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Sven MEINHARDT

An Introduction to (Motivic) Donaldson-Thomas Theory

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AN INTRODUCTION TO (MOTIVIC)
DONALDSON-THOMAS THEORY

SVEN MEINHARDT

Abstract. The aim of the paper is to provide a rather gentle introduction into Donaldson-Thomas theory using quivers with potential. The reader should be familiar with some basic knowledge in algebraic or complex geometry. The text contains many examples and exercises to support the process of understanding the main concepts and ideas.

Contents

1. Introduction

The theory of Donaldson-Thomas invariants started around 2000 with the seminal work of R. Thomas [36]. He associated integers to those moduli spaces of sheaves on a compact Calabi-Yau 3-fold which only contain stable sheaves. After some years, K. Behrend realized in [1] that these numbers, originally written as "integrals" over algebraic cycles or characteristic classes, can also be obtained by an integral over a constructible function, the so-called Behrend function, with

respect to the measure given by the Euler characteristic. This new point of view did not only extend the theory to non-compact moduli spaces but revealed also the "motivic nature" of this new invariant. It has also been realized that quivers with potential provide another class of examples to which Donaldson-Thomas theory applies. Starting around 2006, D. Joyce [11, 12, 13, 14, 15, 16] and Y. Song [17] extended the theory using all kinds of "motivic functions" to produce (possibly rational) numbers even in the presence of semistable objects which is the generic situation when classifying objects in abelian categories. Around the same time, M. Kontsevich and Y. Soibelman [19, 21, 20] independently proposed a theory producing even motives, some sort of refined "numbers", instead of simple numbers, also in the presence of semistable objects. The technical difficulties occurring in their approach disappear in the special situation of representations of quivers with potential. The case of zero potential has been intensively studied by M. Reineke in a series of papers [30, 31, 32]. Despite some computations of motivic or even numerical Donaldson-Thomas invariants for quivers with potential (see [2, 6, 4, 27]), the true nature of Donaldson-Thomas invariants for quiver with potential remained mysterious for quite some time. A full understanding has been obtained recently and is the content of a series of papers [7, 5, 26, 24].

The present text aims at giving a gentle introduction to Donaldson-Thomas theory in the case of quiver with potential. We have two reasons for our restriction to quivers. Firstly, so-called orientation data will not play any role, and secondly, we do not need to touch derived algebraic geometry. Apart from this, many important ideas and concepts are already visible in the case of quiver representations, and since the theory is fully understood, we belief that this is a good starting point for your journey towards an understanding of Donaldson-Thomas theory. There are more survey articles available focusing on different aspects of the theory (see [17, 20, 35]).

Let us give a short outline of the paper. The next section starts very elementary by discussing the problem of classifying objects. The objects which are of interest to us form an abelian category although many ideas of section 2 also apply to "non-linear" moduli problems. We study in detail the difficulties arising from the construction of moduli spaces and develop slowly the concept of a (moduli) stack. Although the theory of stacks is rather rich and complicated, we can restrict ourselves to quotient stacks throughout this paper. Hence, a good understanding of a quotient stacks is inevitable. We try to illustrate this concept by giving important examples. We should mention that only very little knowledge of algebraic or complex geometry is needed. In many cases, you can easily replace "schemes" with "varieties" or "complex manifolds".

Section 3 provides the background on quivers and their representations. The point of view taken here is that quivers are the categorical (noncommutative) analogue of polynomial algebras in ordinary commutative algebra. In other words, they are a useful tool for practical computations when dealing with linear categories, but at the end of the day the result should only depend on the linear category and not its presentation as a quotient of the path category of a quiver by some ideal of relations. The relations important in this paper are given by noncommutative partial derivatives of a so-called potential.

The next two sections provide the framework to formulate Donaldson-Thomas theory in section 6. We start in section 4 with the concept of "motivic theories". The best example the reader should have in mind are constructible functions. It should be clear that constructible functions can be pulled back and multiplied. Using fiberwise integrals with respect to the Euler characteristic, we can even push forward constructible functions. Moreover, every locally closed subscheme/subvariety/submanifold determines a constructible function, namely its characteristic function. In a nutshell, a motivic theory is just a generalization of this associating to every scheme X an abelian group $R(X)$ of "functions" on X which can be pulled back, pushed forward and multiplied. Moreover, to every locally closed subscheme in X there is a "characteristic function" in $R(X)$ such that the characteristic function of a disjoint union is the sum of the characteristic function of its summands. Its is this property what makes a theory of generalized functions "motivic". As usual in algebraic geometry, the term "function" should be used with some care. Every function on say a complex variety X determines a usual function from the set of points in X to the coefficient ring $R(\text{point})$ of our theory, but this is not a one-one correspondence.

In section 5 we introduce vanishing cycles. We do not assume that the reader is familiar with any theory of vanishing cycles. As in the previous section, a vanishing cycle is just an additional structure on motivic theories formalizing the properties of ordinary classical vanishing cycles. The Behrend function mentioned at the beginning of this introduction provides a good example of a vanishing cycle on the theory of constructible functions. In fact, we will construct in a functorial way two vanishing cycles associated to a given motivic theory. The first construction is rather stupid, but the second one essentially covers all known nontrivial examples. At the end of sections 4 and 5 we extend motivic theories and vanishing cycles to quotient stacks as quotient stacks arise naturally in moduli problems. There is a way to circumvent stacks in Donaldson-Thomas theory by considering framed objects, but we belief that the usual approach of using stacks is more conceptual and should be known by anyone who wants to understand Donaldson-Thomas theory seriously.

In the last section 6 we finally introduce Donaldson-Thomas functions and invariants. After stating the main results, we consider many examples to illustrate the theory. Finally, we develop some tools used in Donaldson-Thomas theory such as Ringel-Hall algebras, an important integration map and the celebrated wall-crossing formula.

The reader will realize shortly that the text contains tons of exercises and examples. Most of the exercises are rather elementary and require some elementary computations and standard arguments. Nevertheless, we suggest to the reader to do them carefully in order to get your hands on the subject and to obtain a feeling about the objects involved. There is a lot of material in this text which is not part of the standard graduate courses at universities, and if you are not already familiar with the subject you certainly need some practice as we cannot provide a deep and lengthy discussion of the material presented here.

2. Moduli problems and stacks

Let us start by recalling the general idea of a moduli space. Depending on the situation, mathematicians are trying to classify objects of various types. The general pattern is the following. There is some set (or class) of objects and isomorphisms between two objects. Such a structure is called a groupoid. A groupoid is a category with every morphism being an isomorphism. If the set of objects has cardinality one, a groupoid is just a group. The other extreme is a groupoid such that every morphism is the identity morphism of some object. Such groupoids are in one-to-one correspondence with ordinary sets. Hence, a groupoid interpolates between sets and groups. There are two main sources of groupoids.

Example 2.1. — Let X be a topological space. The fundamental groupoid $\pi_1(X)$ is the groupoid having the points of X as objects, and given two points $x, y \in X$, the set of morphisms from x to y is the set of homotopy classes of paths from x to y. Fixing a base point $x \in X$, the usual fundamental group $\pi_1(X, x)$ is just the automorphism group of x considered as an object in the groupoid $\pi_1(X)$. Denote by $\pi_0(X)$ the set of path connected components, i.e. the set of objects in $\pi_1(X)$ up to isomorphism.

Example 2.2. — Given a category $\mathcal{C}$, one can consider the subcategory $\mathcal{I}so(\mathcal{C})$ of all isomorphisms in $\mathcal{C}$. Thus, $\mathcal{I}so(\mathcal{C})$ is a groupoid, and $\mathcal{C}/_\sim$ denotes the set of objects in $\mathcal{C}$ up to isomorphism.

These two examples are related as follows. To every (small) category one can construct a topological space $X_\mathcal{C}$ - the classifying space of $\mathcal{C}$ - such that $\pi_1(X_\mathcal{C}) \cong \mathcal{I}so(\mathcal{C})$ and $\pi_0(X_\mathcal{C}) = \mathcal{C}/_\sim$.

Let us come back to the classification problem, say of objects in $\mathcal{C}$ up to isomorphism. The problem is to describe the set $\mathcal{C}/_\sim$. If it is discrete in a reasonable sense, one tries to find a parameterization by less complicated (discrete) objects. This applies for instance to the classification of semisimple algebraic groups or finite dimensional representations of the latter. In many other situations, $\mathcal{C}/_\sim$ is uncountable, and one wants to put a geometric structure on the set $\mathcal{C}/_\sim$ to obtain a "moduli space". However, if for instance $\mathcal{C}/_\sim$ has the cardinality of the field of complex numbers, one can always choose a bijection $\mathcal{C}/_\sim \cong M$ to the set of points of any complex variety or manifold M of dimension greater than zero. Pulling back the geometric structure of M along this bijection, we can equip $\mathcal{C}/_\sim$ in many different (non-isomorphic) ways with a structure of a complex manifold. Hence, we should ask:

Question. — Is there a natural geometric structure on $\mathcal{C}/_\sim$? What does "natural" actually mean?

There is a very beautiful idea of what "natural" should mean, and which applies to many situations. Assume there is a notion of a family of objects in $\mathcal{C}$ over some "base" scheme/variety/(complex) manifold S, i.e. some object on S which has "fibers" over $s \in S$, and these fibers should be objects in $\mathcal{C}$.

Example 2.3. — Given a $\mathbb{C}$-algebra A, a family of finite dimensional A-representations is a (holomorphic) vector bundle V on S and an $\mathbb{C}$-algebra homomorphisms $A \to \Gamma(S, \mathcal{E}nd(V))$ from A into the algebra of sections of the endomorphism bundle of V.

Example 2.4. —   Given a scheme/variety/manifold X over $\mathbb{C}$ and some parameter space S, a family of coherent sheaves on X parametrized by S is just a coherent sheaf E on $S \times X$ which is flat over S. The latter condition ensures that taking fibers and pull-backs of families behaves well. If E is a family of zero dimensional sheaves on X, i.e. if the projection $p : \mathrm{Supp}(E) \to S$ has zero-dimensional fibers, flatness of E over S is equivalent to the requirement that E is locally free over S. Using the coherence of E once more, one can show that $p : \mathrm{Supp}(E) \to S$ is a finite morphism and if $X = \mathrm{Spec}\, A$ is affine, E is completely determined by the vector bundle $V := p_* E$ on S together with a $\mathbb{C}$-algebra homomorphism $A \to \Gamma(S, \mathcal{E}nd(V))$. From that perspective, the previous example can be seen as a non-commutative version, namely families of zero dimensional sheaves on the non-commutative affine scheme $\mathrm{Spec}\, A$ for A being a $\mathbb{C}$-algebra.

Example 2.5. —   A G-torsor with respect to some (algebraic) group G is a scheme P with a right G-action such that $P \cong G$ as varieties with right G-action, where G acts on G by right multiplication. A (locally trivial) family of G-torsors over S is defined as a principal G-bundle on S.

Once a family is given, by taking the "fiber" over $s \in S$ we get an object in $\mathcal{C}$ and, hence, a point in $\mathcal{C}/_\sim$. Varying $s \in S$, we end up with a map $u : S \to \mathcal{C}/_\sim$. Moreover, we see that the pull-back of a family on S along a morphism $f : S' \to S$ induces a morphism $u' : S' \to \mathcal{C}/_\sim$ such that $u' = u \circ f$. Coming back to the question formulated above, we can now be more precise by asking:

Question. —   Is there a structure of a variety or scheme on $\mathcal{C}/_\sim$ such that for every family of objects over any S, the induced map $S \to \mathcal{C}/_\sim$ is a morphism of schemes? If so, is there any way to get back the family by knowing only the morphism $S \to \mathcal{C}/_\sim$?

If the first question has a positive answer, we call $\mathcal{M} = (\mathcal{C}/_\sim, \text{scheme structure})$ a coarse moduli space for $\mathcal{C}$. If the second part of the question is also true, we should be able to (re)construct a "universal" family on $\mathcal{M}$ by considering the map $\mathrm{id} : \mathcal{M} \to \mathcal{M}$. Moreover, given a map $u' : S' \to \mathcal{M}$ such that $u' = u \circ f$ for some map $f : S' \to S$, the family on S' should be the pull-back of the family on S associated to u by uniqueness. As every morphism $u : S \to \mathcal{M}$ has an obvious factorization $S \xrightarrow{u} \mathcal{M} \xrightarrow{\mathrm{id}} \mathcal{M}$, we finally see that every family on S must be the pull-back of the "universal" family on $\mathcal{M}$. In such case, we call $\mathcal{M}$ a fine moduli space.

Example 2.6. —   Let $\mathcal{C} = \mathrm{Vect}_\mathbb{C}$ be the category of finite dimensional $\mathbb{C}$-vector spaces. A (locally trivial) family of finite dimensional vector spaces is just a vector bundle on some parameter space S. As a vector space is classified by its dimension, we can put the simplest scheme structure on $\mathrm{Vect}_\mathbb{C}/_\sim = \mathbb{N}$ by thinking of $\mathbb{N}$ as a disjoint union of countably many copies of $\mathrm{Spec}\,\mathbb{C}$. Given a vector bundle V, we obtain a well-defined morphism $S \to \mathbb{N}$ mapping $s \in S$ to the copy of $\mathrm{Spec}\,\mathbb{C}$ indexed by the dimension of the fiber V_s of V at s. The scheme $\mathbb{N}$ is a coarse moduli space, but apart from the zero dimensional case, it can never be a fine moduli space. Indeed, there is an obvious and essentially unique vector bundle on $\mathbb{N}$ inducing the identity map $\mathbb{N} \to \mathbb{N}$, but a vector bundle on S can never be the pull-back of the one on $\mathbb{N}$ unless it is constant. Thus, $\mathbb{N}$ is not a fine moduli space.

These are also bad news for our previous examples concerning representations of an algebra A or sheaves on a variety X. Indeed, for $A = \mathbb{C}$ or $X = \operatorname{Spec}\mathbb{C}$, we are back in the classification problem of finite dimensional $\mathbb{C}$-vector spaces.

Example 2.7. — Similar to the previous example, we see that the classification problem for G-torsors has only a coarse moduli space given by $\operatorname{Spec}\mathbb{C}$.

There are several strategies to overcome the difficulty of constructing a fine moduli space.

Example 2.8 (rigid families). — One possibility is to rigidify families of objects. For example, instead of considering all vector bundles we could also restrict ourselves to constant vector bundles. In this particular case, $\mathbb{N}$ is even a fine moduli space. However, in many situations one wants to glue families together to form families of objects on bigger spaces. This is incompatible with the concept of rigidity, and we will not follow this path.

Example 2.9 (weaker equivalence). — Instead of classifying objects up to isomorphism, we could allow weaker equivalences. For example, we could identify to families $V^{(1)}$ and $V^{(2)}$ (over S) of vector spaces or representations of an algebra A if there is a line bundle L on S such that $V^{(2)} = V^{(1)} \otimes_{\mathcal{O}_S} L$. By doing this, we can always replace a rank one bundle with the trivial rank one bundle. Hence, the moduli space $\operatorname{Spec}\mathbb{C}$ of one-dimensional vector spaces is a fine moduli space.

Example 2.10 (projectivization). — Similar to families of vector spaces of dimension r, one could look at locally trivial families $\mathcal{P}$ of projective spaces $\mathbb{P}^{r-1}$. The transition functions between local trivializations are regular functions with values in $\operatorname{Aut}(\mathbb{P}^{r-1}) = \operatorname{PGL}(r)$. Every vector bundle V of rank r provides such a bundle by taking $\mathcal{P} := \mathbb{P}(V)$, the bundle of lines or hyperplanes in V. Two vector bundles $V^{(1)}$, $V^{(2)}$ define isomorphic bundles $\mathbb{P}(V^{(1)}) \cong \mathbb{P}(V^{(2)})$ if and only if $V^{(2)} = V^{(1)} \otimes_{\mathcal{O}_S} L$ for some line bundle L on S, providing the bridge to the previous example. However, not every $\mathbb{P}^{r-1}$-bundle $\mathcal{P}$ can be realized as $\mathbb{P}(V)$ for some vector bundle V on S. Given a $\mathbb{P}^{r-1}$-bundle $\mathcal{P}$, we can use the transition functions and the isomorphism $\operatorname{PGL}(r) \cong \operatorname{Aut}(\operatorname{Mat}_{\mathbb{C}}(r, r))$ to glue trivial $\operatorname{Mat}_{\mathbb{C}}(r, r)$ bundles on overlaps of charts. Thus we constructed an associated locally trivial bundle $\mathcal{E}_{\mathcal{P}}$ of $\mathbb{C}$-algebras isomorphic to $\operatorname{End}_{\mathbb{C}}(\mathbb{C}^r) \cong \operatorname{Mat}_{\mathbb{C}}(r, r)$. Conversely, every locally trivial bundle $\mathcal{E}$ of $\mathbb{C}$-algebras isomorphic to $\operatorname{Mat}_{\mathbb{C}}(r, r)$ defines an associated $\mathbb{P}^{r-1}$-bundle $\mathcal{P}_{\mathcal{E}}$ as the transition functions of $\mathcal{E}$ must be in $\operatorname{Aut}(\operatorname{Mat}_{\mathbb{C}}(r, r)) = \operatorname{PGL}(r)$. Thus, we have an equivalence of categories between locally trivial $\mathbb{P}^{r-1}$-bundles and locally trivial $\operatorname{Mat}_{\mathbb{C}}(r, r)$-bundles. If the $\mathbb{P}^{r-1}$-bundle $\mathcal{P}$ is given by $\mathbb{P}(V)$ for a vector bundle V of rank r, then $\mathcal{E}_{\mathbb{P}(V)} = \mathcal{E}nd(V)$. Given a $\mathbb{C}$-algebra A, we can study families given by a locally free $\mathbb{P}^{r-1}$-bundle $\mathcal{P}$ or equivalently a locally free $\operatorname{Mat}_{\mathbb{C}}(r, r)$-bundle $\mathcal{E}$ and a homomorphism of algebras $A \to \Gamma(S, \mathcal{E})$. If $A = \mathbb{C}$, there is only a fine moduli space for $r = 1$ as every $\mathbb{P}^0$-bundle must be constant. If the algebra A is more complicated, there are also fine moduli spaces for $r > 1$, but only for objects which are simple in a suitable sense. For $A = \mathbb{C}$ there are no simple vector spaces of dimension $r > 1$.

As we have seen, the construction of fine moduli spaces can only be done in a few cases and severe restrictions. But even if we were only interested in coarse moduli spaces, a standard problem will occur as the following example shows.

Example 2.11. — Instead of looking at representations of $A = \mathbb{C}$, we enter the next level of complexity by looking at finite dimensional representations of $A = \mathbb{C}[z]$. A one-dimensional representation V is determined by the value of z in $\text{End}_{\mathbb{C}}(V) \cong \mathbb{C}$. In other words, a coarse moduli space is given by the complex affine line $\mathbb{A}^1$. Still, we have to face the problem discussed before that a line bundle on S with z acting by multiplication with a fixed number $c \in \mathbb{C}$ could almost never be the pull-back of a universal family under the constant map $S \to \mathbb{A}^1$ mapping $s \in S$ to $c \in \mathbb{A}^1$. Let us ignore the problem of finding a fine moduli space and continue with two-dimensional representations. Consider the trivial rank 2 bundle on $S = \mathbb{A}^1$ with z acting via the unipotent matrix

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

in the fiber over $s \in S = \mathbb{A}^1$. The representations for $s \neq 0$ are all isomorphic to each other, and our "classifying map" $u : S \to \mathcal{M}_2$ to a coarse moduli space $\mathcal{M}_2$ of 2-dimensional representations of A must be constant on $S \backslash \{0\}$. For $s = 0$ we obtain a different representation and $u(0)$ must be another point in $\mathcal{M}_2$ if the latter parametrizes isomorphism types. However, such a discontinuous map $u : S \to \mathcal{M}_2$ cannot exist, and we have to abandon the idea of finding a coarse moduli space parameterizing isomorphism classes. One can show that a "reasonable" coarse moduli space is given by the GIT-quotient $\text{Mat}_{\mathbb{C}}(2,2) /\!\!/ \, \text{GL}(2)$ which is realized as $\text{Spec}\, \mathbb{C}[\text{Mat}_{\mathbb{C}}(2,2)]^{\text{GL}(2)} \cong \mathbb{A}^2$ and similar for higher ranks. The classifying map $S \to \mathbb{A}^2$ will map $s \in S$ to the unordered pair of eigenvalues of the z-action in the fiber over s. Such an unordered pair of eigenvalues is determined by the sum (corresponding to the trace) and the product (corresponding to the determinant) of the eigenvalues and similar for higher ranks. Therefore, $\mathcal{M}$ will parametrize unordered direct sums of one-dimensional representations. In other words, by passing from $\mathcal{C}/_\sim$ to $\mathcal{M}$, we identify each representation with the (unordered) direct sum of its simple Jordan–Hölder factors. Representations having the same Jordan–Hölder factors, i.e. corresponding to the same point in $\mathcal{M}$, are often called S-equivalent[1].

Let us summarize the lessons we have learned in the previous examples:

(1) Constructing coarse moduli spaces has only a chance if we do not parametrize objects up to isomorphism but up to the weaker S-equivalence. In other words, classifying objects up to isomorphism is only possible for simple objects, i.e. objects without subobjects.

(2) The construction of a universal family on the moduli space of simple objects might only work if we identify two families under a weaker equivalence (twist with a line bundle) or pass to some projectivization.

We suggest to the reader to check these statements in the previous examples.

