$. Using the relations of Theorem A.2, one may check that $\varphi_{1\#}(\beta^2) = b$.

First assume that $w \in \operatorname{Ker}(\phi_1)$ is of type one. Then $w \in \operatorname{Ker}(\tau_1)$, and by exactness of the first row of (3.12), there exists $w' \in P_1(K)$ such that $\partial_1(w') = w$, and so $w' = \alpha^r\beta^s$, where $r, s \in \mathbb{Z}$ are unique. If $w' \in \langle\beta^2\rangle$ then $\partial_1(w') = 1$ by exactness of the first row of (3.12), and the conclusion clearly holds. So suppose that either s is odd or $r \neq 0$. Then

$$\varphi_{1\#}(w') = (a^{-1}x^2)^r(y\sigma^{-1})^s = \begin{cases} x^{2r}a^{-r}b^{s/2} & \text{if } s \text{ is even} \\ x^{2r}ya^{-r}b^{(s-1)/2}\sigma^{-1} & \text{if } s \text{ is odd,} \end{cases} \tag{3.15}$$

using the fact that $Z(B_2(T)) = \langle a, b\rangle$. Since $(\widetilde{\delta}_2)_\# \circ \varphi_{1\#}(w') = \phi_1 \circ \partial_1(w') = \phi_1(w) = 1$, by observing the induced permutation of $\varphi_{1\#}(w')$, we conclude that s must be even, so $\varphi_{1\#}(w') \in P_2(T)$, and that $\varphi_{1\#}(w') \in \langle a, b\rangle$ by exactness of the second row of (3.12). Using the decomposition (3.14), it follows that $r = 0$, which yields a contradiction. So if $w \in \operatorname{Ker}(\phi_1)$ is of type one then $w = 1$. Now suppose that $w \in \operatorname{Ker}(\phi_1)$ is of type two. Consider the basepoint-preserving homeomorphism h of K illustrated in Figure 3.1. Observe that $\tau_1([h]) = \tau_1(w)$, so by exactness of the first row of (3.12), there exists $w' \in P_1(K)$ such that $w = [h]\,\partial_1(w')$. Let $r, s \in \mathbb{Z}$ be such that $w' = \alpha^r\beta^s$. Now h lifts to an orientation-preserving homeomorphism

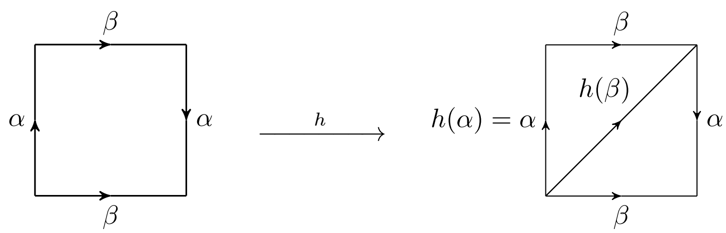

Figure 3.1. The homeomorphism h of K.

$\widetilde{h} \in \operatorname{Top}^+(T, \widetilde{X})$ that is illustrated in Figure 3.2, where $\widetilde{X} = \{\widetilde{x}_1, \widetilde{x}_2\}$ is the inverse image of the basepoint of K under π. It follows that $\widetilde{h} = \rho_1(h)$, and hence $[\widetilde{h}] = [\rho_1(h)] = \phi_1([h])$. On the other hand, relative to the chosen generators of $P_2(T)$, the generator x is represented by the loop c based at $\widetilde{x}_1$, and a is represented by two loops, namely c and a parallel copy c' of c

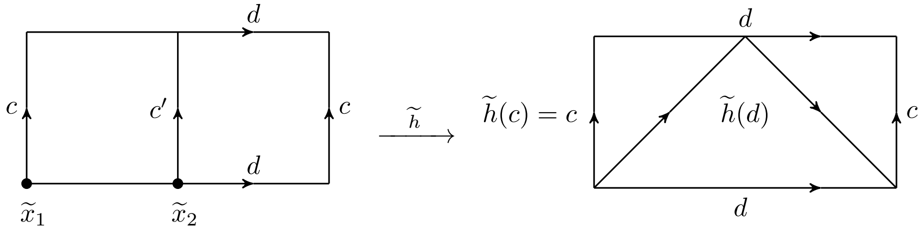

FIGURE 3.2. A lift $\widetilde{h}$ to T of the homeomorphism h.