Even though the set of objects in $\mathcal{C}$ up to isomorphism might be very large, the set of (isomorphism classes of) simple objects can be quite small, even finite. Thus, the "coarse" moduli space would not deliver much insight into the set of isomorphism types in $\mathcal{C}$. However, there is a simple but clever idea to overcome this problem. Instead of looking at $\mathcal{C}$, we should "scan" $\mathcal{C}$ by means of a collection $(\mathcal{C}_\mu)_{\mu \in T}$

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[1] The "S" in "S-equivalent" refers to semisimple, i.e. sums of simples, and should not be confused with our notation of a base of a family.

of "small" full subcategories $\mathcal{C}_\mu \subseteq \mathcal{C}$. An object which might be far away from being simple or semisimple (direct sum of simples) can become semisimple or even simple in $\mathcal{C}_\mu$. By doing this, we can distinguish many S-equivalent objects either because they live in different subcategories or they live in the same subcategory $\mathcal{C}_\mu$ but have different Jordan–Hölder filtrations taken in $\mathcal{C}_\mu$. This brilliant idea is the essence of the concept of stability conditions. The following definition is due to Tom Bridgeland. However, there are more general definitions of stability conditions.

DEFINITION 2.12. —   (1) A central charge on a noetherian abelian category $\mathcal{C}$ is a function Z on the set of objects in $\mathcal{C}$ with values in

$$\mathbb{H}_+ := \{r \exp(\sqrt{-1}\phi) \in \mathbb{C} \mid r \geqslant 0, \phi \in (0, \pi]\}$$

such that $Z(E) = 0$ implies $E = 0$ and $Z(E) = Z(E') + Z(E'')$ for every short exact sequence $0 \to E' \to E \to E'' \to 0$.

(2) Given a central charge Z, we call an object $E \in \mathcal{C}$ semistable if

$$\arg Z(E')) \leqslant \arg Z(E) \quad \text{for all subobjects } E' \subset E.$$

(3) For $\mu \in (-\infty, +\infty]$ we denote by $\mathcal{C}_\mu$ the full subcategory of all semistable objects E of slope $-\cot(\arg Z(E)) = \mu$ and the zero object. It turns out that $\mathcal{C}_\mu$ is an abelian subcategory of $\mathcal{C}$ (cf. Exercise 3.12).

(4) A simple object in $\mathcal{C}_\mu$ is called stable. We assume that every object E in $\mathcal{C}_\mu$ has a Jordan–Hölder filtration with simple subquotients E_i ($i \in I$) and write $\mathrm{gr}(E) := \oplus_{i \in I} E_i$. Two objects E, E' in $\mathcal{C}_\mu$ are called S-equivalent if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$. Semisimple objects E in $\mathcal{C}_\mu$, i.e. those with $E \cong \mathrm{gr}(E)$, are called polystable.

(5) Every object E in $\mathcal{C}$ has a unique filtration $0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E$, the Harder-Narasimhan filtration, with semistable quotients E_i/E_{i-1} of strictly decreasing slopes.

Example 2.13 (The r-Kronecker quiver). —   Let us illustrate this idea with a simple example. Consider the abelian category of r-tuples $\bar{x}$ of linear maps $x_i : V_1 \to V_2$ for $1 \leqslant i \leqslant r$ between finite dimensional vector spaces V_1, V_2. Choosing two complex numbers $\zeta_1, \zeta_2 \in \mathbb{H}_+$, we get a central charge by putting $Z(\bar{x}) = \zeta_1 \dim V_1 + \zeta_2 \dim V_2$. Assume first that $\arg(\zeta_1) = \arg(\zeta_2)$. Then, all objects are semistable of the same slope $\mu = -\cot(\arg \zeta_1)$, and we have to face the old problems. Choose for instance $\dim V_1 = \dim V_2 = 1$. The isomorphism type of such objects is determined by the choice of r complex numbers $x_1, \ldots, x_r$ up to rescaling by $(g_1, g_2) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) = \mathbb{C}^* \times \mathbb{C}^*$ via $g_1 x_i g_2^{-1}$. As the diagonal group $\{(g, g) \mid g \in \mathbb{C}^*\}$ acts trivially, we have the take the GIT quotient of $\mathbb{A}^r$ by $\mathbb{C}^*$ which is just a point as $\mathrm{Spec}\,\mathbb{C}[x_1, \ldots, x_r]^{\mathbb{C}^*} = \mathrm{Spec}\,\mathbb{C}$. This corresponds to the fact that all objects have the same Jordan–Hölder factors $x_i = 0 : V_1 \to 0$ and $x_i = 0 : 0 \to V_2$. Thus, all objects are S-equivalent to "$V_1 \oplus V_2$"= $V_1 \xrightarrow{0} V_2$. If $\arg \zeta_2 > \arg \zeta_1$, non of our objects with $\dim V_1 = \dim V_2 = 1$ are semistable as the central charge ζ_2 of the subobject $0 : 0 \to V_2$ has a bigger argument than the central charge $\zeta_1 + \zeta_2$ of our given object. If, however, $\arg \zeta_2 < \arg \zeta_1$, all objects except for the semisimple $V_1 \oplus V_2$ corresponding to $x_i = 0$ for all $1 \leqslant i \leqslant r$ are semistable of slope $\mu = -\Re e(\zeta_1 + \zeta_2)/\Im m(\zeta_1 + \zeta_2)$, and even stable. The moduli space $\mathcal{M}_{(1,1)}^{\zeta_1, \zeta_2} = \mathbb{A}^r \backslash \{0\}/\mathbb{C}^* = \mathbb{P}^{r-1}$ parameterizing isomorphism classes of simple objects in $\mathcal{C}_\mu$ of dimension vector $(\dim V_1, \dim V_2) = (1, 1)$ is even a fine moduli space if we identify

two families of r-tuples of line bundle morphisms $x_i : V_1 \to V_2$ on a parameter space S as soon as they become isomorphic after twisting V_1 and V_2 with some line bundle L.

Note that coarse moduli spaces parameterizing S-equivalence classes of objects in $\mathcal{C}_\mu$ might not exist for all central charges, but one can show the existence for generic central charges and reasonable abelian categories.

We should also keep in mind that we paid a price for getting a refined version of S-equivalence, namely S-equivalence in subcategories. Indeed, coarse moduli spaces of (S-equivalence classes of) semistable objects can only "see" semistable objects but no objects with a non-trivial Harder-Narasimhan filtration. Hence, the construction of (coarse) moduli spaces remains unsatisfying.

There is, however, an alternative way to overcome all the problems seen in the previous examples. Following this approach, one can construct a fine moduli "space" with a universal family parameterizing all objects — not only simple or stable ones — up to isomorphism. According to the conservative law of mathematical difficulties, we also have to pay a price for getting such a beautiful solution of our moduli problem. It is hidden in the word "space". In fact, we have to leave our comfort zone of varieties or schemes and have to dive into the universe of more general spaces known as "Artin stacks".

Recall that a scheme X is uniquely characterized by its set-valued functor $h_X : S \mapsto \mathrm{Mor}(S, X)$ of points. We have seen many set-valued functors before while studying moduli problems. The general pattern was the following. We considered set-valued contravariant functors $F : S \mapsto \{\text{families of objects in } \mathcal{C}\}/_\sim$ and a fine moduli space would be a scheme $\mathcal{M}$ such that $F \cong h_\mathcal{M}$, while a coarse moduli space is a scheme $\mathcal{M}$ together with a map $F \to h_\mathcal{M}$ which is universal with respect to all maps $F \to h_X$ of functors. One possibility of generalizing the concept of a space is to consider set-valued functors satisfying similar properties like the functor h_X. Note that if one has a collection of morphisms $U_i \to X$ defined on open subsets U_i covering S such that the maps agree on overlaps, one can glue the maps to form a global morphism $S \to X$. This sheaf property should also be satisfied by a general set-valued functor to be a reasonable generalization of a scheme. Such set-valued functors are also often called "spaces". A generalized space is called algebraic if it can be written as the "quotient" $X/_\sim$ of a scheme X by an (étale) equivalence relation. In other words, algebraic spaces are not to far away from schemes and many results for schemes can be generalized to algebraic spaces. In our situation of forming moduli spaces, this is still not the right approach to take, but shows already into the right direction. Indeed, the problems arising in the construction of universal families are related to the presence of (non-trivial) automorphisms. Thus, we should take automorphisms and isomorphisms more seriously into account.

Recall that a set with isomorphisms between points was just a groupoid studied at the beginning of this section. Hence, instead of looking at set-valued functors on the category of schemes, we should consider groupoid-valued contravariant functors. These functors should satisfy some gluing property which looks a bit more complicated than in the set-theoretic context. The best idea of remembering the gluing property is by looking at an example which is — as before — the baby example for all Artin stacks.

Example 2.14. —   Consider the groupoid-valued functor Vect which maps any scheme S to the groupoid of vector bundles (the objects) and isomorphisms between them (the morphisms). By pulling back vector bundles along morphisms $f : S' \to S$, we get indeed a contravariant functor.[2] Given two vector bundles V, V' and an open cover $\bigcup_{i \in I} U_i = S$ of S together with isomorphisms[3] $\phi_i : V|_{U_i} \to V'|_{U_i}$ on the open subsets U_i such that they agree after restriction to the overlaps, i.e. $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$ with $U_{ij} = U_i \cap U_j$, we can always find a unique global isomorphism $\phi : V \to V'$ such that $\phi_i = \phi|_{U_i}$. On the other hand, if we have vector bundles V_i on U_i and isomorphisms $\phi_{ij} : V_i|_{U_{ij}} \to V_j|_{U_{ij}}$ such that the only possible composition $V_i|_{U_{ijk}} \to V_j|_{U_{ijk}} \to V_k|_{U_{ijk}} \to V_i|_{U_{ijk}}$ of their restrictions to the triple overlaps $U_{ijk} = U_i \cap U_j \cap U_k$ is the identity (cocycle condition), one can use the transition isomorphisms ϕ_{ij} to glue the V_i together, i.e. there is a vector bundle V on S and a family of isomorphisms $\phi_i : V|_{U_i} \to V_i$ such that the only possible composition $V|_{U_{ij}} \to V_i|_{U_{ij}} \to V_j|_{U_{ij}} \to V|_{U_{ij}}$ of their restrictions with ϕ_{ij} is the identity. This was the gluing property for isomorphisms and objects, and if we replace the word "vector bundle" with "object", we get the general form of the gluing property for a groupoid-valued functor.

DEFINITION 2.15. —   A stack is a groupoid-valued contravariant functor[4] on the category of schemes satisfying the gluing property for isomorphisms and objects as seen in Example 2.14.

In that perspective, a stack is like a (generalized) space with set-valued functors replaced with groupoid-valued functors.

Exercise 2.16. —   Thinking of a set as a special groupoid with no nontrivial isomorphisms, show that every generalized space is a stack.

Exercise 2.17. —   Fix a $\mathbb{C}$-algebra A. Show that the functor A-Rep associating to every scheme S the groupoid of vector bundles V with algebra homomorphisms $A \to \Gamma(S, \mathcal{E}nd(V))$ (the objects) and isomorphisms of vector bundles compatible with the algebra homomorphisms (the morphisms) is a stack. Prove the same for bundles $\mathcal{E}$ of matrix algebras and algebra homomorphisms $A \to \Gamma(S, \mathcal{E})$ as in Example 2.10.

Exercise 2.18. —   Fix a scheme/variety/manifold X over $\mathbb{C}$. Show that the functor Coh^X associating to every scheme S the groupoid of coherent sheaves E on $S \times X$ flat over S (the objects) and isomorphisms between them (the morphisms) is a stack.

Exercise 2.19. —   Fix an algebraic group G. Show that the functor $\mathrm{Spec}\,\mathbb{C}/G$ associating to every scheme the groupoid of principal G-bundles (the objects) and isomorphisms between them (the morphisms) is a stack.

Example 2.20. —   The following example is a generalization of the previous exercise. Fix an algebraic group G and a scheme X with a (right) G-action. There

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[2]Strictly speaking, we only get a pseudofunctor as $g^* \circ f^*$ is only equivalent to $(f \circ g)^*$, but we will ignore this technical problem as one can always resolve it.

[3]We will always denote the pull-back along an inclusion $U \hookrightarrow S$ of an open subset by $|_U$.

[4]Again, we ignore the fact that $g^* \circ f^*$ might only be equivalent to $(f \circ g)^*$ for a pair $S'' \xrightarrow{g} S' \xrightarrow{f} S$ of composable morphisms.

is a stack X/G associating to every scheme S the groupoid of pairs $(P \to S, m : P \to X)$, where $P \to S$ is a principal G-bundle and $m : P \to X$ is a G-equivariant map, with morphisms being given by G-bundle isomorphisms $u : P \to P'$ satisfying $m' \circ u = m$. The pull-back along a morphism $f : S' \to S$ is given by $(S' \times_S P \to S', m \circ \mathrm{pr}_P)$. The morphism $m : P \to X$ can also be interpreted as a G-equivariant section of the trivial X-bundle $P \times X \to P$ with diagonal G-action on $P \times X$. Such G-equivariant sections are in bijection to sections of the X-bundle $P \times_G X \to S$ obtained by modding out the G-action. Thus, we could also describe X/G as the functor associating to S the groupoid of principal G-bundles P together with a section $m : S \to P \times_G X$. Morphisms are given by isomorphisms $u : P \to P'$ of G-bundles such that the induced isomorphism $P \times_G X \to P' \times_G X$ of X-bundles maps m to m'. The stack X/G is called the quotient stack of X with respect to the G-action.

When is comes to locally trivial families, there is some choice involved, namely the choice of the underlying (Grothendieck) topology. Intuitively, one would start with the Zariski topology, but the étale or even the smooth topology have their advantages, too. In fact, the quotient stack X/G defined above is usually taken with respect to the smooth or, equivalently, étale topology.[5] However, for so-called "special" groups G like $\mathrm{GL}(n)$ we could equivalently take the Zariski topology as every étale locally trivial principal G-bundle is then already Zariski locally trivial. Notice that $\mathrm{PGL}(d)$ is not special and we should better take the étale topology when it comes to principal $\mathrm{PGL}(d)$-bundles and quotient stacks $X/\mathrm{PGL}(d)$.

DEFINITION 2.21. — A 1-morphism (or morphism for short) from a stack F to a stack F' is a natural transformation $\eta : F \to F'$, i.e. a family of functors $\eta_S : F(S) \to F'(S)$ compatible with pull-backs along $f : S' \to S$ up to equivalence of functors. In other words, the functors $F'(f) \circ \eta_S$ and $\eta_{S'} \circ F(f)$ from $F(S)$ to $F'(S')$ are equivalent. A 2-morphism $\alpha : \eta \to \eta'$ between 1-morphisms is an invertible natural transformation $\alpha_S : \eta_S \to \eta'_S$ for every scheme S, compatible with pull-backs. In particular, given two stacks F, F', we get a groupoid of morphisms $\underline{\mathrm{Mor}}(F, F')$ with 1-morphisms being the objects and 2-morphisms being the morphisms. Hence, the category of stacks is a 2-category.

Thinking of a set as being a groupoid having only identity morphisms, we can associate to every scheme X a contravariant functor $h_X : S \mapsto \mathrm{Mor}(S, X)$. As we can glue morphisms, h_X is indeed a stack. The following lemma is very important.

LEMMA 2.22 (Yoneda Lemma). — *The covariant functor $h : X \mapsto h_X$ from schemes to stacks provides a full embedding of the category of schemes into the (2-) category of stacks. Moreover, there is an equivalence of groupoids $\underline{\mathrm{Mor}}(h_X, F) \cong F(X)$ for every scheme X and every stack F, natural in X and F.*

Exercise 2.23. — Try to prove the Yoneda Lemma.

The lemma is basically saying that the 2-category of stacks is an enlargement of the category of schemes, and we will drop the functor h from notation. Though the definition of a stack looks very abstract, the reader should not think of a stack F as a complicated functor, but rather as some object of a bigger 2-category

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[5]The equivalence of the étale and the smooth topology is a consequence of [34, Tag 055V].

containing the category of schemes. The groupoid-valued functor associated to F can be (re)constructed by taking $X \mapsto \underline{\mathrm{Mor}}(X, F)$. In other words, assume that you have a 2-category $\mathcal{C}$ with 2-morphisms being invertible, containing $\mathrm{Sch}_{\mathbb{C}}$ as a full subcategory, and such that 1-morphisms starting at schemes and 2-morphisms between such 1-morphisms can be glued in a natural way. To every object $F \in \mathrm{Obj}(\mathcal{C})$ we can associate the groupoid-valued functor $\underline{\mathrm{Mor}}(-, F)|_{\mathrm{Sch}_{\mathbb{C}}}$ on the category of schemes. It satisfies the gluing axioms given above, and, hence, defines a stack. Thus, we get a covariant functor from $\mathcal{C}$ to the category of stacks showing that stacks form some sort of "natural" enlargement.

Exercise 2.24. —   (1) Let X be a scheme with a right action of an algebraic group G. Consider the trivial principal G-bundle $\mathrm{pr}_X : X \times G \to X$ on X and the G-equivariant map $m : X \times G \to X$ given by the group action. According to our definition of a quotient stack, the pair $(X \times G \to X, m)$ defines an element ρ in $X/G(X)$. Check the Yoneda Lemma by constructing a morphisms $\rho : X \to X/G$ which is called the "standard atlas" of X/G.

(2) Given two schemes X, Y and two algebraic groups G, H acting on X and Y respectively. Assume that $\phi : G \to H$ is a group homomorphism and that $f : X \to Y$ is a morphism of schemes satisfying $f(xg) = f(x)\phi(g)$ for all $g \in G$ and $x \in X$. Construct a natural 1-morphism $\mathfrak{f} : X/G \to Y/H$ of stacks such that

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho_Y} \\
X/G & \xrightarrow{\ \mathfrak{f}\ } & Y/H
\end{array}
$$

commutes.

Warning: Not every morphism $X/G \to Y/H$ is of this form.

(3) Consider the special case $H = \{1\}$, and show that $\mathfrak{f} \mapsto f := \mathfrak{f} \circ \rho_X$ defines an equivalence from $\underline{\mathrm{Mor}}(X/G, Y)$ to the set $\mathrm{Mor}(X, Y)^G$ of G-invariant morphisms, thought as a groupoid.

(4) More generally, given a scheme Y and a stack F, show that $\underline{\mathrm{Mor}}(F, Y)$ is essentially a set, i.e. the only 2-morphisms are the identity morphisms.

Let us come back to moduli spaces. The moduli problem of classifying G-torsors P together with G-equivariant maps $P \to X$ for some fixed scheme X with an action of an algebraic group G, has a natural generalized "moduli space", namely the quotient stack X/G. This is not a deep insight, but just the definition of the associated moduli functor. Note that the isomorphism classes of the $\mathbb{C}$-valued points of X/G, i.e. $X/G(\mathrm{Spec}\,\mathbb{C})/_{\sim}$, is the set of G-orbits in X justifying the notation.

Quotient stacks are also very helpful when it comes to other moduli problems as the following example shows, and their usefulness cannot be overestimated.

Example 2.25. —   Consider the stack of finite dimensional representations of a $\mathbb{C}$-algebra A. Assume that A is finitely presented, i.e. A is generated by a set[6] Q_1 of finitely many elements $\alpha_1, \ldots, \alpha_n$ satisfying a finite set of relations $R =$

---

[6]The notation in this example has been chosen with an eye towards the next section.

$\{r_1, \ldots, r_m\}$. Fix a "dimension" $d \in \mathbb{N}$ and put

$$X_d = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^d, \mathbb{C}^d)^n = \prod_{\alpha \in Q_1} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^d, \mathbb{C}^d)$$

as well as

$$X_d^R = \{(M_\alpha)_{\alpha \in Q_1} \in X_d \mid r_j(M_{\alpha_1}, \ldots, M_{\alpha_n}) = 0 \text{ for all } 1 \leqslant j \leqslant m\}.$$

We claim that the moduli stack $A\text{-Rep}_d$ of d-dimensional representations of A is equivalent to the quotient stack $X_d^R / \operatorname{GL}(d)$ with $\operatorname{GL}(d)$ acting by conjugation on $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^d, \mathbb{C}^d)$. Indeed, a family of d-dimensional representations on $S \in \operatorname{Sch}_{\mathbb{C}}$ is uniquely determined by a vector bundle V of rank d on S and vector bundle endomorphisms $\hat{\alpha}$ associated to $\alpha \in Q_1$ satisfying the relations $r_1, \ldots, r_m$. Consider the frame bundle $\operatorname{Fr}(V) = \{(s, \tau) \mid s \in S, \tau \in \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^d, V_s) \text{ is invertible}\}$ of V parameterizing all possible choices of a basis in all possible fibers of V. It comes with a projection to S and a right action of $\operatorname{GL}(d)$ by composition with $\tau \in \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^d, V_s)$.

Exercise 2.25.1. — Show that $\operatorname{Fr}(V)$ is a principal $\operatorname{GL}(d)$-bundle.

There is also a $\operatorname{GL}(d)$-equivariant map $m(V, (\hat{\alpha})_{\alpha \in Q_1})$ from $\operatorname{Fr}(V)$ into X_d^R mapping a pair (s, τ) to $(M_\alpha := \tau^{-1} \circ \hat{\alpha}|_{V_s} \circ \tau)_{\alpha \in Q_1}$.

Exercise 2.25.2. — Convince yourself that the map

$$(V, (\hat{\alpha})_{\alpha \in Q_1}) \mapsto \big(\operatorname{Fr}(V), m(V, (\hat{\alpha})_{\alpha \in Q_1}) \big)$$

extends to a functor from the groupoid of families of d-dimensional A-representations into the groupoid $X_d^R / \operatorname{GL}(d)(S)$. Show furthermore that this functor is compatible with pull-backs, and, thus, defines a morphism $A\text{-Rep}_d \to X_d^R / \operatorname{GL}(d)$ of stacks.

Conversely, given a principal $\operatorname{GL}(d)$-bundle P on S and a $\operatorname{GL}(d)$-equivariant map $m : P \to X_d^R$, we can consider the trivial vector bundle $P \times \mathbb{C}^d$ on P which comes with a natural $\operatorname{GL}(d)$-action compatible with the projection to P. Moreover, picking the component of m associated to $\alpha \in Q_1$, we get an endomorphism $\hat{\alpha}$ of this trivial bundle. $\operatorname{GL}(d)$-equivariance of m ensures that $\hat{\alpha}$ commutes with the $\operatorname{GL}(d)$-action on $P \times \mathbb{C}^d$. By taking the $\operatorname{GL}(d)$-quotient, we obtain a vector bundle $V = P \times_{\operatorname{GL}(d)} \mathbb{C}^d$ of rank d on S along with vector bundle endomorphisms $\hat{\alpha}$ satisfying the relations $r_1, \ldots, r_m$.

Exercise 2.25.3. — Show that this construction extends to a functor between groupoids, compatible with pull-backs. Hence, we obtain a morphism from the quotient stack $X_d^R / \operatorname{GL}(d)$ to $A\text{-Rep}_d$. Prove that this morphism provides an inverse (up to 2-isomorphism) of the morphism constructed above.

Thus, the claim is proven and the stack $A\text{-Rep}$ of A-representations is isomorphic to $\bigsqcup_{d \in \mathbb{N}} X_d^R / \operatorname{GL}(d)$.

Exercise 2.26. — Use a similar idea of frame bundles parameterizing tuples $(s \in S, \operatorname{Mat}_{\mathbb{C}}(r, r) \xrightarrow{\sim} \mathcal{E}_s)$ for a locally trivial family $\mathcal{E}$ on S of $\mathbb{C}$-algebras isomorphic to $\operatorname{Mat}_{\mathbb{C}}(r, r)$ to show that the stack of projective A-representations is given by $\bigsqcup_{d \in \mathbb{N}} X_d^R / \operatorname{PGL}(d)$. As in Exercise 2.24(2), we obtain a morphism $X_d^R / \operatorname{GL}(d) \to X_d^R / \operatorname{PGL}(d)$ by means of the group homomorphism $\operatorname{GL}(d) \to \operatorname{PGL}(d)$. Show that this morphism is mapping the A-representation on V to the projective A-representation on $\mathbb{P}(V)$, in other words, forget V and keep $\mathcal{E}nd(V)$ together with the algebra homomorphism $A \to \Gamma(S, \mathcal{E}nd(V))$.

Example 2.27. — The "geometry" of the moduli stack Coh^X of coherent sheaves on a smooth projective variety X is more involved. First of all, it decomposes into components Coh^X_c indexed by numerical data like Chern classes similar to the dimension of a representation. Unfortunately, a component can not be written as a quotient stack. However, every component Coh^X_c is the nested union of "open" substacks $\mathrm{Coh}^X_{c,i}, i \in \mathbb{N}$, which can be written as a quotient stack $Y_{c,i}/\mathrm{GL}(n_{c,i})$. Note that $n_{c,i}$ grows with i. More details can be found is section 9 of [11].

The following definition of a fiber product is very important.

DEFINITION 2.28 (fiber product). — Given two morphisms $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Z}$ and $\mathfrak{g} : \mathfrak{Y} \to \mathfrak{Z}$ of groupoid-valued functors, we define the fiber product $\mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}$ as the groupoid-valued functor such that

$$\mathrm{Obj}\, \mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}(S)$$
$$= \{(x, y, w) \mid x \in \mathrm{Obj}\, \mathfrak{X}(S), y \in \mathrm{Obj}\, \mathfrak{Y}(S), w \in \mathrm{Mor}_{\mathfrak{Z}(S)}(\mathfrak{f}_S(x), \mathfrak{g}_S(y))\},$$

and

$$\mathrm{Mor}_{\mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}(S)}\big((x, y, w), (x', y', w')\big)$$
$$= \{(u, v) \in \mathrm{Mor}_{\mathfrak{X}(S)}(x, x') \times \mathrm{Mor}_{\mathfrak{Y}(S)}(y, y') \mid \quad \begin{array}{ccc} \mathfrak{f}_S(x) & \xrightarrow{\ w\ } & \mathfrak{g}_S(y) \\ {\scriptstyle \mathfrak{f}_S(u)}\downarrow & & \downarrow{\scriptstyle \mathfrak{g}_S(v)} \\ \mathfrak{f}_S(x') & \xrightarrow{\ w'\ } & \mathfrak{g}_S(y') \end{array} \quad \text{commutes}\}$$

for every $S \in \mathrm{Sch}_\mathbb{C}$.

Exercise 2.29. — Show the main properties of the fiber product.

(1) Prove that $\mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}$ is a stack if $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ were stacks.

(2) Construct two morphisms $\mathrm{pr}_\mathfrak{X} : \mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y} \to \mathfrak{X}$ and $\mathrm{pr}_\mathfrak{Y} : \mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y} \to \mathfrak{Y}$ of groupoid-valued functors and a 2-morphism $\omega : \mathfrak{f} \circ \mathrm{pr}_\mathfrak{X} \to \mathfrak{g} \circ \mathrm{pr}_\mathfrak{Y}$. Show that the following universal property holds. Given a groupoid-valued functor $\mathfrak{T}$, two morphisms $\mathfrak{p} : \mathfrak{T} \to \mathfrak{X}$, $\mathfrak{q} : \mathfrak{T} \to \mathfrak{Y}$ and a 2-morphism $\eta : \mathfrak{f} \circ \mathfrak{p} \to \mathfrak{g} \circ \mathfrak{q}$, there is a unique morphism $\mathfrak{r} : \mathfrak{T} \to \mathfrak{X} \times_\mathfrak{Z} \mathfrak{Y}$ such that $\mathrm{pr}_\mathfrak{X} \circ \mathfrak{r} = \mathfrak{p}$ and $\mathrm{pr}_\mathfrak{Y} \circ \mathfrak{r} = \mathfrak{q}$.
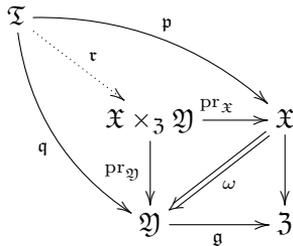
When it comes to quotient stacks, the following examples are very useful.

Exercise 2.30. —   (1) Assume $\mathfrak{X} = X, \mathfrak{Y} = Y \in \mathrm{Sch}_\mathbb{C}$ and $\mathfrak{Z} = Z/G$ for some algebraic group G acting on a scheme Z. The morphisms $\mathfrak{f} : X \to Z/G$ and $\mathfrak{g} : Y \to Z/G$ are given by principal G-bundles $P \to X$ and $Q \to Y$ together with G-equivariant morphisms $f : P \to Z$ and $g : Q \to Z$

respectively. Show that the fiber product $X \times_{Z/G} Y$ is given by the scheme $Iso_{f,g}(P, Q) \subseteq Iso(P, Q)$ over $X \times Y$ given by

$$\{(x, y, w) \mid x \in X, y \in Y, w : P_x \xrightarrow{\sim} Q_y \ G\text{-equivariant such that } f|_{P_x} = g|_{Q_x} \circ w\}.$$

(2) Assume furthermore $X = Z$ and $P = X \times G \xrightarrow{\mathrm{pr_X}} X$ with $f : X \times G \to X$ being the group action. Hence, $\mathfrak{f}$ is the standard atlas $\rho : X \to X/G$. Show that $Iso_{f,g}(P, Q)$ is isomorphic to Q, and

$$
\begin{array}{ccc}
Q & \xrightarrow{\ g\ } & X \\
\downarrow & & \downarrow{\scriptstyle \rho} \\
Y & \xrightarrow{\ \mathfrak{g}\ } & X/G
\end{array}
$$

is the fiber product diagram, i.e. a cartesian square. Hence, $\rho : X \to X/G$ is the universal principal G-bundle.

Example 2.31. — Let $\phi : G \to K$ and $\psi : H \to K$ be homomorphisms between algebraic groups G, H, K acting on X, Y and Z respectively. Moreover, let $f : X \to Z$ and $g : Y \to Z$ be two morphisms such that $f(xg) = f(x)\phi(g)$ and $g(yh) = g(y)\psi(h)$ for all $x \in X$, $y \in Y$, $g \in G$ and $h \in H$. As we have seen in Exercise 2.24, this induces morphisms $\mathfrak{f} : X/G \to Z/K$ and $\mathfrak{g} : Y/H \to Z/K$. Then, $X/G \times_{Z/K} Y/H$ is the quotient stack $(X \times Y) \times_{(Z \times Z)} (Z \times K)/(G \times H)$ using the group actions

$$
\begin{aligned}
(x, y)(g, h) &= (xg, yh), \\
(z, k)(g, h) &= (z\phi(g), \phi(g)^{-1}k\psi(h)), \\
(z_1, z_2)(g, h) &= (z_1\phi(g), z_2\psi(h))
\end{aligned}
$$

of $G \times H$ on $X \times Y$, $Z \times K$ and $Z \times Z$, respectively, and the $G \times H$-equivariant morphisms

$$
\begin{aligned}
X \times Y \ni (x, y) &\mapsto (f(x), g(y)) \in Z \times Z, \\
Z \times K \ni (z, k) &\mapsto (z, zk) \in Z \times Z.
\end{aligned}
$$

Exercise 2.32. — Use the previous example to show that every fiber of the morphism $X_d^R / \mathrm{GL}(d) \to X_d^R / \mathrm{PGL}(d)$ constructed in Exercise 2.26 is isomorphic to $\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m$. Interpret this result in terms of (projective) A-representations. (cf. Example 2.10)

The following definition is slightly stronger than the one used in the literature as we do not have algebraic spaces at our disposal. However, it will be sufficient for our purposes.

DEFINITION 2.33. — A morphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Z}$ is called representable if for every morphisms $\mathfrak{g} : Y \to \mathfrak{Z}$ from a scheme Y into $\mathfrak{Z}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Z}} Y$ is (represented by) a scheme. In such a situation, we call $\mathfrak{f}$ smooth, surjective etc. if $\mathfrak{X} \times_{\mathfrak{Z}} Y \xrightarrow{\mathrm{pr_Y}} Y$ is smooth, surjective etc.

Exercise 2.34. — Here are some examples of representable morphisms.

(1) Show that every morphism between schemes is representable.

(2) Prove that the standard atlas $\rho : X \to X/G$ is representable, smooth and surjective. Hint: Every algebraic group is a smooth scheme.

(3) Use Example 2.31 to show that $\mathfrak{f} : X/G \to Z/K$ is representable if $\phi : G \to K$ is injective. Give a counterexample for the converse statement.

(4) Prove that the diagonal $\Delta_{\mathfrak{Z}} : \mathfrak{Z} \to \mathfrak{Z} \times_{\mathrm{Spec}\,\mathbb{C}} \mathfrak{Z}$ is representable if and only if every morphism $\mathfrak{f} : X \to \mathfrak{Z}$ from a scheme X is representable. Hint: $X \times_{\mathfrak{Z}} Y = (X \times Y)_{(\mathfrak{Z} \times \mathfrak{Z})} \mathfrak{Z}$.

DEFINITION 2.35. — A stack $\mathfrak{X}$ is called algebraic or an Artin stack if

(i) $\Delta_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable (cf. Exercise 2.34(4)) and

(ii) there is a smooth, surjective morphism $\rho : X \to \mathfrak{X}$ from a scheme X.

In such a situation, we call $\rho : X \to \mathfrak{X}$ an atlas of $\mathfrak{X}$.

In a suitable sense, the algebraic stack $\mathfrak{X}$ is a quotient of its atlas X similar to the concept of an algebraic space. However, the quotient is taken in the category of groupoids and not in the category of sets as before. As we have seen in Exercise 2.34, every quotient stack is an Artin stack with standard atlas $\rho : X \to X/G$. By taking $X^R = \bigsqcup_{d \in \mathbb{N}} X_d^R \to A\text{-Rep}$, the moduli stack of finite dimensional representations of a finitely represented $\mathbb{C}$-algebra A is also algebraic. Finally, using $Y = \bigsqcup_{c,i} Y_{c,i} \to \mathrm{Coh}^X$, we see that the moduli stack of coherent sheaves on a smooth projective variety X is also an Artin stack.

3. QUIVERS AND THEIR MODULI

3.1. **Quivers and $\mathbb{C}$-linear categories.** Recall that a groupoid is a category generalizing groups and sets. Similarly, there is a categorical concept interpolating between $\mathbb{C}$-algebras and sets. These are the so-called $\mathbb{C}$-linear categories. A category $\mathcal{A}$ is called $\mathbb{C}$-linear if the morphism sets $\mathrm{Mor}_{\mathcal{A}}(x,y)$ have the structure of a $\mathbb{C}$-vector space such that the composition of morphisms is $\mathbb{C}$-bilinear. As usual, we write $\mathrm{Hom}_{\mathcal{A}}(x,y)$ for the $\mathbb{C}$-vector space of morphisms from x to y and $\mathrm{End}_{\mathcal{A}}(x) = \mathrm{Hom}_{\mathcal{A}}(x,x)$ for the $\mathbb{C}$-algebra of endomorphisms of $x \in \mathrm{Obj}(\mathcal{A})$. A $\mathbb{C}$-linear category with one object is just a $\mathbb{C}$-algebra. On the other hand, $\mathbb{C}$-linear categories with as less morphisms as possible are uniquely classified by their set of objects since any morphism must be zero or a multiple of the identity of some object. Another standard example of a $\mathbb{C}$-linear category is given by the category $\mathrm{Vect}_{\mathbb{C}}$ of finite dimensional $\mathbb{C}$-vector spaces. A finite dimensional representation of a $\mathbb{C}$-linear category $\mathcal{A}$ is simply given by a functor $V : \mathcal{A} \to \mathrm{Vect}_{\mathbb{C}}$. Indeed, if the category $\mathcal{A}$ has only one object $\star$, $V(\star)$ is just a finite dimensional representation of the endomorphism algebra $\mathrm{End}_{\mathcal{A}}(\star)$. As we have seen in the previous section, generators of algebras are very useful when it comes to the construction of moduli stacks. The analogue in the context of $\mathbb{C}$-linear categories is called a quiver. A quiver consists of a set of objects Q_0 and a set of "arrows" Q_1 along with maps $s, t : Q_1 \to Q_0$ indicating the <u>s</u>ource and the <u>t</u>arget of an arrow. We do not require a composition law nor identity morphisms. Given a $\mathbb{C}$-linear category $\mathcal{A}$, a quiver in $\mathcal{A}$ satisfies $Q_0 \subseteq \mathrm{Obj}(\mathcal{A})$, $Q_1 \subseteq \mathrm{Mor}(\mathcal{A})$ and s, t are given by restriction of the corresponding maps on $\mathrm{Mor}(\mathcal{A})$ to Q_1. We say that $\mathcal{A}$ is generated by a quiver Q, if the smallest $\mathbb{C}$-linear category containing Q is $\mathcal{A}$ which implies $Q_0 = \mathrm{Obj}(\mathcal{A})$. There is a biggest $\mathbb{C}$-linear category generated by a given quiver Q, the so-called path category $\mathbb{C}Q$ of Q. A morphism of $\mathbb{C}Q$ from $x \in Q_0$ to $y \in Q_0$ is a $\mathbb{C}$-linear

combination of chains $x = x_1 \to x_2 \to \ldots \to x_{n-1} \to x_n = y$ of composable arrows in Q_1. We also need to add an identity morphism and its $\mathbb{C}$-linear multiples.

Exercise 3.1. — Construct a category of quivers such that $Q \mapsto \mathbb{C}Q$ is a functor from this category to the category of (small) $\mathbb{C}$-linear categories. Construct a right adjoint of this functor.

Exercise 3.2. — Show that there is a bijection between representations of $\mathbb{C}Q$ and representations V of Q associating to every $i \in Q_0$ a vector space V_i and to every arrow $\alpha : i \to j$ in Q_1 a $\mathbb{C}$-linear map $V(\alpha) : V_i \to V_j$.

Given a $\mathbb{C}$-linear category $\mathcal{A}$ and a generating quiver Q in $\mathcal{A}$, we get a full functor $\mathbb{C}Q \twoheadrightarrow \mathcal{A}$ which is a bijection on the set of objects. The kernel is a $\mathbb{C}$-linear subcategory $\mathcal{I}$ in $\mathbb{C}Q$ which has the property $a \circ b \in \mathrm{Mor}(\mathcal{I})$ if $a \in \mathrm{Mor}(\mathcal{I})$ or $b \in \mathrm{Mor}(\mathcal{I})$ categorifying the concept of an ideal. A generating quiver for $\mathcal{I}$ is uniquely determined by its set of arrows $R \subseteq \mathrm{Mor}(\mathbb{C}Q)$ which are called relations. Conversely, every quiver Q with relations gives rise to a $\mathbb{C}$-linear category $\mathbb{C}Q/(R)$ uniquely defined up to isomorphism. Conversely, every $\mathbb{C}$-linear category $\mathcal{A}$ can be written like this (up to isomorphism) in many ways.

Throughout this paper we will only consider finite quivers, i.e. $|Q_0| < \infty$ and $|Q_1| < \infty$ and similarly for the relations. Hence, the $\mathbb{C}$-linear categories $\mathcal{A}$ which can be described by a finite quiver with finitely many relations are exactly the finitely presented $\mathbb{C}$-linear categories.

Exercise 3.3. — Show that the category of $\mathbb{C}$-linear categories $\mathcal{A}$ with finite set of objects is equivalent to the category of $\mathbb{C}$-algebras together with a distinguished finite set $\{e_i\}_{i \in I}$ of mutually orthogonal idempotent elements e_i such that $1 = \sum_{i \in I} e_i$. Hint: Put $A := \oplus_{i,j \in \mathrm{Obj}(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(i,j)$ and $e_i = \mathrm{id}_i$ for all $i \in \mathcal{A}$. Moreover, prove that the category of representations of such a $\mathbb{C}$-linear category is isomorphic to the category of representations of the associated algebra.

Using the last exercise, we can also talk about the path $\mathbb{C}$-algebra of a quiver with finite set Q_0 and its representations. Note that the path $\mathbb{C}$-algebra has a distinguished family $(e_i)_{i \in Q_0}$ of mutually orthogonal idempotent elements summing up to 1.

3.2. **Quiver moduli spaces and stacks.** Generalizing the moduli functor A-Rep of finite dimensional representations of a given $\mathbb{C}$-algebra A (see Example 2.17), we define the moduli functor $\mathcal{A}$-Rep of finite dimensional representations of a $\mathbb{C}$-linear category $\mathcal{A}$ as follows. To every scheme S over $\mathbb{C}$ we associate the isomorphism groupoid $\mathcal{A}$-Rep(S) of the category of functors $\mathcal{A} \to \mathrm{Vect}_S$, where Vect_S is the category of vector bundles on S.

Exercise 3.4. — Show that $\mathcal{A}$-Rep is a stack, i.e. satisfies the gluing axiom for groupoid-valued functors.

If $\mathcal{A}$ is represented by a quiver Q with relations $R \subseteq \mathrm{Mor}(\mathbb{C}Q)$, the category $\mathcal{A}$-Rep(S) is equivalent to the category of families $(V_i)_{i \in Q_0}$ of vector bundles on S together with vector bundle morphisms $\hat{\alpha} = V(\alpha) : V_i \to V_j$ such that $V(r) = r\big((\hat{\alpha})_{\alpha \in Q_1}\big) = 0$ for all $r \in R$, where we extended V from Q_1 to $\mathrm{Mor}(\mathbb{C}Q_1) \supseteq Q_1$.

Let us assume that Q_0, Q_1 and R are finite sets. Using Example 2.25, it should not come as a surprise that $\mathcal{A}$-Rep is isomorphic to a disjoint union $\mathfrak{M}^R :=$

$\bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathfrak{M}_d^R$ of quotient stacks $\mathfrak{M}_d^R := X_d^R / G_d$ with

$$X_d^R := \{(M_\alpha)_{\alpha \in Q_1} \in X_d \mid r\big((M_\alpha)_{\alpha \in Q_1}\big) = 0 \,\forall\, r \in R\}$$
$$\subseteq X_d := \prod_{Q_1 \ni \alpha : i \to j} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

and $G_d = \prod_{i \in Q_0} \operatorname{GL}(d_i)$ acting on X_d by simultaneous conjugation. The "dimension vector" $d \in \mathbb{N}^{Q_0}$ is fixing $\dim V = (\operatorname{rk} V_i)_{i \in Q_0}$.

Similarly, given a sequence of dimension vectors $d^{(1)}, \ldots, d^{(r)} \in \mathbb{N}^{Q_0}$ we denote by $X_{d^{(1)}, \ldots, d^{(r)}} \subseteq X_{d^{(1)} + \ldots + d^{(r)}}$ the affine subvariety parameterizing linear maps preserving the standard flag

$$0 \subseteq \mathbb{C}^{d_i^{(1)}} \subseteq \mathbb{C}^{d_i^{(1)}} \oplus \mathbb{C}^{d_i^{(2)}} \subseteq \ldots \subseteq \mathbb{C}^{d_i^{(1)}} \oplus \ldots \oplus \mathbb{C}^{d_i^{(r)}}$$

for every $i \in Q_0$. The subgroup $G_{d^{(1)}, \ldots, d^{(r)}} \subseteq G_{d^{(1)} + \ldots + d^{(r)}}$ is defined in the same way. Finally, we put $X_{d^{(1)}, \ldots, d^{(r)}}^R := X_{d^{(1)}, \ldots, d^{(r)}} \cap X_{d^{(1)} + \ldots + d^{(r)}}^R$.