based at $\widetilde{x}_2$. The loop c' represents the element $x^{-1}a$ of $P_2(T)$. Using the definition of $\widetilde{\partial}_2$, we must find a homotopy H of T starting at the identity, *i.e.* $H(\cdot, 0) = \mathrm{Id}_T$, for which the evaluation of the homotopy at the points x_1 and x_2 yields the braid $x^{-1}a$. Let H be the homotopy obtained by pushing $\widetilde{x}_2$ along the loop c' and that satisfies $H(\cdot, 1) = \widetilde{h}$. This homotopy clearly has the desired trace (or evaluation), and so $\widetilde{\partial}_2(x^{-1}a) = [\widetilde{h}]$. Since $w \in \mathrm{Ker}(\phi_1)$, by commutativity of the diagram (3.12) we have:

$$1 = \phi_1(w) = \phi_1([h]\,\partial_1(w')) = [\widetilde{h}]\,\widetilde{\partial}_2 \circ \varphi_{1\#}(w') = \widetilde{\partial}_2(x^{-1}a\varphi_{1\#}(w')),$$

where $\varphi_{1\#}(w')$ is given by equation (3.15). Once more, by considering the induced permutation of $x^{-1}a\varphi_{1\#}(w')$, s must be even, and so

$$\widetilde{\partial}_2(x^{2r-1}a^{-r+1}b^{s/2}) = 1,$$

where $x^{2r-1}a^{-r+1}b^{s/2} \in P_2(T)$ is in the normal form of equation (3.14). Since $r \in \mathbb{Z}$, we thus obtain a contradiction. We conclude that $\phi_1\,\big|_{MCG(K;1)}$ is injective, and this completes the proof of the theorem. ☐

4. COHOMOLOGICAL ASPECTS OF MAPPING CLASS GROUPS OF PUNCTURED SURFACES

In this section, we study the (virtual) cohomological dimension of surface braid groups and mapping class groups with marked points, and we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. — Let $g, k \geqslant 1$, and let $S = S_g$ or N_{g+1}. Since $P_k(S)$ is of (finite) index $k!$ in $B_k(S)$, and $F_k(S)$ is a finite-dimensional CW-complex and an Eilenberg-Mac Lane space of type $K(\pi, 1)$ [8], the cohomological dimensions of $P_k(S)$ and $B_k(S)$ are finite and equal, so it suffices to determine $\mathrm{cd}(P_k(S))$. Let us prove by induction on k that $\mathrm{cd}(P_k(S)) = k+1$ and that $H^{k+1}(P_k(S), \mathbb{Z}) \neq 0$. The result is true if $k = 1$ since then $F_1(S) = S$, $H^2(\pi_1(S), \mathbb{Z}) \neq 0$ and $\mathrm{cd}(P_1(S)) = \mathrm{cd}(\pi_1(S)) = 2$. So suppose that the induction hypothesis holds for some $k \geqslant 1$. The Fadell-Neuwirth fibration $p\colon F_{k+1}(S) \longrightarrow F_k(S)$ given by forgetting the last coordinate gives rise to the following short exact sequence of braid groups:

$$1 \longrightarrow N \longrightarrow P_{k+1}(S) \overset{p_\#}{\to} P_k(S) \longrightarrow 1, \tag{4.1}$$

where $N = \pi_1(S \setminus \{x_1, \ldots, x_k\}, x_{k+1})$, $(x_1, \ldots, x_k)$ being an element of $F_k(S)$, and $p_\#$ is defined geometrically by forgetting the last string. Since $S \setminus \{x_1, \ldots, x_k\}$ has the homotopy type of a bouquet of circles, $H^i(S \setminus \{x_1, \ldots, x_k\}), A)$ is trivial for all $i \geqslant 2$ and for any choice of local coefficients A, and $H^1(S \setminus \{x_1, \ldots, x_k\}), \mathbb{Z}) \neq 0$, hence $\mathrm{cd}(N) = 1$. By [6, Chapter VIII], it follows that $\mathrm{cd}(P_{k+1}(S)) \leqslant \mathrm{cd}(P_k(S)) + \mathrm{cd}(N) \leqslant k+2$. To conclude the proof of the theorem, it remains to show that there exist local coefficients A such that $H^{k+2}(P_{k+1}(S), A) \neq 0$. We will show that this is the case for $A = \mathbb{Z}$. By the induction hypothesis, we have $H^{k+1}(P_k(S), \mathbb{Z}) \neq 0$. Consider the Serre spectral sequence with integral coefficients associated to the fibration p. Then we have that:

$$E_2^{p,q} = H^p\big(P_k(S), H^q(S \setminus \{x_1, \ldots, x_k\})\big).$$

Since $\mathrm{cd}(P_k(S)) = k + 1$ and $\mathrm{cd}(S \setminus \{x_1, \ldots, x_k\}) = 1$ from above, this spectral sequence has two horizontal lines whose possible non-vanishing terms occur for $0 \leqslant p \leqslant k+1$ and $0 \leqslant q \leqslant 1$. We claim that the group $E_2^{k+1,1}$ is non trivial. To see this, first note that $H^1(S \setminus \{x_1, \ldots, x_k\}), \mathbb{Z})$ is isomorphic to the free Abelian group of rank r, where $r = 2g + k - 1$ if $S = S_g$ and $r = g + k$ if $S = N_{g+1}$, so $r \geqslant 2$, and hence $E_2^{k+1,1} = H^{k+1}\big(P_k(S), \mathbb{Z}^r\big)$, where we identify $\mathbb{Z}^r$ with (the dual of) N^{Ab}, the Abelianisation of N. The action of $P_k(S)$ on N by conjugation induces an action of $P_k(S)$ on N^{Ab}. Let H be the subgroup of N^{Ab} generated by the elements of the form $\alpha(x) - x$, where $\alpha \in P_k(S)$, $x \in N^{\mathrm{Ab}}$, and $\alpha(x)$ represents the action of α on x. Then we obtain a short exact sequence $0 \longrightarrow H \longrightarrow N^{\mathrm{Ab}} \longrightarrow N^{\mathrm{Ab}}/H \longrightarrow 0$ of Abelian groups, and the long exact sequence in cohomology applied to $P_k(S)$ yields:

$$\cdots \to H^{k+1}(P_k(S), N^{\mathrm{Ab}}) \to H^{k+1}(P_k(S), N^{\mathrm{Ab}}/H) \to H^{k+2}(P_k(S), H) \to \cdots. \tag{4.2}$$

The last term is zero since $\mathrm{cd}(P_k(S)) = k+1$, and so the map between the remaining two terms is surjective. So to prove that $E_2^{k+1,1}$ is non trivial, it suffices to show that $H^{k+1}(P_k(S), N^{\mathrm{Ab}}/H)$ is non trivial. To do so, we first determine N^{Ab}/H. Suppose first that $S = S_g$, and consider the presentation of $P_k(S)$ on N^{Ab} given in [12, Corollary 8]. With the notation of that paper, a basis $\mathcal{B}$ of N is given by:

$$\{\, \rho_{k+1,r},\ \tau_{k+1,r},\ C_{i,k+1} \,|\, 1 \leqslant r \leqslant g,\ 1 \leqslant i \leqslant k-1 \}\,,$$