Exercise 3.5. — Show that the stack of all successive extensions

$$0 \to V^{(1)} \to \hat{V}^{(2)} \to V^{(2)} \to 0,$$
$$0 \to \hat{V}^{(2)} \to \hat{V}^{(3)} \to V^{(3)} \to 0,$$
$$\vdots$$
$$0 \to \hat{V}^{(r-1)} \to \hat{V}^{(r)} \to V^{(r)} \to 0$$

of quiver representations satisfying the relations R and with $\dim V^{(j)} = d^{(j)}$ for all $1 \leqslant j \leqslant r$ is given by the quotient stack $\mathfrak{M}_{d^{(1)}, \ldots, d^{(r)}}^R = X_{d^{(1)}, \ldots, d^{(r)}}^R / G_{d^{(1)}, \ldots, d^{(r)}}$. Hint: The standard flag introduced above defines a standard successive extension of Q_0-graded vector spaces of prescribed dimension vectors. Given a family of successive extensions, consider the principal $G_{d^{(1)}, \ldots, d^{(r)}}$-bundle parameterizing all isomorphism from the standard extension to the fibers of the family, and proceed as usual.

We are mainly interested in the following type of relations. A potential W is an element of the vector space $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, where $[\mathbb{C}Q, \mathbb{C}Q]$ denotes the $\mathbb{C}$-linear span (and not the spanned ideal) of all commutators. Note that $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ is the 0-th Hochschild homology of the $\mathbb{C}$-linear category $\mathbb{C}Q$. Convince yourself that W is essentially just a $\mathbb{C}$-linear combination of equivalence classes of cycles in Q with two cycles being equivalent if they can be transformed into each other by a cyclic permutation.

Example 3.6. — The three elements $[x, y]z = xyz - yxz$, $[z, x]y$ and $[y, z]x$ in $\mathbb{C}Q^{(3)}$ of the 3-loop quiver $Q^{(3)}$
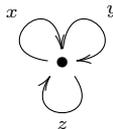
define the same potential W.

For a fixed potential $W = \sum_{l=1}^{L} a_l \cdot [C_l]$ we define relations $\partial W/\partial \alpha \in \mathrm{Hom}_{\mathbb{C}Q}(j, i)$ for every $\alpha : i \to j$ in Q_1 as follows.

$$\frac{\partial W}{\partial \alpha} := \sum_{l=1}^{L} a_l \cdot \sum_{C_l = u\alpha v} vu$$

with $a_l \in \mathbb{C}$, where the second sum is over all occurrences of α in a fixed representative of an equivalence class $[C_l]$ of cycles in Q.

Exercise 3.7. — Show that the definition of $\partial W/\partial \alpha$ is independent of the choice of the representative $C_l \in [C_l]$ for all $1 \leqslant l \leqslant L$.

Example 3.8. — Using the potential $W = [x, y]z = xyz - yxz$ from the previous example, we compute

$$\frac{\partial W}{\partial x} = yz - zy = [y, z],$$
$$\frac{\partial W}{\partial y} = zx - xz = [z, x],$$
$$\frac{\partial W}{\partial z} = xy - yx = [x, y].$$

Convince yourself that $W = [z, x]y$ and $W = [y, z]x$ provide the same relations.

Given a dimension vector $d \in \mathbb{N}^{Q_0}$ and a potential $W = \sum_{l=1}^{L} a_l \cdot [C_l]$ with $C_l = \alpha_l^{(1)} \circ \ldots \circ \alpha_l^{(n_l)}$, we define the following function

$$\mathrm{Tr}(W)_d : X_d \ni (M_\alpha)_{\alpha \in Q_1} \mapsto \sum_{l=1}^{L} a_l \cdot \mathrm{Tr}\left(M_{\alpha_l^{(1)}} \cdot \ldots \cdot M_{\alpha_l^{(n_l)}}\right) \in \mathbb{A}^1$$

which is independent of the choice of the representative $C_l \in [C_l]$ as the trace is invariant under cyclic permutation. By the same argument, $\mathrm{Tr}(W)_d$ is G_d-invariant, and induces a function $\mathfrak{Tr}(W)_d : \mathfrak{M}_d \to \mathbb{A}^1$ on the quotient stack.

Exercise 3.9. — Let us take the relations $R = \{\partial W/\partial \alpha \mid \alpha \in Q_1\}$. Show that $X_d^R = \mathrm{Crit}(\mathrm{Tr}(W)_d)$ is the critical locus of $\mathrm{Tr}(W)_d$, and similarly $\mathfrak{M}_d^R = \mathrm{Crit}(\mathfrak{Tr}(W)_d)$.

Throughout the paper we will use the superscript W instead of the superscript R for $R = \{\partial W/\partial \alpha \mid \alpha \in Q_1\}$, and no superscript if $W = 0$. We will also use the notation $\mathrm{Jac}(Q, W)$ for the so-called Jacobi algebra $\mathbb{C}Q/(R)$.

The moduli stack $\mathfrak{M}_d^R$ has a coarse moduli space $\mathcal{M}_d^{R, ssimp}$ parameterizing semisimple (direct sums of simple) representations of dimension vector d. It is an affine scheme given by $\mathrm{Spec}\,\mathbb{C}[X_d^R]^{G_d}$ with $\mathbb{C}[X_d^R]^{G_d}$ denoting the G_d-invariant regular functions on the affine scheme X_d^R.

Example 3.10. — For the 3-loop quiver $Q^{(3)}$ with potential $W = [x, y]z$, the scheme X_d^W parametrizes triples of commuting $d \times d$-matrices M_x, M_y, M_z. Hence, a simple representation of the Jacobi algebra $\mathrm{Jac}(Q^{(3)}, W) = \mathbb{C}[x, y, z]$ is one-dimensional and determined by $(M_x, M_y, M_z) \in \mathbb{A}^3$. Therefore, $\mathcal{M}_d^{W, ssimp} = \mathrm{Sym}^d(\mathbb{A}^3) = (\mathbb{A}^3)^d/S_d$.

Let us finally introduce a stability condition by choosing a tuple $\zeta \in \mathbb{H}_+^{Q_0}$ of complex numbers in the (extended) upper half plane $\mathbb{H}_+$ giving rise to the "central charge" $Z(V) := \zeta \cdot \dim V = \sum_{i \in Q_0} \zeta_i \dim V_i \in \mathbb{H}_+$ for every representation V of Q.

DEFINITION 3.11. — A representation $V \neq 0$ of a quiver Q (with relations) is called ζ-semistable if

$$\arg Z(V') \leqslant \arg Z(V)$$

for all proper subrepresentations $V' \subset V$. If the inequality is strict, V is called ζ-stable. The real number $\mu(V) := -\cot(\arg Z(V))$ is called the slope of V. Hence, V is semistable if and only if $\mu(V') \leqslant \mu(V)$ for all proper subrepresentations $V' \subset V$.

Exercise 3.12. — Let us show that semistable representations of the same slope μ form a nice full subcategory.

(1) Consider a morphism $f : V^{(1)} \to V^{(2)}$ of semistable representations of slopes $\mu(V^{(1)}) > \mu(V^{(2)})$. Show that $f = 0$. Hint: Relate the slope of $V/\ker(f) = \operatorname{im}(f)$ to $\mu(V^{(1)})$ and to $\mu(V^{(2)})$ by drawing the central charges of all objects involved.

(2) Using the notation of the first part, let us assume $\mu(V^{(1)}) = \mu(V^{(2)})$ for the semistable representations $V^{(1)}, V^{(2)}$. Show that $\ker(f)$ and $\operatorname{coker}(f)$ are also semistable of the same slope $\mu(V^{(1)})$. In particular, the semistable representations of a fixed slope μ form a full abelian subcategory.

(3) Show that the stable objects of slope μ are the simple objects in the full abelian subcategory of semistable representations of slope μ.

(4) Prove that the extension of two semistable representations of slope μ is again semistable of the same slope.

Every representation V of a quiver (with relations) has a unique Harder-Narasimhan filtration, i.e. a finite filtration $0 \subset V^{(1)} \subset \ldots \subset V^{(r)} = V$ such that the subquotients $V^{(i)}/V^{(i-1)}$ are semistable of slope $\mu^{(i)}$ satisfying $\mu^{(1)} > \ldots > \mu^{(r)}$.

Exercise 3.13. — Let us prove the last statement in three steps.

(1) Show that V has a maximal nonzero subrepresentation of maximal slope. Hint: Show that the set of slopes of subrepresentations of V has a maximal element. Use Exercise 3.12(4) to construct a maximal subrepresentation of maximal slope.

(2) Use Exercise 3.12(1) to construct a Harder-Narasimhan filtration. Hint: Let $V^{(1)}$ be the subrepresentation constructed in the first step, and let $V^{(2)}$ be the preimage of a maximal subrepresentation in $V/V^{(1)}$ of maximal slope. Proceed in this way, and use the previous exercise to estimate the slopes.

(3) Prove the uniqueness of this filtration by applying Exercise 3.12(1) once more.

We denote by $X_d^{R,\zeta-ss}$ the subscheme of linear maps $(M_\alpha)_{\alpha \in Q_1}$ such that the induced quiver representation on $(\mathbb{C}^{d_i})_{i \in Q_0}$ is ζ-semistable. As shown in [25], it is open and stable under the G_d-action. Hence, we can form the quotient stack $\mathfrak{M}_d^{R,\zeta-ss} = X_d^{R,\zeta-ss}/G_d$ of ζ-semistable representations of dimension vector d. The open subscheme $X_d^{R,\zeta-st} \subseteq X_d^{R,\zeta-ss}$ and the open substack $\mathfrak{M}_d^{R,\zeta-st} \subset \mathfrak{M}_d^{R,\zeta-ss}$ of ζ-stable representations are defined accordingly.

If $\zeta_i = -\theta_i + \sqrt{-1}$ with $\theta_i \in \mathbb{Z}$ for all $i \in Q_0$, one can lift the G_d-action to the total space of the trivial line bundle $X_d \times \mathbb{A}^1 \to X_d$ using the character

$$\chi_\theta : G_d \ni (g_i)_{i \in Q_0} \to \prod_{i \in Q_0} \det(g_i)^{\theta \cdot d - |d|\theta_i} \in \mathbb{G}_m$$

for the G_d-action on the fiber $\mathbb{A}^1$ with $\theta \cdot d = \sum_{i \in Q_0} \theta_i d_i$ and $|d| := \sum_{i \in Q_0} d_i$. We denote this G_d-equivariant line bundle $X_d \times \mathbb{A}^1 \to X_d$ by $\mathcal{L}$. Notice that the tensor product $\mathcal{L}^m$ is again the trivial line bundle but with G_d acting by the m-th power χ_θ^m of the character χ_θ on the fibers. A section s of $\mathcal{L}^m$ can be interpreted as a regular function on X_d transforming as $s(g \cdot p) = \chi_\theta^m(g^{-1})s(p)$ for $p \in X_d$ and $g \in G_d$. A point $p \in X_d$ is called semistable with respect to $\mathcal{L}$ if there is a G_d-equivariant section s of $\mathcal{L}^m$ for some $m > 0$ with $s(p) \neq 0$. A. King showed in [18] that $X_d^{R,\zeta-ss}$ is the subscheme of points in X_d^R which are semistable with respect to $\mathcal{L}$. Hence, a GIT-quotient $X_d^{R,\zeta-ss} /\!\!/ G_d = \mathcal{M}_d^{R,\zeta-ss}$ with stable sublocus $X_d^{R,\zeta-st}/G_d = \mathcal{M}_d^{R,\zeta-st}$ exists. The points of the GIT-quotient parametrize closed G_d-orbits in $X_d^{R,\zeta-ss}$ corresponding to isomorphism classes of polystable objects (see Definition 2.12), or equivalence classes of orbits in $X_d^{R,\zeta-ss}$, where two orbits corresponding to semistable representations E, E' are equivalent if their closure contains the same closed orbit which then corresponds to $\mathrm{gr}(E) \cong \mathrm{gr}(E')$. Thus, the points of $\mathcal{M}_d^{R,\zeta-ss}$ also parametrize S-equivalence classes (see Definition 2.12) of representations.

According to the main result of the paper [25], the picture just described generalizes for arbitrary ζ. More precisely, given a stability condition ζ and a dimension vector d, there is a character $\chi_{\zeta,d}$ for G_d giving rise to a G_d-equivariant trivial line bundle $\mathcal{L}$ on X_d as above such that $X_d^{R,\zeta-ss}$ is the open subscheme of points in X_d^R which are semistable with respect to $\mathcal{L}$. Hence, a GIT-quotient $X_d^{R,\zeta-ss} /\!\!/ G_d = \mathcal{M}_d^{R,\zeta-ss}$ with stable sublocus $X_d^{R,\zeta-st}/G_d = \mathcal{M}_d^{R,\zeta-st}$ exists and the points in $\mathcal{M}_d^{R,\zeta-ss}$ correspond to isomorphism classes of polystable objects or S-equivalence classes of ζ-semistable representations of dimension vector d satisfying the relations R.[7]

In case $\zeta_i = \sqrt{-1}$ for all $i \in Q_0$, i.e. $\theta = 0$, we write $\mathcal{M}_d^{R,ssimp}$ for $\mathcal{M}_d^{R,\zeta-ss}$ as its points correspond to isomorphism classes of semisimple $\mathbb{C}Q$-representations satisfying the relations R.

Remark 3.14. — Notice that $\mathbb{G}_m$, embedded into G_d diagonally, acts trivially on X_d, and G_d induces a $PG_d := G_d/\mathbb{G}_m$-action on X_d. The character given above descends to a character on PG_d and $X_d^{R,\zeta-ss}$ is also the semistable locus of X_d^R with respect to the PG_d-equivariant trivial line bundle $X_d \times \mathbb{A}^1 \to X_d$. Hence, $\mathcal{M}_d^{R,\zeta-ss} = X_d^{R,\zeta-ss} /\!\!/ PG_d$ is also the coarse moduli space for $X_d^{R,\zeta-ss}/PG_d$, the stack of "d-dimensional" projective semistable quiver representations satisfying the relations R. It is not difficult to see that $\mathcal{M}_d^{R,\zeta-st}$ is in fact a fine moduli space for $X_d^{R,\zeta-st}/PG_d$, in other words, there is an isomorphism $X_d^{R,\zeta-st}/PG_d \cong \mathcal{M}_d^{R,\zeta-st}$ of stacks. In particular, $\mathcal{M}_d^{R,\zeta-st}$ carries a universal family $\mathcal{P}$ of projective stable quiver representations satisfying our relations R. The reader should compare this with our final remarks in Example 2.10 and the two lessons we have mentioned after

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[7]See also Section 2.2 of [26] for stability conditions not lying on a wall.

Example 2.11. The morphism $X_d^{R,\zeta-st}/G_d \to X_d^{R,\zeta-st}/PG_d \cong \mathcal{M}_d^{R,\zeta-st}$ is not an isomorphism. It is not hard to see that this map has a right inverse, i.e. a section, if $\gcd(d) := \gcd(d_i : i \in Q_0) = 1$ (see [29], Section 5.4 for more details). Such a section is nothing else than a family $V = \bigoplus_{i \in Q_0} V_i$ of stable quiver representations on $\mathcal{M}_d^{R,\zeta-st}$ such that $\mathcal{P} = \mathbb{P}(V)$. The section, i.e. the family, is not unique. Any two sections corresponding to $V^{(1)}$ and $V^{(2)}$ differ (up to isomorphism) by a line bundle L on $\mathcal{M}_d^{R,\zeta-st}$ with

$$V^{(2)} \cong V^{(1)} \otimes_{\mathcal{O}_{\mathcal{M}_d^{R,\zeta-st}}} L$$

as in Example 2.10 (see also Exercise 2.32). Therefore, V on $\mathcal{M}^{R,\zeta-ss}$ is only universal up to this weaker equivalence. It has been shown in [33], Thm. 3.4 that under some mild conditions on the pair (d, ζ) the moduli space $\mathcal{M}_d^{\zeta-st}$ has no "universal family" V, i.e. $X_d^{\zeta-st}/G_d \to \mathcal{M}_d^{\zeta-st}$ has no section, if $\gcd(d) > 1$.

DEFINITION 3.15. — A stability condition ζ is called generic if $\langle d, d' \rangle = 0$ for all $d, d' \in \Lambda_\mu^\zeta := \{e \in \mathbb{N}^{Q_0} \mid e = 0 \text{ or } e \text{ has slope } \mu\}$ and all $\mu \in \mathbb{R}$, where $\langle d, d' \rangle = (d, d') - (d', d)$ denotes the antisymmetrized Euler pairing

$$(d, d') = \sum_{i \in Q_0} d_i d_i' - \sum_{\alpha : i \to j} d_i d_j'$$

satisfying $(\dim V, \dim V') = \dim \text{Hom}_{\mathbb{C}Q}(V, V') - \dim \text{Ext}_{\mathbb{C}Q}^1(V, V')$ for all $\mathbb{C}Q$-representations V, V'.

4. FROM CONSTRUCTIBLE FUNCTIONS TO MOTIVIC THEORIES

4.1. **Constructible functions.** Let us start by recalling some facts about constructible functions. A constructible function is a function $a : X(\mathbb{C}) \to \mathbb{Z}$ on the set of (closed) points of a scheme[8]/variety/manifold X over $\mathbb{C}$ with only finitely many values on each connected component of X and such that the level sets of a are the (closed) points of locally closed subsets in each connected component of X. We denote by $\text{Con}(X)$ the group of constructible functions on X.

Exercise 4.1. — Assume that X is connected. Show that the map associating to every irreducible closed subset V of X its characteristic function extends to an isomorphism $\bigoplus_{x \in X} \mathbb{Z}x \cong \bigoplus_{V \subset X} \mathbb{Z}V \xrightarrow{\sim} \text{Con}(X)$, where the first sum is over all not necessarily closed points $x \in X$, and the second sum is taken over all irreducible closed subsets $V \subset X$.

Apparently, we can pull back constructible functions and also multiply them pointwise. Contrary to the usual notation, we denote the pointwise product by $a \cap b$, i.e. $(a \cap b)(x) = a(x)b(x)$. The constant function $\mathbb{1}_X(x) = 1$ for all $x \in X(\mathbb{C})$ is the unit for the $\cap$-product. There is another product, the external product $a \boxtimes b = \text{pr}_X^*(a) \cap \text{pr}_Y^*(b)$ of two functions $a \in \text{Con}(X)$ and $b \in \text{Con}(Y)$ on $X \times Y$ such that $a \cap b = \Delta_X^*(a \boxtimes b)$ if $Y = X$. The unit for the $\boxtimes$-product is $1 \in \mathbb{Z} = \text{Con}(\text{Spec } \mathbb{C})$.

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[8]In case of schemes we assume that each connected component is of finite type over $\mathbb{C}$.

Moreover, we can define a push-forward of a constructible function $a \in \mathrm{Con}(X)$ along a morphism $u : X \to Y$ of finite type by[9]

$$u_!(a)(y) := \int_{u^{-1}(y)^{an}} a \, d\chi_c := \sum_{m \in \mathbb{Z}} m \chi_c \{x \in X \mid u(x) = y, a(x) = m\}^{an}$$

for $y \in Y$. Here χ_c denotes the Euler characteristic with compact support, i.e. the alternating sum of the dimensions of the compactly supported cohomology. As shown in Prop. 1 of [23], $u_!(a)$ is indeed a constructible function. One can think of χ_c as a signed measure χ_c^X on X, even though it is only additive and not σ-additive. Given a constructible function a on X, we get a new measure $a \cdot \chi_c^X$ on X of density a with respect to χ_c^X. A push-forward of a measure is well-defined and $u_!(a\chi_c^X)$ has density $u_!(a)$ with respect to χ_c^Y. Using the push-forward, one can define a third product for constructible functions on a monoidal scheme, i.e. a scheme X with two maps $0 : \mathrm{Spec}\,\mathbb{C} \to X$ and $+ : X \times X \to X$ of finite type satisfying an associativity and unit law. The convolution product is given by $ab = +_!(a \boxtimes b)$ and is commutative if $+$ is commutative. The unit is given by $0_!(1)$ with $1 \in \mathrm{Con}(\mathrm{Spec}\,\mathbb{C}) = \mathbb{Z}$ being the unit for the $\boxtimes$-product. If we had taken $X = \mathbb{A}^1$, the convolution product is just the "constructible version" of the usual convolution product of integrable functions. The free commutative monoid generated by a scheme X is given by $\mathrm{Sym}(X) = \bigsqcup_{n \in \mathbb{N}} \mathrm{Sym}^n(X)$ with $\mathrm{Sym}^n(X) = X^n /\!\!/ S_n$, and $\oplus : \mathrm{Sym}(X) \times \mathrm{Sym}(X) \to \mathrm{Sym}(X)$ is just the concatenation of unordered tuples of (geometric) points of X. The unit $0 : \mathrm{Spec}\,\mathbb{C} =: \mathrm{Sym}^0(X) \hookrightarrow \mathrm{Sym}(X)$ is given by the "empty tuple". We can apply the definition of the convolution product $ab := \oplus_!(a \boxtimes b)$ to $\mathrm{Con}(\mathrm{Sym}(X))$ making it into a commutative ring. This (convolution) ring has even more structure. Indeed, there is a family of maps $\sigma^n : \mathrm{Con}(X) \to \mathrm{Con}(\mathrm{Sym}^n(X))$ mapping the characteristic function of $V \subseteq X$ to the characteristic function of $\mathrm{Sym}^n(V) \subseteq \mathrm{Sym}^n(X)$.

Example 4.2. — Consider the example $X = \mathrm{Spec}\,\mathbb{C}$. Then $\mathrm{Sym}(X) \cong \mathbb{N}$, and $\mathrm{Con}(X) \cong \mathbb{Z}[[t]]$ follows. The convolution product is just the ordinary product of power series and $\mathrm{Sym}^n(at) = \binom{a+n-1}{n} t^n$. The pointwise $\cap$-product is known as the Hadamard product of two power series.

Let us collect the main properties of the structures described above.