and a set of coset representatives in $P_{k+1}(S)$ of a generating set $\mathcal{S}$ of $P_k(S)$ is given by $\{\, \rho_{m,r},\ \tau_{m,r},\ C_{i,j} \,|\, 1 \leqslant r \leqslant g,\ 1 \leqslant i < j \leqslant k,\ 1 \leqslant m \leqslant k\}$. Using [12, Table 1 and Theorem 7 or Corollary 8], one sees that the commutators of the elements of $\mathcal{S}$ with those of $\mathcal{B}$ project to the trivial element of N^{Ab}, with the exception of $[\rho_{m,r}, \tau_{k+1,r}]$ and $[\tau_{m,r}, \rho_{k+1,r}]$ that project to (the coset of) $C_{m,k+1}$ for all $m = 1, \ldots, k-1$ (we take the opportunity here to correct a couple of small misprints in these results: relation (19) of Theorem 7 should read as in relation (XII_n) of Corollary 8; and in Table 1, in each of the three rows, the first occurrence of $j > k$ should read $j < k$). Note that we obtain a similar result for $m = k$, but using the surface relation [12, equation (1)] in $P_{k+1}(S)$ and taking $i = k+1$ yields no new information. It follows that H is the subgroup of N^{Ab} generated by the $C_{m,k+1}$, where $1 \leqslant m \leqslant k-1$, and that N^{Ab}/H is the free Abelian group generated by the coset representatives of the $\rho_{k+1,r}$ and $\tau_{k+1,r}$, where $1 \leqslant r \leqslant g$. In particular, $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^{2g}$. Since the induced action of $P_k(S)$ on N^{Ab}/H is trivial, using the induction hypothesis, we conclude that:

$$H^{k+1}(P_k(S), N^{\mathrm{Ab}}/H) = \big(H^{k+1}(P_k(S), \mathbb{Z})\big)^{2g} \neq 0.$$

It then follows from (4.2) that $E_2^{k+1,1} = H^{k+1}(P_k(S), N^{\mathrm{Ab}}) \neq 0$. Since $E_2^{p,q} = 0$ for all $p > k+1$ and $q > 1$, we have $E_2^{k+1,1} = E_\infty^{k+1,1}$, thus $E_\infty^{k+1,1}$ is non trivial, and hence $H^{k+2}(P_{k+1}(S), \mathbb{Z}) \neq 0$. This proves the result in the orientable case.

Now let us turn to the non-orientable case. The idea of the proof is the same as in the orientable case, but the computations for N^{Ab} and H are a little different. We use the presentation of $P_k(S)$ given in [13]. With the notation of that paper, a basis $\mathcal{B}$ of N is given by $\{\rho_{k+1,r}, B_{i,k+1} \mid 1 \leqslant r \leqslant g, 1 \leqslant i \leqslant k-1\}$, and a set of coset representatives in $P_{k+1}(S)$ of a generating set $\mathcal{S}$ of $P_k(S)$ is given by $\{\rho_{m,r}, B_{i,j} \mid 1 \leqslant r \leqslant g, 1 \leqslant i < j \leqslant k, 1 \leqslant m \leqslant k\}$. Using [13, Theorem 3], one sees that the commutators of the elements of $\mathcal{S}$ with those of $\mathcal{B}$ project to the trivial element of N^{Ab}, with the exception of $[\rho_{m,r}, \rho_{k+1,r}]$ that projects to (the coset of) $B_{m,k+1}^{-1}$ for all $m = 1, \ldots, k-1$ and $1 \leqslant r \leqslant g$. We obtain a similar result for $m = k$, and using the surface relation [13, relation (c), Theorem 3] in $P_{k+1}(S)$ and taking $i = k+1$ implies that the element $2\sum_{l=1}^{g} \rho_{k+1,l}$ also belongs to H using additive notation. It follows that H is the subgroup of N^{Ab} generated by the $B_{m,k+1}$, where $1 \leqslant m \leqslant k-1$, and the element $2\sum_{l=1}^{g} \rho_{k+1,l}$. We conclude that $N^{\mathrm{Ab}}/H \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. The argument then goes through as in the orientable case. □

The exact sequences given by (1.4) and (3.6) also allow us to obtain information about the virtual cohomological dimension of $MCG(N_g; k)$ if $g \geqslant 2$ and $k \geqslant 1$. The virtual cohomological dimension for mapping class groups of non-orientable surfaces was calculated by Ivanov [23] and is given in (1.5). The result in the case of the Klein bottle may also be recovered using our methods. We first prove the following lemma.

Lemma 4.1. — *Let $k \in \mathbb{N}$, and consider the homomorphism*

$$(\delta_k)_\#\colon \pi_1 Top(K) \longrightarrow P_k(K) \ (resp.\ (\widetilde{\delta}_{2k})_\#\colon \pi_1 Top^+(T) \longrightarrow B_{2k}^{(2)}(T)),$$

where the map δ_k (resp. $\widetilde{\delta}_{2k}$) is as defined in (3.5). Then the image of $(\delta_k)_\#$ (resp. $(\widetilde{\delta}_{2k})_\#$) is equal to the the centre of $P_k(K)$ (resp. of $P_{2k}(T)$).

Proof. — Let us first consider the case of the Klein bottle. If $k = 1$, with the notation of (3.13), it follows from the proof of [19, Theorem 4.1] that the image of $(\delta_k)_\#$ is equal to $\langle \beta^2 \rangle$, which is equal to the centre of $\pi_1(K)$. Furthermore, a generator γ of $\pi_1 \mathrm{Top}(K)$ may be taken to be the loop class $[(T_s)_{s\in[0,1]}]$, where with respect to the geometric representation of K given in Figure 3.1 as the quotient of the unit square $[0,1] \times [0,1]$ by the relations $(x,0) \sim (x,1)$ and $(0, 1-y) \sim (1,y)$ for all $x, y \in [0,1]$, T_s is the translation defined by $T_s(x,y) = (x + 2s, y)$. Now let $k \geqslant 2$. By [17, proof of Proposition 5.2] or [32, Proposition 2.2.4], the centre of $P_k(K)$ is infinite cyclic, generated by the braid illustrated in Figure 4.1. By pushing the crossings in the rightmost square into the middle square, this braid may be deformed in $P_k(K)$ to the braid shown in Figure 4.2 that is clearly equal to $(\delta_k)_\#(\gamma)$. This proves the statement of the lemma for K. The proof in the case of the torus is similar, and is obtained by noting that the centre of $P_{2k}(T)$ is isomorphic to $\mathbb{Z}^2$ by [31, Proposition 4.2], where a basis is given by sets of parallel strings in the x- and y-directions respectively, and that by the proof of [18, Theorem 2], $\pi_1 \mathrm{Top}^+(T)$ is a free Abelian group of rank two generated by the loop classes of unit translations in the x- and y-directions respectively. □

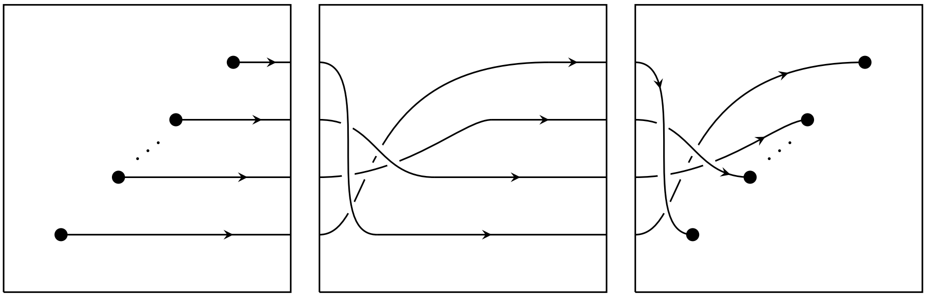

FIGURE 4.1. A generator of the centre of $P_n(K)$. The right-hand side of each square should be identified with the left-hand side of the following square according to the identification $(1, y) \sim (0, 1 - y)$.

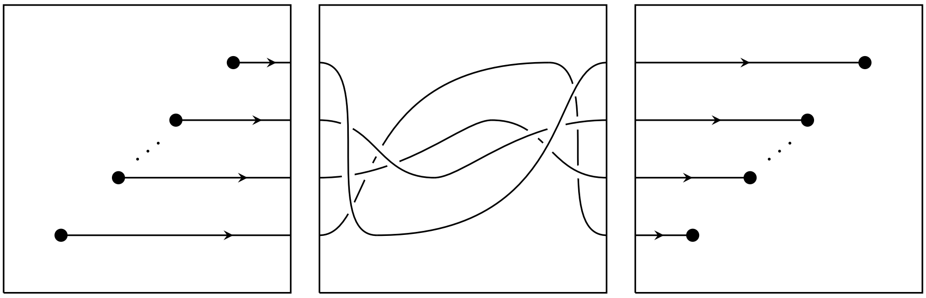

FIGURE 4.2. A deformation of the braid of Figure 4.1 that is equal to $(\delta_k)_\#(\gamma)$ in $P_n(K)$.