PROPOSITION 4.3. — *By taking pull-backs and push-forwards of constructible functions, we obtain a functor* Con *from the category* $\mathrm{Sch}_\mathbb{C}$ *to the category of abelian groups which is both contravariant with respect to all morphisms and covariant with respect to all morphisms of finite type, i.e. for every morphism* $u : X \to Y$ *there is a group homomorphism* $u^* : \mathrm{Con}(Y) \to \mathrm{Con}(X)$, *and if* u *is of finite type, there is also a group homomorphism* $u_! : \mathrm{Con}(X) \to \mathrm{Con}(Y)$. *Moreover, there is an "exterior" product*

$$\boxtimes : \mathrm{Con}(X) \otimes_\mathbb{Z} \mathrm{Con}(Y) \to \mathrm{Con}(X \times Y)$$

---

[9]For every scheme X locally of finite type over $\mathbb{C}$, we denote by X^{an} the "analytification" of X which is an analytic space locally isomorphic to the vanishing locus of holomorphic functions on $\mathbb{C}^n$. If X is smooth, X^{an} is a complex manifold. In any case X^{an} carries the analytic topology which is much finer than the Zariski topology on X.

defined for every pair $X, Y \in \mathrm{Sch}_{\mathbb{C}}$ which is associative, symmetric[10] and has a unit $1 \in \mathrm{Con}(\mathrm{Spec}\,\mathbb{C})$. Finally, there are also operations

$$\sigma^n : \mathrm{Con}(X) \to \mathrm{Con}(\mathrm{Sym}^n(X))$$

for $n \in \mathbb{N}$ such that $\sigma^n(1) = 1$ holds for all $n \in \mathbb{N}$. Additionally, we have the following properties.

(i) Considered as a functor from $\mathrm{Sch}_{\mathbb{C}}^{op}$ to abelian groups, Con commutes with all (not necessarily finite) products, i.e. the morphism

$$\mathrm{Con}(X) \to \prod_{X_i \in \pi_0(X)} \mathrm{Con}(X_i)$$

given by restriction to connected components is an isomorphism for all $X \in \mathrm{Sch}_{\mathbb{C}}$.

(ii) "Base change" holds, i.e. for every cartesian diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\tilde{v}} & X \\ \tilde{u} \downarrow & & \downarrow u \\ Y & \xrightarrow{v} & Z \end{array}$$

with u and, therefore, also $\tilde{u}$ of finite type, we have $\tilde{u}_! \circ \tilde{v}^* = v^* \circ u_!$.

(iii) The functor Con commutes with exterior products and σ^n, i.e.

$$(u \times v)^*(a \boxtimes b) = u^*(a) \boxtimes v^*(b)$$

for all $u : X \to X'$, $v : Y \to Y'$, $a \in \mathrm{Con}(X')$ and $b \in \mathrm{Con}(Y')$. If u, v are of finite type, then

$$(u \times v)_!(a \boxtimes b) = u_!(a) \boxtimes v_!(b)$$

and

$$\mathrm{Sym}^n(u)_!(\sigma^n(a)) = \sigma^n(u_!(a))$$

for all $a \in \mathrm{Con}(X)$, $b \in \mathrm{Con}(Y)$ and $n \in \mathbb{N}$.

(iv) Using the convolution product $ab = \oplus_!(a \boxtimes b)$ and viewing $\mathrm{Con}(\mathrm{Sym}^n(X))$ as a subgroup of $\mathrm{Con}(\mathrm{Sym}(X))$ by means of $\big(\mathrm{Sym}^n(X) \hookrightarrow \mathrm{Sym}(X)\big)_!$, we have

$$\sigma^n(a + b) = \sum_{l=0}^{n} \sigma^l(a)\sigma^{n-l}(b)$$

with $\sigma^1(a) = a$ and $\sigma^0(a) = 1 \in \mathrm{Con}(\mathrm{Spec}\,\mathbb{C}) \hookrightarrow \mathrm{Con}(\mathrm{Sym}(X))$ for all $a, b \in \mathrm{Con}(X)$.

(v) The "motivic property" holds, i.e. for every X and every closed subscheme $Z \subseteq X$ giving rise to inclusions $i : Z \hookrightarrow X$ and $j : X \setminus Z \hookrightarrow X$, we have $i^*i_! = \mathrm{id}_{\mathrm{Con}(Z)}, j^*j_! = \mathrm{id}_{\mathrm{Con}(X \setminus Z)}, j^*i_! = i^*j_! = 0$ and

$$a = i_!i^*(a) + j_!j^*(a) \quad \forall a \in \mathrm{Con}(X).$$

(vi) The equation $\sigma^n(\mathbb{1}_X) = \mathbb{1}_{\mathrm{Sym}^n(X)}$ holds for all X and all $n \in \mathbb{N}$ with $\mathbb{1}_X = (X \to \mathrm{Spec}\,\mathbb{C})^*(1)$ and similarly for $\mathbb{1}_{\mathrm{Sym}^n(X)}$.

---

[10]If $\tau : X \boxtimes Y \xrightarrow{\sim} Y \boxtimes X$ is the transposition, being symmetric means $\tau_!(a \boxtimes b) = \tau^*(a \boxtimes b) = b \boxtimes a$.

Exercise 4.4. — Show that the projection formula $u_!(a \cap u^*(b)) = u_!(a) \cap b$ holds for all $u : X \to Y$ of finite type and all $a \in R(X)$, $b \in R(Y)$ by using the properties mentioned in Proposition 4.3 and $a \cap b = \Delta_X^*(a \boxtimes b)$. Hint: Consider the diagram

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{\mathrm{id}_X \times u} X \times Y$$

$$\left\downarrow{u}\qquad\qquad\qquad\qquad\qquad\left\downarrow{u \times \mathrm{id}_Y}\right.$$

$$Y \xrightarrow{\qquad\qquad \Delta_Y \qquad\qquad} Y \times Y.$$

4.2. **Motivic theories for schemes.** Generalizing constructible functions, we define a motivic theory[11] to be a rule associating to every scheme X an abelian group $R(X)$, like $\mathrm{Con}(X)$, along with pull-backs $u^* : R(Y) \to R(X)$ for all morphisms $u : X \to Y$ and push-forwards $u_! : R(X) \to R(Y)$ if u is of finite type. Moreover, there should be some associative, symmetric exterior product $\boxtimes : R(X) \otimes_{\mathbb{Z}} R(Y) \to R(X \times Y)$ with unit element $1 \in R(\mathrm{Spec}\,\mathbb{C})$, and some operations $\sigma^n : R(X) \to R(\mathrm{Sym}^n(X))$ for all $n \in \mathbb{N}$, satisfying exactly the same properties as $\mathrm{Con}(X)$ given in Proposition 4.3. Similar to the case Con, we can construct a $\cap$-product $a \cap b = \Delta_X^*(a \boxtimes b)$ with unit $\mathbb{1}_X = (X \to \mathrm{Spec}\,\mathbb{C})^*(1)$ and a convolution product $ab = +_!(a \boxtimes b)$ with unit $1_0 = 0_!(1)$ if, additionally, $(X, +, 0)$ is a (commutative) monoid with $+$ being of finite type. Note that all these products coincide on $R(\mathrm{Spec}\,\mathbb{C})$.

Exercise 4.5. — Given a motivic theory R and a scheme X with morphism $c : X \to \mathrm{Spec}\,\mathbb{C}$, we define $[X]_R := c_! c^*(1) \in R(\mathrm{Spec}\,\mathbb{C})$.

(i) Show that $[X]_R = [Z]_R + [X \backslash Z]_R$ (cut and paste relation) for every closed subscheme $Z \subseteq X$ and $[X \times Y]_R = [X]_R [Y]_R$ by applying the defining properties of Proposition 4.3. In particular, $X \mapsto [X]_R \in R(\mathrm{Spec}\,\mathbb{C})$ is a generalization of the classical Euler characteristic

$$\chi_c : \mathrm{Sch}_{\mathbb{C}} \to \mathbb{Z} = \mathrm{Con}(\mathrm{Spec}\,\mathbb{C}).$$

(ii) Use (i) to show

$$[\mathbb{P}^n]_R = \mathbb{L}_R^n + \ldots + \mathbb{L}_R + 1 = (\mathbb{L}_R^{n+1} - 1)/(\mathbb{L}_R - 1)$$

with $\mathbb{L}_R := [\mathbb{A}^1]_R$.

Exercise 4.6. — We use the notation introduced in the previous exercise.

(i) Assume $Y \to X$ is a Zariski locally trivial fiber bundle with fiber F. Use the cut and paste relation to prove $[Y]_R = [F]_R [X]_R$.
(ii) Use (i) applied to the projection onto the first column and induction over $n \in \mathbb{N}$ to show $[\mathrm{GL}(n)]_R = \prod_{i=0}^{n-1}(\mathbb{L}_R^n - \mathbb{L}_R^i)$.
(iii) Use (i) to prove

$$[\mathrm{Gr}(k, n)]_R = [\mathrm{GL}(n)]_R / [\mathrm{GL}(k)]_R [\mathrm{GL}(n-k)]_R.$$

A morphism $\eta : R \to R'$ between motivic theories is a collection of group homomorphisms $\eta_X : R(X) \to R'(X)$ commuting with pull-backs, push-forwards and exterior products. It is called a λ-morphism, if it additionally commutes with the

---

[11]Motivic theories are special cases of reduced motivic λ-ring (Sch, ft)-theories defined in [5]. Every reduced motivic λ-ring (Sch, ft)-theory is a motivic theory in our sense if $\sigma^n(\mathbb{1}_X) = \mathbb{1}_{\mathrm{Sym}^n(X)}$ holds for all X and all $n \in \mathbb{N}$. (cf. Proposition 4.3.(vi))

σ^n-operations. Thus, we obtain a category of motivic theories containing the sub-category of motivic theories with λ-morphisms.

The rule $X \mapsto R(X) = 0$ is the terminal object in the category of motivic theories. Moreover, the following holds.

LEMMA 4.7. — *The category of motivic theories has an initial object given by the (completed) relative Grothendieck group* $\underline{K}_0(\mathrm{Sch}) : X \mapsto \underline{K}_0(\mathrm{Sch}_X)$ *as constructed below. The unique morphisms starting at* $\underline{K}_0(\mathrm{Sch})$ *are also* λ-morphisms.

Instead of giving an ad hoc definition of $\underline{K}_0(\mathrm{Sch})$, let us motivate the construction by looking at constructible functions. Starting with the constant function 1 on $\mathrm{Spec}\,\mathbb{C}$, we take the pull-back $c^*(1) =: \mathbb{1}_X$ for the constant map $c : X \to \mathrm{Spec}\,\mathbb{C}$ which is the constant function with value 1 on X, but more importantly, it is the unit object for the $\cap$-product. For any morphism $v : V \to X$ of finite type, consider the function $v_!(\mathbb{1}_V) \in \mathrm{Con}(X)$ and denote it by $[V \xrightarrow{v} X]_{\mathrm{Con}}$. If $v : V \hookrightarrow X$ is the embedding of a locally closed subscheme, $[V \hookrightarrow X]_{\mathrm{Con}}$ is just the characteristic function of V.

Using Proposition 4.3, we get

(1) $[V \to X]_{\mathrm{Con}} = [Z \to X]_{\mathrm{Con}} + [V \backslash Z \to X]_{\mathrm{Con}}$ for all closed $Z \subset V$,

(2) $1 = [\mathrm{Spec}\,\mathbb{C} \xrightarrow{\mathrm{id}} \mathrm{Spec}\,\mathbb{C}]_{\mathrm{Con}}$,

(3) $u^*([W \to Y]_{\mathrm{Con}}) = [X \times_Y W \to X]_{\mathrm{Con}}$ for all $u : X \to Y$,

(4) $u_!([V \xrightarrow{v} X]_{\mathrm{Con}}) = [V \xrightarrow{u \circ v} Y]_{\mathrm{Con}}$ if $u : X \to Y$ is of finite type,

(5) $[V \xrightarrow{v} X]_{\mathrm{Con}} \boxtimes [W \xrightarrow{w} Y]_{\mathrm{Con}} = [V \times W \xrightarrow{v \times w} X \times Y]_{\mathrm{Con}}$,

(6) $\sigma^n([V \to X]_{\mathrm{Con}}) = [\mathrm{Sym}^n(V) \to \mathrm{Sym}^n(X)]_{\mathrm{Con}}$,

(7) $[V \xrightarrow{v} X]_{\mathrm{Con}} = [V' \xrightarrow{v'} X]_{\mathrm{Con}}$ if there is an isomorphism $v'' : V \to V'$ such that $v = v' \circ v''$.

The definition of $[V \xrightarrow{v} X]_{\mathrm{Con}} = v_!(\mathbb{1}_V)$ was purely formal, and we can define the symbols $[V \xrightarrow{v} X]_R \in R(X)$ by $v_!(\mathbb{1}_V)$ for any motivic theory R.[12] Obviously, equations (1)-(7) must hold in every motivic theory as they share the same properties, and so the same applies to the initial object if it exists. Moreover, for connected X the group $\mathrm{Con}(X)$ is generated by all classes $[V \to X]_{\mathrm{Con}}$ satisfying relation (1). The same must be true for the initial motivic theory since otherwise the subgroup spanned by the elements $[V \to X]_{init}$ for connected X and extended by Property 4.3(i) for non-connected X is a proper subtheory of the initial theory which will lead to a contradiction. However, there are more relations in $\mathrm{Con}(X)$, such as

$$[\mathbb{G}_m \xrightarrow{z^d} \mathbb{G}_m]_{\mathrm{Con}} = d[\mathbb{G}_m \xrightarrow{\mathrm{id}} \mathbb{G}_m]_{\mathrm{Con}},$$

which might not hold in other motivic theories (for instance the initial one). Dropping the subscript "init" we will, therefore, define our (hopefully) initial theory by associating to every connected scheme X the group $\mathrm{K}_0(\mathrm{Sch}_X)$ generated by symbols $[V \xrightarrow{v} X]$ for every isomorphism class (due to equation (7)) of morphisms $v : V \to X$ of finite type, subject to the relation (1). For non-connected X we simply put

$$\underline{K}_0(\mathrm{Sch}_X) := \prod_{X_i \in \pi_0(X)} \mathrm{K}_0(\mathrm{Sch}_{X_i}).$$

---

[12]We already introduced the shorthand $[V]_R$ for $[V \to \mathrm{Spec}\,\mathbb{C}]_R$ in Exercise 4.5.

To obtain a motivic theory $\underline{K}_0(\mathrm{Sch})$, we must define $1 \in K_0(\mathrm{Sch}_\mathbb{C}), u^*, u_!, \boxtimes$ and σ^n as in equations (2)–(6), at least over connected components. It has been shown in [10] that σ^n-operations satisfying these properties do indeed exist. Moreover, the authors prove that $\sigma^n(a\mathbb{L}) = \sigma^n(a)\mathbb{L}^n$ holds for every $a \in \underline{K}_0(X)$ and every $n \in \mathbb{N}$, where $\mathbb{L} = c_! c^*(1) = [\mathbb{A}^1 \xrightarrow{c} \mathrm{Spec}\,\mathbb{C}] \in K_0(\mathrm{Sch}_\mathbb{C})$ is considered as an element of $\underline{K}_0(\mathrm{Sch}_{\mathrm{Sym}(X)})$ via the embedding $0_! : K_0(\mathrm{Sch}_\mathbb{C}) \hookrightarrow \underline{K}_0(\mathrm{Sch}_{\mathrm{Sym}(X)})$.

Exercise 4.8. — Use the properties of a morphism between motivic theories to show that $[V \to X] \mapsto [V \to X]_R$ defines a homomorphism $\eta_X : K_0(\mathrm{Sch}_X) \to R(X)$ for connected X which extends to a morphism $\eta : \underline{K}_0(\mathrm{Sch}) \to R$ of motivic theories. Prove that this morphism is the only possible one. Hence, $\underline{K}_0(\mathrm{Sch})$ is the initial object in the category of motivic theories. Moreover, show that η is a λ-morphism.

The initial property of $\underline{K}_0(\mathrm{Sch})$ is just a generalization of the well-known property that $X \to [X] \in K_0(\mathrm{Sch}_\mathbb{C})$ is the universal Euler characteristics.

Let $R^{gm}(X) \subset R(X)$ be the subgroup generated by all elements $[V \to X]_R$ if X is connected and $R^{gm}(X) := \prod_{X_i \in \pi_0(X)} R^{gm}(X_i)$ for general X. One should think of elements in $R^{gm}(X)$ as "geometric" since they are $\mathbb{Z}$-linear combinations of elements obtained by geometric constructions, namely pull-backs and push-forwards of the unit $1 \in R(\mathrm{Spec}\,\mathbb{C})$.

Exercise 4.9. — Show that $X \mapsto R^{gm}(X)$ defines a subtheory of R. By construction, it is the image of the λ-morphism $\eta : \underline{K}_0(\mathrm{Sch}) \to R$ obtained in the previous exercise. Show that $\mathrm{Con}^{gm} = \mathrm{Con}$ and $\underline{K}_0(\mathrm{Sch})^{gm} = \underline{K}_0(\mathrm{Sch})$. Prove that $\sigma^n(a\mathbb{L}_R) = \sigma^n(a)\mathbb{L}_R^n$ holds for every $a \in R^{gm}(X)$ and every $n \in \mathbb{N}$.

4.3. **Motivic theories for quotient stacks.** In the previous section we generalized constructible functions and the classical Euler characteristic to more refined "functions" and invariants. When it comes to moduli problems, we should also be able to compute refined invariants of quotient stacks as they occur naturally in moduli problems. Hence, we need to extend motivic theories to disjoint unions of quotient stacks X/G for schemes X locally of finite type over $\mathbb{C}$ and linear algebraic groups G. This is the topic of this subsection.

Exercise 4.10. — Given a closed embedding $G \hookrightarrow \mathrm{GL}(n)$ of a linear algebraic group G and a G-action on a scheme X. Show that the morphism

$$X/G \to (X \times_G \mathrm{GL}(n))/\mathrm{GL}(n)$$

induced by $X \ni x \mapsto (x,1) \in X \times_G \mathrm{GL}(n)$ and $G \hookrightarrow \mathrm{GL}(n)$ as in Exercise 2.24 is in fact an isomorphism of quotient stacks. Hint: Given a principal $\mathrm{GL}(n)$-bundle $P \to S$ and a $\mathrm{GL}(n)$-equivariant morphism $\psi : P \to X \times_G \mathrm{GL}(n)$, show that $\psi^{-1}(X) \to S$ for $X \hookrightarrow X \times_G \mathrm{GL}(n)$ is a principal G-bundle over S. To prove local triviality of $\psi^{-1}(X) \to S$ one has to construct local sections of $\psi^{-1}(X) \to S$. For this, one can take a local section $\nu : U \to P$ of $P \to S$ and a lift $(f,g) : \tilde{U} \to X \times \mathrm{GL}(n)$ of $\psi \circ \nu : U \to X \times_G \mathrm{GL}(n)$ on a possibly smaller étale neighborhood $s \in \tilde{U} \subset U$ of $s \in S$. Then $\tilde{\nu} : \tilde{U} \ni t \mapsto \nu(t)g(t)^{-1} \in \psi^{-1}(X)$ is a local section of $\psi^{-1}(X) \to S$. Note that if G is special, these étale neighborhoods U and $\tilde{U}$ can even be replaced with Zariski neighborhoods.

DEFINITION 4.11. — A stacky motivic theory R is a rule associating to every disjoint union $\mathfrak{X} = \bigsqcup_{i \in I} X_i/G_i$ of quotient stacks with linear algebraic groups

G_i an abelian group $R(\mathfrak{X})$ along with pull-backs $u^* : R(\mathfrak{Y}) \to R(\mathfrak{X})$ for all (1-) morphisms $u : \mathfrak{X} \to \mathfrak{Y}$ and push-forwards $u_! : R(\mathfrak{X}) \to R(\mathfrak{Y})$ if u is of finite type. Moreover, there should be some associative, symmetric exterior product $\boxtimes : R(\mathfrak{X}) \times R(\mathfrak{Y}) \to R(\mathfrak{X} \times \mathfrak{Y})$ with unit element $1 \in R(\operatorname{Spec} \mathbb{C})$, and some operations $\sigma^n : R(X) \to R(\operatorname{Sym}^n(X))$ for all $n \in \mathbb{N}$ and all schemes X, satisfying the stacky analogue of the properties of $\operatorname{Con}(-)$ given in Proposition 4.3.

Remark 4.12. — There are two technical difficulties to overcome when we try to generalize Proposition 4.3, which serves as our definition of a (stacky) motivic theory, to disjoint unions of quotient stacks. First of all, we need to explain what the correct generalization of a finite type morphism ought to be. For us, this is a (1-)morphism $u : \mathfrak{X} \to \mathfrak{Y}$ of algebraic stacks such the preimage of each "connected component" Y/H (with connected Y) consists of only finitely many connected components X_i/G_i of $\mathfrak{X}$. Secondly, we need to define $\operatorname{Sym}^n(X/G)$ for quotient stacks. There is an obvious candidate given by the quotient stack $X^n/(S_n \ltimes G^n)$. However, if $G = \{1\}$ is the trivial group, we get the quotient stack X^n/S_n which is different from its coarse "moduli space" $\operatorname{Sym}^n(X) = X^n /\!\!/ S_n$. To avoid these problems, we only require the existence of σ^n-operations for schemes $\mathfrak{X} = X$ and not for general disjoint unions of quotient stacks.

Example 4.13. — There is no stacky motivic theory R with $R|_{\operatorname{Sch}_{\mathbb{C}}} = \operatorname{Con}$ such that the pull-back $\rho^* : R(X/G) \to R(X) = \operatorname{Con}(X)$ is an embedding. Indeed, consider the case $X = \operatorname{Spec} \mathbb{C}$ and $G = \mathbb{G}_m$. Then $X \xrightarrow{\rho} X/G \to \operatorname{Spec} \mathbb{C}$ is the identity, and $\rho_!(\mathbb{1}_X)$ cannot be zero. By assumption, $\rho^* \rho_!(\mathbb{1}_X)$ is also nonzero. However, applying base change to the diagram

$$
\begin{array}{ccc}
X \times G & \xrightarrow{\ m\ } & X \\
{\scriptstyle \operatorname{pr}_X} \downarrow & & \downarrow {\scriptstyle \rho} \\
X & \xrightarrow{\ \rho\ } & X/G
\end{array}
$$

with $m : X \times G \to X$ denoting the (trivial) group action, $\rho^* \rho_!(\mathbb{1}_X) = \operatorname{pr}_{X\,!}(\mathbb{1}_{X \times G}) = \chi_c(G)\mathbb{1}_X = 0$, a contradiction.