Proof of Corollary 1.3. —

(a) Since $PMCG(K; k)$ is a subgroup of finite index of $MCG(K; k)$, it suffices to work with the pure mapping class group $PMCG(K; k)$. Moreover, using the exact sequence given by the third row of (3.6):

$$1 \longrightarrow \pi_1 \mathrm{Top}(K) \xrightarrow{(\delta_k)_\#} P_k(K) \xrightarrow{\partial_k} PMCG(K; k) \xrightarrow{\tau_k} MCG(K) \longrightarrow 1,$$

Lemma 4.1 and the fact that $MCG(K)$ is finite, $PMCG(K; k)$ has the same vcd as the quotient $P_k(K)/Z(P_k(K))$. We claim that $\mathrm{vcd}(PMCG(K; k)) = k$. To prove the claim, we proceed by induction on k. First, the result is true for $k = 1$ since $P_1(K)/Z(P_1(K)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Now suppose that the result holds for some $k \geqslant 1$. By taking the image of the generator γ of $\pi_1 \mathrm{Top}(K)$ of the proof of Lemma 4.1 by $(\delta_{k+1})_\#$ and $(\delta_k)_\#$, the projection $P_{k+1}(K) \longrightarrow P_k(K)$ maps the centre $Z(P_{k+1}(K))$ of $P_{k+1}(K)$ isomorphically onto the centre $Z(P_k(K))$ of $P_k(K)$, and we have the following short exact sequence of groups:

$$1 \longrightarrow P_1(K \setminus \{x_0\}) \longrightarrow P_{k+1}(K)/Z(P_{k+1}(K)) \longrightarrow P_k(K)/Z(P_k(K)) \longrightarrow 1.$$

Now $\mathrm{vcd}(P_k(K)/Z(P_k(K))) = k$ by induction, hence

$$\mathrm{vcd}(P_{k+1}(K)/Z(P_{k+1}(K))) \leqslant k + 1,$$

because $\operatorname{vcd}(P_1(K \setminus \{x_0\})) = 1$. Using the short exact sequence

$$1 \longrightarrow Z(P_{k+1}(K)) \longrightarrow P_{k+1}(K) \longrightarrow P_{k+1}(K)/Z(P_{k+1}(K)) \longrightarrow 1,$$

and the fact that $\operatorname{cd}(P_{k+1}(K)) = k+2 \leqslant 1+\operatorname{vcd}(P_{k+1}(K)/Z(P_{k+1}(K)))$ by Theorem 1.2, we see that $\operatorname{vcd}(P_{k+1}(K)/Z(P_{k+1}(K))) \geqslant k+1$, so

$$\operatorname{vcd}(P_{k+1}(K)/Z(P_{k+1}(K))) = k+1,$$

and the result follows.

(b) Let $g \geqslant 3$, and consider the following short exact sequence

$$1 \longrightarrow B_k(N_g) \longrightarrow MCG(N_g;k) \longrightarrow MCG(N_g) \longrightarrow 1$$

given by equation (1.4). From [22, Corollary 2.2] and Theorem 1.2, we have:

$$\operatorname{vcd}(MCG(N_g;k)) \leqslant \operatorname{cd}(B_k(N_g)) + \operatorname{vcd}(MCG(N_g)) \leqslant k+1+4g-9 = 4g+k-8. \quad \square$$

Appendix

In this appendix, we provide presentations of $P_2(T)$ and $B_2(T)$ that are adapted to our situation. From [7, Section 4], we have:

Theorem A.1. — *The group $P_2(T)$ possesses a presentation with generators $B_{1,2}$, $\rho_{1,1}$, $\rho_{1,2}$, $\rho_{2,1}$ and $\rho_{2,2}$ subject to the following relations:*

(a) $[\rho_{1,1}, \rho_{1,2}^{-1}] = [\rho_{2,1}, \rho_{2,2}^{-1}] = B_{1,2}$.
(b) $\rho_{2,1}\rho_{1,1}\rho_{2,1}^{-1} = B_{1,2}\rho_{1,1}B_{1,2}^{-1}$.
(c) $\rho_{2,1}\rho_{1,2}\rho_{2,1}^{-1} = B_{1,2}\rho_{1,2}[\rho_{1,1}^{-1}, B_{1,2}]$.
(d) $\rho_{2,2}\rho_{1,1}\rho_{2,2}^{-1} = \rho_{1,1}B_{1,2}^{-1}$.
(e) $\rho_{2,2}\rho_{1,2}\rho_{2,2}^{-1} = B_{1,2}\rho_{1,2}B_{1,2}^{-1}$.