Applying the functoriality of the pull-back to the previous diagram, we obtain $\operatorname{pr}_X^*(b) = m^*(b)$ for $b = \rho^*(a)$. In other words, for every stacky motivic theory R, the image of ρ^* is contained in the subgroup
$$
R(X)^G := \{a \in R(X) \mid \operatorname{pr}_X^*(b) = m^*(b)\}
$$
of "G-invariant" elements. Despite the negative result given by the previous example, we will provide a functorial construction which associates to every motivic theory (for schemes) R satisfying
$$
\sigma^n(a\mathbb{L}_R) = \sigma^n(a)\mathbb{L}_R^n \quad \forall\, a \in R(X) \tag{4.1}
$$
another motivic theory R^{st} such that R^{st} extends to a stacky motivic theory, also denoted by R^{st}, for which $\rho^* : R^{st}(X/G) \to R^{st}(X)^G$ is an isomorphism. Moreover, there is a morphism $R \to R^{st}|_{\operatorname{Sch}_{\mathbb{C}}}$ of motivic theories satisfying the property that every morphism $R \to R'|_{\operatorname{Sch}_{\mathbb{C}}}$ with R' being a stacky motivic theory satisfying $\rho^* : R'(X/G) \xrightarrow{\sim} R'(X)^G$ must factorize though $R \to R^{st}$. In particular, the restriction functor from the category of stacky motivic theories R' satisfying (4.1)

and $\rho^* : R'(X/G) \xrightarrow{\sim} R'(X)^G$ has a left adjoint given by $R \mapsto R^{st}$. As we will see, $\mathrm{Con}^{st} = 0$.

Recall that a linear algebraic group G was called special if every étale locally trivial principal G-bundle is already Zariski locally trivial. In particular, given a closed embedding $G \hookrightarrow \mathrm{GL}(n)$, the map $\mathrm{GL}(n) \to \mathrm{GL}(n)/G$ must be a Zariski locally trivial principal G-bundle. On can show that this property is already sufficient for being special. Hence, $\mathrm{GL}(n)$ is special for every $n \in \mathbb{N}$. As a result of Exercise 4.6 we get $[\mathrm{GL}(n)]_R = [G]_R[\mathrm{GL}(n)/G]_R$ in $R(\mathrm{Spec}\,\mathbb{C})$ for every motivic theory R (for schemes). In particular, $[G]_R$ is invertible for every special group G if and only if $[\mathrm{GL}(n)]_R$ is invertible for every $n \in \mathbb{N}$.

DEFINITION 4.14. — Given a group G and a stack $\mathfrak{P}$ with a right G-action, a G-invariant (1-)morphism $u : \mathfrak{P} \to \mathfrak{X}$ of stacks is called a principal G-bundle on $\mathfrak{X}$ if u is representable and the pull-back $\tilde{u} : X \times_{\mathfrak{X}} \mathfrak{P} \to X$ of u along every morphism $X \to \mathfrak{X}$ with X being a scheme is a principal G-bundle on X. Notice that $X \times_{\mathfrak{X}} \mathfrak{P}$ carries an induced right G-action and $\tilde{u}$ is G-invariant.

Exercise 4.15. — Given a stacky motivic theory R. We want to show in several steps that the condition $\rho^* : R(X/G) \to R(X)^G$ being an isomorphism for every special group G is equivalent to the condition that $[\mathfrak{P} \xrightarrow{u} \mathfrak{X}]_R := u_!(\mathbb{1}_{\mathfrak{P}}) = [G]\mathbb{1}_{\mathfrak{X}}$ for every special group G and every principal G-bundle $u : \mathfrak{P} \to \mathfrak{X}$ in the category of disjoint unions of quotient stacks.

(1) Assume for simplicity $\mathfrak{X} = X/\mathrm{GL}(n)$. Given a principal G-bundle $\mathfrak{P} \to \mathfrak{X}$, consider the cartesian diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\tilde{\rho}} & \mathfrak{P} \\
{\scriptstyle \tilde{u}}\downarrow & & \downarrow{\scriptstyle u} \\
X & \xrightarrow{\rho} & \mathfrak{X}.
\end{array}
$$

By assumption, $\tilde{u} : P \to X$ is a principal G-bundle. Use injectivity of ρ^* to prove $u_!(\mathbb{1}_{\mathfrak{P}}) = [G]_R \mathbb{1}_{\mathfrak{X}}$ (Hint: Use base change and Exercise 4.6.) Extend this result to arbitrary disjoint unions of quotient stacks.

(2) Conversely, assume that $u_!(\mathbb{1}_{\mathfrak{P}}) = [G]_R \mathbb{1}_{\mathfrak{X}}$ for every principal G-bundle in the category of quotient stacks. Show first that $[G]_R$ is invertible in $R(\mathrm{Spec}\,\mathbb{C})$ with inverse $[\mathrm{Spec}\,\mathbb{C}/G]_R$. (Hint: Consider the principal G-bundle $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,\mathbb{C}/G$.)

(3) Secondly, prove that $\rho^* : R(X/G) \to R(X)^G$ is invertible by showing that $\rho_!(-)/[G]_R$ is an inverse. (see [5], Lemma 5.13 if you need help)

For connected X we define $R^{st}(X) := R(X)[[\mathrm{GL}(n)]_R^{-1} \mid n \in \mathbb{N}]$ using the $R(\mathrm{Spec}\,\mathbb{C})$-module structure of $R(X)$ by means of the $\boxtimes$-product. We extend it via $R^{st}(X) = \prod_{X_i \in \pi_0(X)} R^{st}(X_i)$ to non-connected X. The morphism $R(X) \to R^{st}(X)$ is obvious, and it is also easy to see how to extend $u_!, u^*$ for $u : X \to Y$ and the $\boxtimes$-product. The only nontrivial part is the extension of σ^n. For $X \in \mathrm{Sch}_{\mathbb{C}}$ and $a \in R(X)$ define

$$
\sigma_t(a) := \sum_{n \in \mathbb{N}} \sigma^n(a)t^n \in R(\mathrm{Sym}(X))[[t]].
$$

The Adams operations $\psi^n : R(X) \to R(\mathrm{Sym}^n(X)) \subseteq R(\mathrm{Sym}(X))$ are defined by means of the series

$$\psi_t(a) := \sum_{n \geqslant 1} \psi^n(a)t^n := \frac{d\log \sigma_t(a)}{d\log t} = t\sigma_t(-a)\frac{d\sigma_t(a)}{dt},$$

where the product is the convolution product in $R(\mathrm{Sym}(X))$. Using the properties of σ^n, we observe $\sigma_t(0) = 1$ and $\sigma_t(a + b) = \sigma_t(a)\sigma_t(b)$. Thus, $\psi_t(0) = 0$ as well as $\psi_t(a + b) = \psi_t(a) + \psi_t(b)$ follows. Property (4.1) implies $\psi^n(aP(\mathbb{L}_R)) = \psi^n(a)P(\mathbb{L}_R^n)$ for every polynomial $P(x) \in \mathbb{Z}[x]$. Due to Exercise 4.6(ii), we have to extend ψ^n to $R^{st}(X)$ by means of

$$\psi^n\Big(\frac{a}{\prod_{i \in I}[\mathrm{GL}(m_i)]_R}\Big) := \frac{\psi^n(a)}{\prod_{i \in I}P_{m_i}(\mathbb{L}_R^n)}$$

using the polynomial $P_m(x) = \prod_{j=0}^{m-1}(x^m - x^j)$ satisfying $[\mathrm{GL}(m)]_R = P_m(\mathbb{L}_R)$. Having extended the Adams operations, we can also extend the σ^n-operations by putting

$$\sigma_t(a) = \exp\Big(\int \psi_t(a)\frac{dt}{t}\Big).$$

Note that the last expression involves rational coefficient, but one can show that the rational coefficients disappear in the expression for

$$\sigma_t\Big(\frac{a}{\prod_{i \in I}[\mathrm{GL}(m_i)]_R}\Big) = \exp\Big(\sum_{n \geqslant 1}\frac{\psi^n(a)t^n}{n\prod_{i \in I}P_{m_i}(\mathbb{L}_R^n)}\Big)$$

if we express $\psi^n(a)$ in terms of $\sigma^m(a)$ for $1 \leqslant m \leqslant n$. (See [5], Appendix B for more details.)

Exercise 4.16. — Show that $\mathrm{Con}^{st}(X) = 0$ for all X.

Now, as we have constructed R^{st} on schemes, we will put $R^{st}(\mathfrak{X}) := R^{st}(X)^G$ for a quotient stack $\mathfrak{X} = X/G$ with special group G and connected X. We have to show that this definition is independent of the presentation of the quotient stack. For this let $\mathfrak{X} \cong Y/H$ be another presentation with a special group H. Let us form the cartesian square
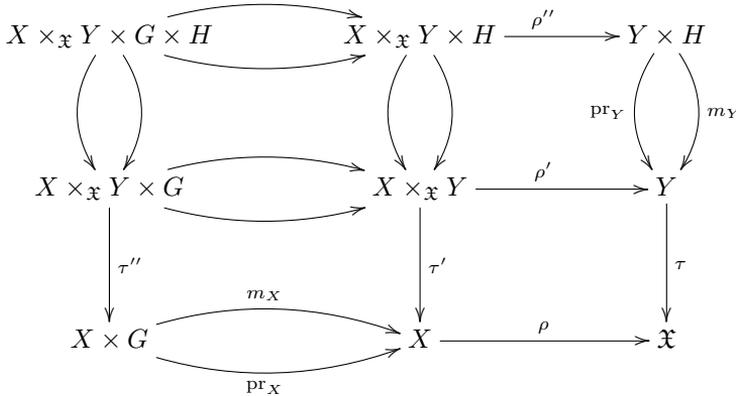
with ρ', ρ'' and τ', τ'' being G- respectively H-principal bundles. The other maps are either projections or actions of G or H. Applying R^{st}, we get the following

diagram with exact rows and columns by construction of R^{st}, where K denotes the kernel of say $\mathrm{pr}_Y^* - m_Y^*$:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
K & \longrightarrow & R^{st}(X) & \xrightarrow{\mathrm{pr}_X^* - m_X^*} & R^{st}(X \times G) \\
\downarrow & & \downarrow{\scriptstyle \tau'^*} & & \downarrow{\scriptstyle \tau''^*} \\
0 \longrightarrow R^{st}(Y) & \xrightarrow{\rho'^*} & R^{st}(X \times_{\mathfrak{X}} Y) & \longrightarrow & R^{st}(X \times_{\mathfrak{X}} Y \times G) \\
\downarrow{\scriptstyle \mathrm{pr}_Y^* - m_Y^*} & & \downarrow & & \downarrow \\
0 \longrightarrow R^{st}(Y \times H) & \xrightarrow{\rho''^*} & R^{st}(X \times_{\mathfrak{X}} Y \times H) & \longrightarrow & R^{st}(X \times_{\mathfrak{X}} Y \times G \times H).
\end{array}
$$

Hence, $R^{st}(X)^G \cong K \cong R^{st}(Y)^H$ showing that $R^{st}(\mathfrak{X})$ is independent of the choice of a presentation. Given a morphism $u : X/G \to Y/H$ of quotient stacks, we form the cartesian diagram

$$
\begin{array}{ccccc}
X \times_{Y/H} Y & \xrightarrow{\tilde{\rho}} & X/G \times_{Y/H} Y & \xrightarrow{\tilde{u}} & Y \\
\downarrow{\scriptstyle \tilde{\tau}} & & \downarrow & & \downarrow{\scriptstyle \tau} \\
X & \xrightarrow{\rho} & X/G & \xrightarrow{u} & Y/H.
\end{array}
$$

For $a \in R^{st}(X/G) \cong R^{st}(X)^G$ and $b \in R^{st}(Y)^H$ we put

$$u_!(a) := (\tilde{u} \circ \tilde{\rho})_! \tilde{\tau}^*(a)/[G]_R \in R^{st}(Y)^H \text{ and}$$
$$u^*(b) := \tilde{\tau}_! (\tilde{u} \circ \tilde{\rho})^*(b)/[H]_R \in R^{st}(X)^G$$
$$a \boxtimes b := a \boxtimes b \in R^{st}(X \times Y)^{G \times H}.$$

For disjoint unions $\mathfrak{X} = \bigsqcup_{i \in I} X_i/G_i$ of connected quotient stacks, we can always assume that G_i is special for all $i \in I$ due to Exercise 4.10. Then, we need to define $R^{st}(\mathfrak{X}) := \prod_{i \in I} R^{st}(X_i/G_i)$ according to Proposition 4.3(i), and extend $u_!$, u^* and $\boxtimes$ in the natural way.

Given a morphism $\eta : R \to R'|_{\mathrm{Sch}_{\mathbb{C}}}$ with $\rho^* : R'(X/G) \to R'(X)^G$ being an isomorphism, we define $\eta_{X/G} : R^{st}(X/G) = R(X)^G \xrightarrow{\eta_X} R'(X)^G \xrightarrow{\rho^{*-1}} R'(X/G)$ with $\rho^{*-1}(a) = \rho_!(a)/[G]_R$ which is the only possible choice to extend η to a morphism $R^{st} \to R'$ of stacky motivic theories. More details in a more general context are given in [5], Section 5.

5. Vanishing cycles

The aim of this section is to introduce the notion of a vanishing cycle taking values in a (stacky) motivic theory R. We start by considering vanishing cycles of morphisms $f : X \to \mathbb{A}^1$ defined on smooth schemes X.

5.1. **Vanishing cycles for schemes.**

DEFINITION 5.1. — Given a motivic theory R, a vanishing cycle[13] (with values in R) is a rule associating to every regular function $f : X \to \mathbb{A}^1$ on a smooth scheme/variety or complex manifold X an element $\phi_f \in R(X)$ such that the following holds.

(1) If $u : Y \to X$ is a smooth, then $\phi_{f \circ u} = u^*(\phi_f)$.
(2) Let X be a smooth variety containing a smooth closed subvariety $i : Y \hookrightarrow X$. Denote by $j : E \hookrightarrow \mathrm{Bl}_Y X$ the exceptional divisor of the blow-up $\pi : \mathrm{Bl}_Y X \to X$ of Y in X. Then the formula

$$\pi_! \big(\phi_{f \circ \pi} - j_! \phi_{f \circ \pi \circ j} \big) = \phi_f - i_! \phi_{f \circ i}$$

holds for every $f : X \to \mathbb{A}^1$.

(3) Given two morphisms $f : X \to \mathbb{A}^1$ and $g : Y \to \mathbb{A}^1$ on smooth X and Y, we introduce the notation $f \boxtimes g : X \times Y \xrightarrow{f \times g} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1$. Then $\phi_{f \boxtimes g} = \phi_f \boxtimes \phi_g$ in $R(X \times Y)$. Moreover, $\phi_{\mathrm{Spec}\, \mathbb{C} \xrightarrow{0} \mathbb{A}^1} = 1$.

Notice that instead of $f \boxtimes g$ the notation $f \oplus g$ or $f \boxplus g$ is more common in the literature. Our notation should remind the reader that ϕ maps (box)products into (box)products. There is also a more general theory allowing arbitrary commutative monoids replacing $\mathbb{A}^1$ as the target of maps. For the trivial monoid $\mathrm{Spec}(\mathbb{C})$ the $\boxtimes$-product is just the cartesian product. See [5] for more details.

LEMMA 5.2. — *A collection of elements $\phi_f \in R(X)$ for regular functions $f : X \to \mathbb{A}^1$ on smooth schemes X satisfying the properties (1), (2) and (3) is equivalent to a collection of group homomorphisms[14] $\phi_f : \underline{\mathrm{K}}_0(\mathrm{Sch}_X) \to R(X)$ for all regular functions $f : X \to \mathbb{A}^1$ on arbitrary schemes X such that the following diagrams commute:*

$$
\begin{array}{ccc}
\underline{\mathrm{K}}_0(\mathrm{Sch}_X) & \xrightarrow{\ \phi_f\ } & R(X) \\
{\scriptstyle u^*}\big\downarrow & & \big\downarrow{\scriptstyle u^*} \\
\underline{\mathrm{K}}_0(\mathrm{Sch}_Y) & \xrightarrow{\ \phi_{f \circ u}\ } & R(Y)
\end{array}
\qquad \text{if } u : Y \to X \text{ is smooth,}
$$

$$
\begin{array}{ccc}
\underline{\mathrm{K}}_0(\mathrm{Sch}_Y) & \xrightarrow{\ \phi_{f \circ u}\ } & R(Y) \\
{\scriptstyle u_!}\big\downarrow & & \big\downarrow{\scriptstyle u_!} \\
\underline{\mathrm{K}}_0(\mathrm{Sch}_X) & \xrightarrow{\ \phi_f\ } & R(X)
\end{array}
\qquad \text{if } u : Y \to X \text{ is proper,}
$$

$$
\begin{array}{ccc}
\underline{\mathrm{K}}_0(\mathrm{Sch}_X) \otimes \underline{\mathrm{K}}_0(\mathrm{Sch}_Y) & \xrightarrow{\ \phi_f \otimes \phi_g\ } & R(X) \otimes R(Y) \\
{\scriptstyle \boxtimes}\big\downarrow & & \big\downarrow{\scriptstyle \boxtimes} \\
\underline{\mathrm{K}}_0(\mathrm{Sch}_{X \times Y}) & \xrightarrow{\ \phi_{f \boxtimes g}\ } & R(X \times Y).
\end{array}
$$

---

[13] The definition given here differs from the one given in [5] for the sake of simplicity. We do not require the support property but a blow-up formula.

[14] The collection of group homomorphisms ϕ_f is what is called a morphisms of (motivic) ring $(Sm, prop)$-theories over $\mathbb{A}^1$ in [5].

and $\phi_{\operatorname{Spec} \mathbb{C} \xrightarrow{0} \mathbb{A}^1}(1) = 1$.

Exercise 5.3. — Proof the lemma using the following fact (see [3], Thm. 5.1). The group $K_0(\operatorname{Sch}_Z)$ can also be written as the abelian group generated by symbols $[X \xrightarrow{p} Z]$ with smooth X and proper p subject the "blow-up relation": If $i : Y \hookrightarrow X$ is a smooth subvariety and $\pi : \operatorname{Bl}_Y X \to X$ the blow-up of Y in X with exceptional divisor $j : E \hookrightarrow \operatorname{Bl}_Y X$, then $[\operatorname{Bl}_Y X \xrightarrow{p\pi} Z] - [E \xrightarrow{p\pi j} Z] = [X \xrightarrow{p} Z] - [Y \xrightarrow{pi} Z]$. Hint: Given a function $f : Z \to \mathbb{A}^1$, try the Ansatz $\phi_f([X \xrightarrow{p} Z]) := p_! \phi_{f \circ p}$ for a proper morphism $X \xrightarrow{p} Z$ on a smooth scheme X, where $\phi_{f \circ p} \in R(X)$ on the right hand side is given by our family of elements. In particular, $\phi_f(\mathbb{1}_X) = \phi_f \in R(X)$ for a regular function f on a smooth scheme X.

We need to apologize for using the same symbol ϕ_f with two different meanings. However, with a bit of practice it should be clear from the context which interpretation is used.

Example 5.4. — For every motivic theory there is a canonical vanishing cycle such that $\phi_{can,f}^R = \mathbb{1}_X \in R(X)$ for every $f : X \to \mathbb{A}^1$. Hence, it does not depend on f, and the map $\phi_f : \underline{K}_0(\operatorname{Sch}_X) \to R(X)$ is just the morphism constructed in Lemma 4.7.

Let us look at the following more interesting examples.

Example 5.5. — Let $R = \operatorname{Con}$. For $x \in X$ we fix a metric on an analytic neighborhood of $x \in X^{an}$ for example by embedding such a neighborhood into $\mathbb{C}^n$. We form the so-called Milnor fiber $\operatorname{MF}_f(x) := f^{-1}(f(x) + \delta) \cap B_\varepsilon(x)$, where $B_\varepsilon(x)$ is a small open ball around $x \in X$ and $0 < \delta \ll \varepsilon \ll 1$ are small real parameters. Notice, that the Milnor fiber depends on the choice of the metric and the choice of δ, ε. However, its reduced cohomology and its Euler characteristic $\chi(\operatorname{MF}_f(x))$ are independent of the choices made. We finally define $\phi_f^{con}(x) := 1 - \chi(\operatorname{MF}_f(x))$ for sufficiently small δ, ε. One can show that the properties listed above are satisfied. Due to [1], ϕ_f^{con} agrees with the Behrend function of $\operatorname{Crit}(f)$ up to the sign $(-1)^{\dim X}$.

Example 5.6. — The previous example has a categorification. For a closed point $t \in \mathbb{A}^1(\mathbb{C})$ consider the cartesian diagram with the induced projections $\tilde{f}$ and q_t

$$\begin{array}{ccc} X^{an} & \xleftarrow{\quad q_t \quad} & X^{an} \times_{\mathbb{C}} \mathbb{C} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ \mathbb{C} & \xleftarrow{\quad t+\exp(-) \quad} & \mathbb{C}. \end{array}$$

Denoting the inclusion of the fiber $X_t = f^{-1}(t)$ into X by ι_t, the (classical) vanishing cycle ϕ_f^{perv} is defined via

$$\oplus_{t \in \mathbb{A}^1(\mathbb{C})} \iota_{t!} \operatorname{Cone}(\mathbb{Q}_{X_t} \to \iota_t^* q_{t*} q_t^* \mathbb{Q}_X),$$

in the derived category $D^b(X, \mathbb{Q})$ of sheaves of $\mathbb{Q}$-vector spaces on X. Here, $\mathbb{Q}_X$ respectively $\mathbb{Q}_{X_t} = \iota_t^* \mathbb{Q}_X$ denotes the locally constant sheaf on X respectively X_t with stalk $\mathbb{Q}$. The morphism is the restriction to X_t of the adjunction $\mathbb{Q}_X \to$

$q_{t*}q_t^*\mathbb{Q}_X$. Spelling out the definition we see that the stalk of ϕ_f^{perv} at x is given by the reduced cohomology of the Milnor fiber $\mathrm{MF}_f(x)$ shifted by -1.

Associating to every connected X the Grothendieck group $\mathrm{K}_0(D_{con}^b(X,\mathbb{Q}))$ of the triangulated subcategory $D_{con}^b(X,\mathbb{Q}) \subset D^b(X,\mathbb{Q})$ consisting of complexes of sheaves of $\mathbb{Q}$-vector spaces with constructible cohomology, we obtain a motivic theory $\underline{\mathrm{K}}_0(D_{con}^b(-,\mathbb{Q}))$ with

$$\underline{\mathrm{K}}_0(D_{con}^b(X,\mathbb{Q})) := \prod_{X_i \in \pi_0(X)} \mathrm{K}_0(D_{con}^b(X_i,\mathbb{Q})).$$

Notice that $D_{con}^b(X,\mathbb{Q})$ has a bounded t-structure whose heart is the abelian category $\mathrm{Perv}(X)$ of perverse "sheaves" on X.[15] Thus, $\mathrm{K}_0(D_{con}^b(X,\mathbb{Q})) = \mathrm{K}_0(\mathrm{Perv}(X))$ motivating the shorthand $\underline{\mathrm{K}}_0(\mathrm{Perv})$ for our motivic theory. Moreover, up to a shift, ϕ_f^{perv} turns out to be a perverse "sheaf", i.e. a very special complex of sheaves with constructible cohomology, justifying our notation. By taking its class in the Grothendieck group, we get a vanishing cycle with values in $\underline{\mathrm{K}}_0(\mathrm{Perv})$ satisfying all required properties.

Example 5.7. — There is a refinement $\phi_f^{mon} \in D^b(\mathrm{MMHM}(X))$ of the previous example involving (complexes of) monodromic mixed Hodge modules with quasi-unipotent monodromy. Forgetting the Hodge and the monodromy structure, we get a functor $D^b(\mathrm{MMHM}(X)) \to D_{con}^b(X,\mathbb{Q})$ mapping ϕ_f^{mon} to ϕ_f^{perv}. By passing to the associated Grothendieck groups, we get a vanishing cycle ϕ_f^{mon} with values in

$$\underline{\mathrm{K}}_0(D^b(\mathrm{MMHM}(X))) = \underline{\mathrm{K}}_0(\mathrm{MMHM}(X)) := \prod_{X_i \in \pi_0(X)} \mathrm{K}_0(\mathrm{MMHM}(X_i)).$$

The motivic theory $\underline{\mathrm{K}}_0(\mathrm{MMHM})$ has a subtheory $\underline{\mathrm{K}}_0(\mathrm{MHM})$ defined similarly using the Grothendieck group of the category of ordinary mixed Hodge modules, i.e. monodromic mixed Hodge modules with trivial monodromy. See [7], Section 2.1 for more details.

In the remaining part of this subsection we will construct vanishing cycles depending functorialy on R. First of all we need to enlarge R by defining a new motivic theory $R(- \times \mathbb{A}^1)$ mapping X to $R(X \times \mathbb{A}^1)$ and using the exterior product

$$R(X \times \mathbb{A}^1) \otimes R(Y \times \mathbb{A}^1) \xrightarrow{\boxtimes} R(X \times Y \times \mathbb{A}^2) \xrightarrow{(\mathrm{id} \times +)_!} R(X \times Y \times \mathbb{A}^1)$$

with unit $1' := 0_!(1) \in R(\mathrm{Spec}\,\mathbb{C} \times \mathbb{A}^1) = R(\mathbb{A}^1)$ and the σ^n-operations

$$R(X \times \mathbb{A}^1) \xrightarrow{\sigma^n} R(\mathrm{Sym}^n(X \times \mathbb{A}^1)) \xrightarrow{s_!} R(\mathrm{Sym}^n(X) \times \mathbb{A}^1),$$

where s maps an unordered tuple $[(x_1,z_1),\ldots,(x_n,z_n)]$ to $([x_1,\ldots,x_n], z_1 + \cdots + z_n)$.

Exercise 5.8. — Check the properties of a motivic theory given in Proposition 4.3.

----

[15]We put the word sheaf in quotes as a perverse sheaf is often not a sheaf but a complex of sheaves with constructible cohomology. In particular, this t-structure is different from the standard t-structure on $D_{con}^b(X,\mathbb{Q})$ whose heart is the abelian category of constructible sheaves of $\mathbb{Q}$-vector spaces.

Given a scheme X, let $\mathbb{G}_m$ act on $X \times \mathbb{A}^1$ via $g(x,z) = (x,gz)$. For connected X we denote by $R_{\mathbb{G}_m}^{gm}(X \times \mathbb{A}^1)$ the subgroup of $R(X \times \mathbb{A}^1)$ generated by elements $[Y \xrightarrow{f} X \times \mathbb{A}^1]_R$ such that Y carries a good[16] $\mathbb{G}_m$-action for which f is homogeneous of some degree $d > 0$, i.e. $f(gy) = g^d f(y)$. Notice, that such a Y will carry many actions for which f is homogeneous. Indeed, given $0 \neq n \in \mathbb{N}$, let $\mathbb{G}_m$ act on Y via $g \star y := g^n y$ using the old action on the right hand side. Then, f is homogeneous of degree dn with respect to the new action. In particular, given finitely many generators $[Y_i \xrightarrow{f_i} X \times \mathbb{A}^1]$, we can always assume that the degrees of f_i are equal. Finally, we put $R_{\mathbb{G}_m}^{gm}(X \times \mathbb{A}^1) := \prod_{X_i \in \pi_0(X)} R_{\mathbb{G}_m}^{gm}(X_i \times \mathbb{A}^1)$ for non-connected X.

LEMMA 5.9. — *The subgroup $R_{\mathbb{G}_m}^{gm}(X \times \mathbb{A}^1) \subset R(X \times \mathbb{A}^1)$ is invariant under pull-backs, push-forwards, exterior products and the σ^n-operations. Moreover, pr_X^* maps $R^{gm}(X)$ onto a "λ-ideal" $I_X^{gm} \subseteq R_{\mathbb{G}_m}^{gm}(X \times \mathbb{A}^1)$, i.e.*

$$a \boxtimes b \in I_{X \times Y}^{gm} \qquad \text{for } a \in I_X^{gm} \text{ and } b \in R_{\mathbb{G}_m}^{gm}(Y \times \mathbb{A}^1), \text{ and}$$
$$\sigma^n(a) \in I_{\mathrm{Sym}^n(X)}^{gm} \qquad \text{for } a \in I_X^{gm}.$$

Exercise 5.10. —   Check the first sentence of the previous lemma.

Proof. — To show that I_X^{gm} is a λ-ideal, it suffices to look at generators

$$[V \times \mathbb{A}^1 \xrightarrow{f \times \mathrm{id}_{\mathbb{A}^1}} X \times \mathbb{A}^1] \quad \text{and} \quad [W \xrightarrow{(g,h)} Y \times \mathbb{A}^1]$$

of I_X^{gm} and $R_{\mathbb{G}_m}^{gm}(Y \times \mathbb{A}^1)$ respectively with $[V \xrightarrow{f} X] \in R^{gm}(X)$ (cf. Exercise 5.11). For the $\boxtimes$-product

$$[V \times \mathbb{A}^1 \to X \times \mathbb{A}^1] \boxtimes [W \to Y \times \mathbb{A}^1] = [V \times \mathbb{A}^1 \times W \xrightarrow{u} X \times Y \times \mathbb{A}^1]$$

with $u(v,z,w) = (f(v), g(w), z + h(w))$ we use the isomorphism

$$V \times \mathbb{A}^1 \times W \ni (v,z,w) \mapsto (v,w,z+h(w)) \in V \times W \times \mathbb{A}^1$$

to show

$$[V \times \mathbb{A}^1 \times W \xrightarrow{u} X \times Y \times \mathbb{A}^1] = \mathrm{pr}_{X \times Y}^*([V \times W \xrightarrow{f \times g} X \times Y]) \in I_{X \times Y}^{gm}.$$

We also have

$$\sigma^n[V \times \mathbb{A}^{1\,1} \xrightarrow{f \times \mathrm{id}_{\mathbb{A}^1}} X \times \mathbb{A}^1] = [\mathrm{Sym}^n(V \times \mathbb{A}^1) \xrightarrow{\tilde{p}} \mathrm{Sym}^n(X) \times \mathbb{A}^1]$$

with $\tilde{p}$ being induced by the S_n-invariant morphism $p : (V \times \mathbb{A}^1)^n \to \mathrm{Sym}^n(X) \times \mathbb{A}^1$ with $p((v_1,z_1), \dots, (v_n,z_n)) = ((f(v_1), \dots, f(v_n)), z_1 + \dots + z_n)$. We define

$$(V^n \times \mathbb{A}^n)^0 := \{(v_1, \dots, v_n, z_1, \dots, z_n) \in V^n \times \mathbb{A}^n \mid z_1 + \dots z_n = 0\}$$

for all $n > 0$. There is a S_n-equivariant isomorphism

$$\psi : (V^n \times \mathbb{A}^n)^0 \times \mathbb{A}^1 \to (V \times \mathbb{A}^1)^n,$$

$$((v_1, \dots, v_n, z_1, \dots, z_n), z) \mapsto ((v_1, z_1 + z/n), \dots, (v_n, z_n + z/n)).$$

Then, $p \circ \psi = q \times \mathrm{id}_{\mathbb{A}^1} : (V^n \times \mathbb{A}^n)^0 \times \mathbb{A}^1 \to \mathrm{Sym}^n(X) \times \mathbb{A}^1$ for the S_n-invariant morphism

$$q : (V^n \times \mathbb{A}^n)^0 \twoheadrightarrow (V^n \times \mathbb{A}^n)^0 /\!\!/ S_n \xrightarrow{\tilde{q}} \mathrm{Sym}^n(X)$$

---

[16]An action of $\mathbb{G}_m$ on Y is called good if every point $y \in Y$ has an affine $\mathbb{G}_m$-invariant neighborhood.

with $q(v_1, \ldots, v_n, z_1, \ldots, z_n) = (f(v_1), \ldots, f(v_n))$. Modding out the S_n-action, we see that

$$[\mathrm{Sym}^n(V \times \mathbb{A}^1) \xrightarrow{\tilde{p}} \mathrm{Sym}^n(X) \times \mathbb{A}^1] = \mathrm{pr}^*_{\mathrm{Sym}^n}([(V^n \times \mathbb{A}^{1^n})^0 /\!\!/ S_n \xrightarrow{\tilde{q}} \mathrm{Sym}^n(X)])$$

is indeed in $I^{gm}_{\mathrm{Sym}^n(X)}$.   □

Exercise 5.11. — Convince yourself using the formula

$$\sigma^n(a+b) = \sum_{l=0}^{n} \pi^{(l)}_! (\sigma^l(a) \boxtimes \sigma^{n-l}(b))$$

for the motivic theory $R(- \times \mathbb{A}^1)$ with $\pi^{(l)} : \mathrm{Sym}^l(X) \times \mathrm{Sym}^{n-l}(X) \to \mathrm{Sym}^n(X)$ being the natural map, that $a \in I^{gm}_X$ implies $\sigma^n(a) \in I^{gm}_{\mathrm{Sym}^n(X)}$ is indeed true if it already holds for generators $a = \mathrm{pr}^*_X([V \xrightarrow{f} X])$ of I^{gm}_X.

Due to Lemma 5.9, we can form the quotient

$$R^{gm}_{mon}(X) = R^{gm}_{\mathbb{G}_m}(X \times \mathbb{A}^1) / \mathrm{pr}^*_X R^{gm}(X)$$

and obtain a new motivic theory together with a morphism $R^{gm} \to R^{gm}_{mon}$ of motivic theories (cf. Exercise 4.9) given by

$$R^{gm}(X) \xrightarrow{(\mathrm{id} \times 0)_!} R^{gm}_{\mathbb{G}_m}(X \times \mathbb{A}^1) \twoheadrightarrow R^{gm}_{mon}(X).$$

Exercise 5.12. — Fix a motivic theory R and consider the map $R^{gm}_{\mathbb{G}_m}(X \times \mathbb{A}^1) \to R^{gm}(X)$ given by $a \mapsto (\mathrm{id} \times 0)^*(a) - (\mathrm{id} \times 1)^*(a)$, where $0, 1 : \mathrm{Spec}\,\mathbb{C} \to \mathbb{A}^1$ are the obvious maps. As $\mathrm{pr}^*_X R^{gm}(X)$ is in the kernel, we obtain a well-defined group homomorphism $R^{gm}_{mon}(X) \to R^{gm}(X)$. Note that the composition

$$R^{gm}(X) \to R^{gm}_{mon}(X) \to R^{gm}(X)$$

is the identity. Hence, $R^{gm}(X)$ is a direct summand of $R^{gm}_{mon}(X)$. However, show that the retraction $R^{gm}_{mon}(X) \to R^{gm}(X)$ is not a morphism of motivic theories and R^{gm} is not a direct summand of the motivic theory R^{gm}_{mon}.

Exercise 5.13. — Using the notation of the previous exercise, show that the kernel of $\mathrm{Con}^{gm}_{mon}(X) \to \mathrm{Con}^{gm}(X) = \mathrm{Con}(X)$ is trivial, i.e. $\mathrm{Con}(X) = \mathrm{Con}^{gm}_{mon}(X)$. On the other hand, show that the kernel is nonzero for $X = \mathrm{Spec}\,\mathbb{C}$ and $R = \underline{\mathrm{K}}_0(\mathrm{Perv})$, $R = \underline{\mathrm{K}}_0(\mathrm{MHM})$ and $R = \underline{\mathrm{K}}_0(\mathrm{Sch})$.

If $R = R^{gm}$, we suppress the superscript "gm" from notation. This applies for instance to $\underline{\mathrm{K}}_0(\mathrm{Sch})$ but also to R^{gm} as $(R^{gm})^{gm} = R^{gm}$.

Exercise 5.14. — Check that $R^{gm}_{mon} = (R^{gm})^{gm}_{mon} = (R^{gm})_{mon}$ using our convention for the last equation.

If $R^{gm} \subsetneq R$ is a proper subtheory, as for example for $\underline{\mathrm{K}}_0(\mathrm{Perv})$ or for $\underline{\mathrm{K}}_0(\mathrm{MHM})$, we can nevertheless define a theory R_{mon} under the assumption that the formula

$$\sigma^n(a \boxtimes b) = \sum_{\lambda \vdash n} \pi^{(n)}_! \Big(P^\lambda\big(\sigma^1(a), \ldots, \sigma^n(a)\big) \boxtimes P^\lambda\big(\sigma^1(b), \ldots, \sigma^n(b)\big) \Big)$$

holds in $R(\mathrm{Sym}^n(X \times Y))$ for all $0 \neq n \in \mathbb{N}$, all $a \in R(X)$, $b \in R(Y)$ and all X, Y, where the sum is taken over all partitions $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_n \geqslant 0)$ of n and

$$P^\lambda(x_1, \ldots, x_n) = \det(x_{\lambda_i + j - i})_{1 \leqslant i, j \leqslant n} = \begin{vmatrix} x_{\lambda_1} & x_{\lambda_1+1} & \cdots & x_{\lambda_1+n-1} \\ x_{\lambda_2-1} & x_{\lambda_2} & \cdots & x_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\lambda_n-n+1} & x_{\lambda_n-n+2} & \cdots & x_{\lambda_n} \end{vmatrix}$$

is the polynomial from the Jacobi-Trudi formula with the convention $x_0 = 1$ and $x_m = 0$ for $m < 0$ or $m > n$. Here, $\pi^{(n)} : \mathrm{Sym}^n(X) \times \mathrm{Sym}^n(Y) \to \mathrm{Sym}^n(X \times Y)$ is the obvious map. The expression $P^\lambda(\sigma^1(a), \ldots, \sigma^n(a))$ is computed in $R(\mathrm{Sym}(X))$ with respect to the convolution product, and is an element of $R(\mathrm{Sym}^n(X))$ since $\lambda_1 + \ldots + \lambda_n = n$. Similarly for $P^\lambda(\sigma^1(b), \ldots, \sigma^n(b))$. It can be show that this formula holds[17] whenever R has a "categorification" as for example $\underline{\mathrm{K}}_0(\mathrm{MHM})$ or $\underline{\mathrm{K}}_0(\mathrm{Perv})$. In this case, we may replace $R^{gm}_{\mathbb{G}_m}(X \times \mathbb{A}^1)$ with the $R(X)$-submodule $R_{\mathbb{G}_m}(X \times \mathbb{A}^1)$ of $R(X \times \mathbb{A}^1)$ generated by $R^{gm}_{\mathbb{G}_m}(X \times \mathbb{A}^1)$. Note that $R(X \times \mathbb{A}^1)$ is an $R(X)$-module using the convolution product and the embedding $R(X) \hookrightarrow R(X \times \mathbb{A}^1)$ provided by the "zero section" $0_X = \mathrm{id}_X \times 0 : X \hookrightarrow X \times \mathbb{A}^1$. The formula for $\sigma^n(a \boxtimes b)$ ensures that $R_{\mathbb{G}_m}(X \times \mathbb{A}^1)$ is still closed under taking

$$\sigma^n : R(X \times \mathbb{A}^1) \to R(\mathrm{Sym}^n(X) \times \mathbb{A}^1)$$

and, thus, defines another motivic theory containing $R^{gm}_{\mathbb{G}_m}(- \times \mathbb{A}^1)$ as a subtheory. Moreover, $\mathrm{pr}_X^*(R(X)) =: I_X$ is a λ-ideal and the quotient

$$R_{mon}(X) := R_{\mathbb{G}_m}(X \times \mathbb{A}^1)/I_X$$

is a well-defined motivic theory which contains R as a subtheory such that $R(X) \hookrightarrow R_{mon}(X)$ is a retract for every X. Moreover, the following diagram is cartesian

$$\begin{array}{ccc} R^{gm} & \hookrightarrow & R^{gm}_{mon} \\ \cap \downarrow & & \downarrow \\ R & \hookrightarrow & R_{mon}. \end{array}$$

Exercise 5.15. — Show that $R^{gm}_{\mathbb{G}_m}(X \times \mathbb{A}^1)$ is already an $R(X)$-module under the assumption $R = R^{gm}$. Also $I_X^{gm} = \mathrm{pr}_X^*(R(X))$ in this case. Hence, we do not get anything new by the previous construction whenever it applies, and putting $R_{mon} := R^{gm}_{mon}$ for theories $R = R^{gm}$ will not cause any confusion.

Example 5.16. — There is a morphism $\underline{\mathrm{K}}_0(\mathrm{MMHM}) \to \underline{\mathrm{K}}_0(\mathrm{MHM})_{mon}$ of motivic theories. Roughly speaking, $\mathrm{MMHM}(X)$ is obtained by a categorification of the construction just described.

Example 5.17. — There is a morphism $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Sch}_X) \to \underline{\mathrm{K}}_0(\mathrm{Sch}_X)_{mon}$ for every (connected) X with $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Sch}_X)$ being defined in [8]. In a nutshell, $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Sch}_X)$ is constructed very similar to $\underline{\mathrm{K}}_0(\mathrm{Sch}_X)_{mon}$ by considering generators $[Y \xrightarrow{f} X \times \mathbb{A}^1, \rho]$ with Y carrying a $\mathbb{G}_m$-action $\rho : \mathbb{G}_m \times Y \to Y$ such that f is homogeneous of degree $d > 0$. In contrast to our previous definition, the $\mathbb{G}_m$-action ρ is part of the

---

[17]The formula is a direct consequence of the assumption that $R(\mathrm{Sym}(X \times Y))$ is a special λ-ring which is true for any "decategorification".

data, and $K_0^{\hat{\mu}}(\mathrm{Sch}_X) \to \underline{K}_0(\mathrm{Sch}_X)_{mon}$ forgets the $\mathbb{G}_m$-action ρ. In particular, given another homogeneous map $Y' \xrightarrow{f'} X \times \mathbb{A}^1$ with Y' carrying a $\mathbb{G}_m$-action ρ' and an isomorphism $\theta : Y' \xrightarrow{\sim} Y$ such that $f\theta = f'$, then $[Y \xrightarrow{f} X \times \mathbb{A}^1] = [Y' \xrightarrow{f'} X \times \mathbb{A}^1]$ in $\underline{K}_0(\mathrm{Sch}_X)_{mon}$, but unless θ is $\mathbb{G}_m$-equivariant, the generators

$$[Y \xrightarrow{f} X \times \mathbb{A}^1, \rho] \qquad \text{and} \qquad [Y' \xrightarrow{f'} X \times \mathbb{A}^1, \rho']$$

of $K_0^{\hat{\mu}}(\mathrm{Sch}_X)$ might be different. The relations in $K_0^{\hat{\mu}}(\mathrm{Sch}_X)$ are the cut and paste relation for $\mathbb{G}_m$-invariant closed subschemes $Z \subset Y$ and $[Y \times \mathbb{A}^1 \xrightarrow{u \times \mathrm{id}_{\mathbb{A}^1}} X \times \mathbb{A}^1, \rho] = 0$ for every $\mathbb{G}_m$-invariant morphism $u : Y \to X$ from a scheme Y with a good $\mathbb{G}_m$-action. Here, ρ is given by $g(y, z) = (gy, gz)$ using the $\mathbb{G}_m$-action on Y. There is a third relation dealing with linear actions of $\mu_d \subset \mathbb{G}_m$, the group of d-th roots of unity, which is also fulfilled in $\underline{K}_0(\mathrm{Sch}_X)_{mon}$ as étale locally trivial vector bundles are already Zariski locally trivial. Indeed, vector bundles are always induced by principal $\mathrm{GL}(n)$-bundles and $\mathrm{GL}(n)$ is special for all $n \in \mathbb{N}$.