If necessary, the generator $B_{1,2}$ may be suppressed from the list of generators of $P_2(T)$. Using Theorem A.1, we may obtain the following useful relations in $P_2(T)$:

$$\rho_{2,1}B_{1,2}\rho_{2,1}^{-1} = B_{1,2}\rho_{1,1}^{-1}B_{1,2}\rho_{1,1}B_{1,2}^{-1} \text{ and } \rho_{2,2}B_{1,2}\rho_{2,2}^{-1} = B_{1,2}\rho_{1,2}^{-1}B_{1,2}\rho_{1,2}B_{1,2}^{-1}.$$

Setting $\delta_{1,1} = \rho_{1,1}$, $\tau_{1,1} = \rho_{1,2}$, $\delta_{2,1} = B_{1,2}^{-1}\rho_{2,1}$ and $\tau_{2,1} = B_{1,2}^{-1}\rho_{2,2}$, we obtain a new presentation of $P_2(T)$ whose generators are $B_{1,2}$, $\delta_{1,1}$, $\tau_{1,1}$, $\delta_{2,1}$ and $\tau_{2,1}$, that are subject to the following relations:

(a) $[\delta_{1,1}, \tau_{1,1}^{-1}] = [B_{1,2}\delta_{2,1}, \tau_{2,1}^{-1}B_{1,2}^{-1}] = B_{1,2}$.
(b) $[\delta_{2,1}, \delta_{1,1}] = [\tau_{2,1}, \tau_{1,1}] = 1$.
(c) $\delta_{2,1}\tau_{1,1}\delta_{2,1}^{-1} = \tau_{1,1}\delta_{1,1}^{-1}B_{1,2}\delta_{1,1}$.
(d) $\tau_{2,1}\delta_{1,1}\tau_{2,1}^{-1} = B_{1,2}^{-1}\delta_{1,1}$.

If we let $a = \delta_{1,1}\delta_{2,1}$ $b = \tau_{1,1}\tau_{2,1}$, $x = \delta_{1,1}$ and $y = \tau_{1,1}$, then it is not hard to show that $P_2(T)$ has a presentation whose generators are x, y, a and b that are subject to the following relations:

(a) $[a,b] = 1$.
(b) $[a,x] = [b,x] = 1$.
(c) $[a,y] = [b,y] = 1$.

It follows from this presentation that $P_2(T)$ is isomorphic to $\mathbb{F}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$, where $\{x,y\}$ is a basis of the free group $\mathbb{F}_2$, and $\{a,b\}$ is a basis of the free Abelian group $\mathbb{Z} \oplus \mathbb{Z}$. We then obtain the following presentation of $B_2(T)$.

Theorem A.2. — *The group $B_2(T)$ has a presentation with generators $B = B_{1,2}, \sigma, x, y, a$ and b that are subject to the following relations:*

(a) $\sigma^2 = [x, y^{-1}] = B$.
(b) $[a, b^{-1}] = 1$.
(c) $axa^{-1} = x$ *and* $aya^{-1} = y$.
(d) $bxb^{-1} = x$ *and* $byb^{-1} = y$.
(e) $\sigma x \sigma^{-1} = Bx^{-1}a$ *and* $\sigma y \sigma^{-1} = By^{-1}b$.
(f) $\sigma a \sigma^{-1} = a$ *and* $\sigma b \sigma^{-1} = b$.

The generator σ given in the statement of Theorem A.2 is the Artin generator σ_1. The proof of the theorem is standard, and makes use of the above presentation of $P_2(T)$, the short exact sequence:

$$1 \longrightarrow P_2(T) \longrightarrow B_2(T) \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

standard results on presentations of group extensions [25, Proposition 1, p. 139], and computations of relations between the given generators by means of geometric arguments.

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