Exercise 5.18. — Show that the construction $R \mapsto R_{mon}^{gm}$ is functorial in R.

The motivic theory R_{mon}^{gm} will be the target of our vanishing cycle which we are going to construct now. For this let $f : X \to \mathbb{A}^1$ be a regular function on a smooth connected scheme and let $\mathcal{L}_n(X)$ be the scheme parameterizing all arcs of length n in X, i.e. the scheme representing the set-valued functor

$$Y \mapsto \mathrm{Mor}(Y \times \mathrm{Spec}\,\mathbb{C}[z]/(z^{n+1}), X).$$

The standard action of $\mathbb{G}_m$ on $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[z]$ given by $z \mapsto gz$ induces an action on $\mathrm{Spec}\,\mathbb{C}[z]/(z^{n+1})$ and, hence, also on $\mathcal{L}_n(X)$. By functoriality applied to $f : X \to \mathbb{A}^1$, we also get a $\mathbb{G}_m$-equivariant morphism $\mathcal{L}_n(X) \xrightarrow{\mathcal{L}_n(f)} \mathcal{L}_n(\mathbb{A}^1) \cong \mathbb{A}^{n+1}$, where $\mathbb{G}_m$ acts coordinatewise on the latter arc space with weights $0, 1, \ldots, n$. Fix $t \in \mathbb{A}^1(\mathbb{C})$ and recall that $\iota_t : X_t \hookrightarrow X$ was the inclusion of the fiber of f over t. Consider the map

$$f_n = \mathrm{pr}_{n+1} \circ \mathcal{L}_n(f) : \mathcal{L}_n(X) \to \mathbb{A}^1,$$

a $\mathbb{G}_m$-equivariant map of degree n, and the projection $\pi_n : \mathcal{L}_n(X) \to X$ mapping an arc to its base point. Denote by $\mathcal{L}_n(X)|_{X_t}$ the preimage of X_t under π_n, i.e. the subscheme of arcs going through X_t, and, abusing notation, we will also write f resp. π_n for their restrictions to $\mathcal{L}_n(X)|_{X_t}$. By $\mathbb{G}_m$-equivariance,

$$[\mathcal{L}_n(X)|_{X_t} \xrightarrow{\pi_n \times f_n} X \times \mathbb{A}^1]_R$$

is in $R_{\mathbb{G}_m}^{gm}(X_t \times \mathbb{A}^1)$ and defines an element in $R_{mon}^{gm}(X_t)$. We form the generating series

$$Z_{f,t}^R(T) := \sum_{n \geq 1} \mathbb{L}_R^{-n \dim X} [\mathcal{L}_n(X)|_{X_t} \xrightarrow{\pi_n \times f_n} X \times \mathbb{A}^1]_R \, T^n \text{ in } R_{mon}^{gm}(X_t)[[T]].$$

The following result is a consequence of Thm. 3.3.1 in the article [8] of Denef and Loeser or of Thm. 5.4 in Looijenga's paper [22].

THEOREM 5.19. — *The series $Z_{f,t}^R(T)$ is a Taylor expansion of a rational function in T. The latter has a regular value at $T = \infty$ which is $-\mathbb{1}_{X_t}$ if X_t is smooth, i.e. $X_t \cap \mathrm{Crit}(f) = \emptyset$.*

For the last statement see also Exercise 5.26. As X is smooth and connected, X_t is singular only for finitely many values $t \in \mathbb{A}^1(\mathbb{C})$. In particular, the following object is well-defined.

DEFINITION 5.20. — Using the same notation for the rational function, we define $\phi_{f,t}^R := \mathbb{1}_{X_t} + Z_{f,t}^R(\infty) \in R_{mon}^{gm}(X_t)$ and, finally, let

$$\phi_f^R := \sum_{t \in \mathbb{A}^1(\mathbb{C})} \iota_{t!} \phi_{f,t}^R \in R_{mon}^{gm}(X)$$

be the vanishing cycle of $f : X \to \mathbb{A}^1$. For non-connected X we apply the definition to every connected component X_i and define ϕ_f^R to be the family $(\phi_{f|_{X_i}}^R)_{X_i \in \pi_0(X)}$ in $R_{mon}^{gm}(X) = \prod_{X_i \in \pi_0(X)} R_{mon}^{gm}(X_i)$.

Example 5.21. — One can show $\phi_f^{\mathrm{Con}} = \phi_f^{con}$ for all $f : X \to \mathbb{A}^1$ on smooth X.

Example 5.22. — Let $f : X \to \mathbb{A}^1$ be a regular function on a smooth connected scheme X. Using the morphism $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Sch}_X) \to \underline{\mathrm{K}}_0(\mathrm{Sch}_X)_{mon}$, the vanishing cycle ϕ_f^{mot} constructed by Denef and Loeser maps to $\phi_f^{\underline{\mathrm{K}}_0(\mathrm{Sch})}$ up to the normalization factor $(-1)^{\dim X}$, and we will keep the shorter notation ϕ_f^{mot} for $\phi_f^{\underline{\mathrm{K}}_0(\mathrm{Sch})}$.

Example 5.23. — Let f be a regular function on a smooth scheme X as before. Using the map $\underline{\mathrm{K}}_0(\mathrm{MMHM}(X)) \to \underline{\mathrm{K}}_0(\mathrm{MHM}(X))_{mon}$ discussed earlier, the vanishing cycle ϕ_f^{mon} maps to $\phi_f^{\underline{\mathrm{K}}_0(\mathrm{MHM})}$, and we will keep the shorter notation ϕ_f^{mon}. Similarly for ϕ_f^{perv}.

Exercise 5.24. — Prove that ϕ_f^R is functorial in R. In particular, the diagram

$$
\begin{array}{ccc}
\underline{\mathrm{K}}_0(\mathrm{Sch}_X) & =\!=\!=\!= & \underline{\mathrm{K}}_0(\mathrm{Sch}_X) \\
\Big\downarrow{\scriptstyle \phi^{mot}} & & \Big\downarrow{\scriptstyle \phi_f^R} \\
\underline{\mathrm{K}}_0(\mathrm{Sch}_X)_{mon} & \longrightarrow & R_{mon}^{gm}(X)
\end{array}
$$

commutes where we used Lemma 5.2, Lemma 4.7 and Exercise 5.18 to construct the corresponding morphisms. Conclude that ϕ_f^R is a vanishing cycle using the known fact that ϕ^{mot} is a vanishing cycle.

In order to compute the vanishing cycle in practice, we choose an embedded resolution of $X_t \subset X$, i.e. a smooth variety Y together with a proper morphism $\pi : Y \to X$ such that $Y_t = (f \circ \pi)^{-1}(a) = \pi^{-1}(X_t)$ is a normal crossing divisor and $\pi : Y \backslash Y_t \xrightarrow{\sim} X \backslash X_t$. Denote the irreducible components of Y_t by E_i with $i \in J$ and let $m_i > 0$ be the multiplicity of $f \circ \pi$ at E_i. Since $f \circ \pi$ is a section in $\mathcal{O}_Y(-\sum_{i \in J} m_i E_i)$, it induces a regular map to $\mathbb{A}^1$ from the total space of $\mathcal{O}_Y(\sum_{i \in I} m_i E_i)$ for any $\emptyset \neq I \subset J$. The latter space restricted to $E_I^\circ := \bigcap_{i \in I} E_i \backslash \bigcup_{i \notin I} E_i$ is just $\bigotimes_{i \in I} N_{E_i|Y}^{\otimes m_i}|_{E_I^\circ}$. By composition with the tensor product we get a regular map

$$f_I : N_I := \prod_{i \in I} (N_{E_i|Y} \backslash E_i)|_{E_I^\circ} \to \mathbb{A}^1$$

which is obviously homogeneous of degree m_i with respect to the $\mathbb{G}_m$-action on the factor $(N_{E_i|Y} \backslash E_i)|_{E_I^\circ}$ and homogeneous of degree $m_I := \sum_{i \in I} m_i$ with respect to

the diagonal $\mathbb{G}_m$-action. By composing with $\pi : Y \to X$, the projection $N_I \to E_I^\circ$ induces a map $\pi_I : N_I \to X_t$.

THEOREM 5.25 ([8] or [22]). — *Let $f : X \to \mathbb{A}_k^1$ be a regular map and $\pi : Y \to X$ be an embedded resolution of X_t. In the notation just explained we have*[18]

$$Z_{f,t}^R(\infty) \;=\; \sum_{\emptyset \neq I \subset J} (-1)^{|I|-1} [N_I \xrightarrow{\pi_I \times f_I} X_t \times \mathbb{A}^1]_R \;\in\; R_{mon}^{gm}(X_t).$$

Exercise 5.26. — Assume that X_t is a smooth fiber. Use Theorem 5.25 and the construction of $R_{mon}^{gm}(X_t)$ to show

$$Z_{f,t}^R(\infty) = -\mathbb{1}_{X_t} = -[X_t \xrightarrow{\text{id} \times 0} X_t \times \mathbb{A}^1]_R.$$

COROLLARY 5.27 (support property). — *Given a regular function $f : X \to \mathbb{A}^1$ on a smooth scheme X, the vanishing cycle ϕ_f^R is supported on $\mathrm{Crit}(f)$, i.e. ϕ_f^R is in the image of the embedding $R(\mathrm{Crit}(f))_{mon}^{gm} \hookrightarrow R(X)_{mon}^{gm}$, where*

$$\mathrm{Crit}(f) = \{df = 0\} \subset X$$

is the critical locus of f.

The following result might also be helpful when it comes to actual computations. Let $\int_X a \in R(\mathrm{Spec}\,\mathbb{C})$ be the short notation for $(X \to \mathrm{Spec}\,\mathbb{C})_!(a)$ for $a \in R(X)$.

THEOREM 5.28 ([6]). — *Let X be a smooth variety with $\mathbb{G}_m$-action such that every point $x \in X$ has a neighborhood $U \subset X$ isomorphic to $\mathbb{A}^{n(x)} \times U^{\mathbb{G}_m}$ with $\mathbb{G}_m$ acting by multiplication (with weight one) on $\mathbb{A}^{n(x)}$. Let $f : X \to \mathbb{A}^1$ be a homogeneous function of degree $d > 0$. Then, $\int_X \phi_f^R = [X \xrightarrow{f} \mathbb{A}^1]_R$ in $R_{mon}^{gm}(\mathrm{Spec}\,\mathbb{C})$.*

Exercise 5.29. — (1) Prove that the scheme $\{(x,y) \in \mathbb{A}^2 \mid xy \neq 0\} \ni (x,y) \mapsto xy \in \mathbb{A}^1$ over $\mathbb{A}^1$ is isomorphic to the scheme $\mathbb{G}_m \times \mathbb{G}_m \ni (x,y) \to y \in \mathbb{A}^1$ over $\mathbb{A}^1$ and conclude $[\{xy \neq 0\} \xrightarrow{xy} \mathbb{A}^1]_R = -[\mathbb{G}_m]_R = 1 - \mathbb{L}_R$ in $R_{mon}^{gm}(\mathrm{Spec}\,\mathbb{C})$.

(2) Use this and any of the two previous theorems to show $\phi_{f,0}^R = \int_{\mathbb{A}^2} \phi_f^R = \mathbb{L}_R$ for the function $f : \mathbb{A}^2 \ni (x,y) \mapsto xy \in \mathbb{A}^1$. Note that $\mathrm{Crit}(f) \cong \mathrm{Spec}\,\mathbb{C}$ is the origin $0 \in \mathbb{A}^2$. Hence, ϕ_f^R is located at the origin.

(3) Use the result of the second part and the product formula for vanishing cycles to prove that $\phi_{z^2}^R$ for the function $\mathbb{A}^1 \ni z \mapsto z^2 \in \mathbb{A}^1$ is a square root $\mathbb{L}_R^{1/2}$ of $\mathbb{L}_R$.

(4) Show that $\sigma^n(\mathbb{L}_R^{1/2}) = 0$ for all $n \geqslant 2$. Hint: It is a well-known fact that $\mathbb{A}^n \ni (z_1, \ldots, z_n) \mapsto \left(\sum_{i=1}^n z_i^k\right)_{k=1}^n \in \mathbb{A}^n$ has a factorization $\mathbb{A}^n \to \mathrm{Sym}^n(\mathbb{A}^1) \xrightarrow{\sim} \mathbb{A}^n$ into the quotient map for the natural S_n-action on $\mathbb{A}^n$ and an isomorphism. Remember that for every scheme X the equation $[X \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}_{\mathbb{A}^1}} \mathbb{A}^1] = 0$ holds in $R(\mathrm{Spec}\,\mathbb{C})_{mon}^{gm}$ by construction.

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[18]This formula seems to differs from the one given in [8] or [22] by a sign. However, this is not true as the authors work with schemes over X with good μ_d-action. Given such a scheme $Y \xrightarrow{u} X$ with μ_d-invariant u, the associated generator of $R_{mon}^{gm}(X)$ is $-[Y \times_{\mu_d} \mathbb{G}_m \ni (y,z) \mapsto (u(y), z^d) \in X \times \mathbb{A}^1]_R$. The sign here is chosen in such a way that if Y carries a trivial μ_d-action, then the generator is equivalent to $[Y \ni y \mapsto (u(y), 0) \in X \times \mathbb{A}^1]_R$ in $R^{gm}(X) \subset R_{mon}^{gm}(X)$.

5.2. **Vanishing cycles for quotient stacks.** The theory of vanishing cycles for regular functions $f : X \to \mathbb{A}^1$ on smooth schemes generalizes straight forward to functions $\mathfrak{f} : \mathfrak{X} \to \mathbb{A}^1$ on (disjoint unions of) smooth quotient stacks. Note that a quotient stack X/G is called smooth if X is smooth. A closed substack $\mathfrak{P} \subset \mathfrak{X}$ is given by Y/G for a G-invariant closed subset $Y \subset X$. The blow-up of $\mathfrak{P}$ in $\mathfrak{X}$ is then simply given by the quotient stack $\mathrm{Bl}_{\mathfrak{P}} \mathfrak{X} = \mathrm{Bl}_Y X/G$ having exceptional divisor $\mathfrak{E} = E/G$. Given a quotient stack X/G with smooth X and a regular function $\mathfrak{f} : X/G \to \mathbb{A}^1$, we denote by $\mathrm{Crit}(\mathfrak{f})$ the quotient stack $\mathrm{Crit}(\mathfrak{f}\rho)/G$ with $\rho : X \to X/G$. The generalization to disjoint unions of quotient stacks is at hand.

DEFINITION 5.30. — Given a stacky motivic theory R, a stacky vanishing cycle (with values in R) is a rule associating to every regular function $\mathfrak{f} : \mathfrak{X} \to \mathbb{A}^1$ in a disjoint union of smooth quotient stacks an element $\phi_{\mathfrak{f}} \in R(\mathfrak{X})$ such that the following holds.

(1) If $u : \mathfrak{P} \to \mathfrak{X}$ is a smooth, then $\phi_{\mathfrak{f} \circ u} = u^*(\phi_{\mathfrak{f}})$.
(2) Let $\mathfrak{X}$ be a disjoint union of smooth quotients containing a smooth closed substack $i : \mathfrak{P} \hookrightarrow \mathfrak{X}$. Denote by $j : \mathfrak{E} \hookrightarrow \mathrm{Bl}_{\mathfrak{P}} \mathfrak{X}$ the exceptional divisor of the blow-up $\pi : \mathrm{Bl}_{\mathfrak{P}} \mathfrak{X} \to \mathfrak{X}$ of $\mathfrak{P}$ in $\mathfrak{X}$. Then the formula

$$\pi_!\big(\phi_{\mathfrak{f} \circ \pi} - j_! \phi_{\mathfrak{f} \circ \pi \circ j}\big) = \phi_{\mathfrak{f}} - i_! \phi_{\mathfrak{f} \circ i}$$

holds for every $\mathfrak{f} : \mathfrak{X} \to \mathbb{A}^1$.
(3) Given two morphisms $\mathfrak{f} : \mathfrak{X} \to \mathbb{A}^1$ and $\mathfrak{g} : \mathfrak{Y} \to \mathbb{A}^1$ with smooth $\mathfrak{X}$ and $\mathfrak{Y}$, we introduce the notation $\mathfrak{f} \boxtimes \mathfrak{g} : \mathfrak{X} \times \mathfrak{P} \xrightarrow{\mathfrak{f} \times \mathfrak{g}} \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1$. Then $\phi_{\mathfrak{f} \boxtimes \mathfrak{g}} = \phi_{\mathfrak{f}} \boxtimes \phi_{\mathfrak{g}}$ in $R(\mathfrak{X} \times \mathfrak{P})$. Moreover, $\phi_{\mathrm{Spec}\,\mathbb{C} \xrightarrow{0} \mathbb{A}^1}(1) = 1$.

Recall, that we constructed a correspondence between motivic theories R with $\mathbb{L}_R^{-1}, (\mathbb{L}_R^n - 1)^{-1} \in R(\mathrm{Spec}\,\mathbb{C})$ for all $0 \neq n \in \mathbb{N}$ and stacky motivic theories satisfying $\rho^* : R(X/G) \xrightarrow{\sim} R(X)^G$ for every special group G.

LEMMA 5.31. — *Let R be a motivic theory such that $\mathbb{L}_R^{-1}$ and $(\mathbb{L}_R^n - 1)^{-1}$ are in $R(\mathrm{Spec}\,\mathbb{C})$ for all $0 \neq n \in \mathbb{N}$. The restriction to schemes provides a bijection between stacky vanishing cycles with values in R^{st} and vanishing cycles with values in R.*

Proof. — By applying the first property of a stacky vanishing cycle to the smooth map $\rho : X \to X/G$, we see that $\phi_{\mathfrak{f}}$ is uniquely determined by $f = \mathfrak{f} \circ \rho : X \to \mathbb{A}^1$ as $\rho^* : R(X/G) \to R(X)$ is injective for special G. Moreover, applying the first property once more to

$$X \times G \xrightarrow{m} X$$
$$\downarrow{\scriptstyle \mathrm{pr}_X}$$
$$X$$

we observe that the vanishing cycle ϕ_f of a G-invariant function $f : X \to \mathbb{A}^1$ is G-invariant. Hence, given a vanishing cycle with values in R, we can define $\phi_{\mathfrak{f}}^{st}$ to be the unique element in $R^{st}(X/G)$ mapping to $\phi_f = \phi_{\mathfrak{f} \circ \rho}$ under the isomorphism $\rho^* : R^{st}(X/G) \cong R(X)^G$. Alternatively, we can write $\phi_{\mathfrak{f}}^{st} = \rho_!(\phi_f)/[G]_R$ since $[G]_R^{-1}\rho_!$ is the inverse of ρ^* on $R(X)^G$. If $\mathfrak{f} : \mathfrak{X} \to \mathbb{A}^1$ is a function on a disjoint union of quotient stacks $\mathfrak{X}_i = X_i/G_i$, we may assume that G_i is special for all i (see

Exercise 4.10) and define $\phi_{\mathfrak{f}}^{st}$ by means of the family $\phi_{\mathfrak{f}|_{\mathfrak{X}_i}}^{st}$ using the first property of a stacky motivic theory. $\qquad\qquad\square$

Exercise 5.32. — Complete the proof of the previous lemma by checking that $\phi_{\mathfrak{f}}^{st}$ on a quotient stack $\mathfrak{X} = X/G$ is independent of the choice of a presentation, i.e. if $X/G \cong Y/H$ for special groups G and H, then X is smooth if and only if Y is smooth, and $\phi_{\mathfrak{f} \circ \rho_X}$ corresponds to $\phi_{\mathfrak{f} \circ \rho_Y}$ under the isomorphism $R(X)^G \cong R(Y)^H$ in this case. Moreover, show that ϕ^{st} satisfies the properties of a stacky vanishing cycle.

Example 5.33. — Given a motivic theory R satisfying equation (4.1) for all $a \in R(X), n \in \mathbb{N}$ and a vanishing cycle ϕ with values in R, we can adjoin inverses of $\mathbb{L}_R$ and $\mathbb{L}_R^n - 1$ for all $n > 0$. By applying the previous lemma to the new motivic theory and the induced vanishing cycle, we get a motivic theory R^{st} and a vanishing cycle ϕ^{st} such that $\eta_X(\phi_f) = \phi_f^{st}$ for every $f : X \to \mathbb{A}^1$, where $\eta_X : R(X) \to R^{st}(X)$ is the adjunction morphism $R \to R^{st}|_{\mathrm{Sch}_{\mathbb{C}}}$ from the previous section. In particular, we can apply this to $\phi_{can}^{R^{gm}}$ with values in R^{gm} and also to ϕ^R with values in R_{mon}^{gm} as equation (4.1) holds in both cases (cf. Exercise 4.9). If R satisfies equation (4.1), we can also extend ϕ_{can}^R with values in R. Note that $\phi_{can,\mathfrak{f}}^{R,st} = \mathbb{1}_{\mathfrak{X}} \in R^{st}(\mathfrak{X})$ for all $\mathfrak{f} : \mathfrak{X} \to \mathbb{A}^1$.

6. Donaldson-Thomas theory

After introducing a lot of technical notation, we are now in the position to provide the definition of Donaldson-Thomas functions and to state a couple of results in Donaldson-Thomas theory. We close this section by given a list of examples. There are basically three approaches to define Donaldson-Thomas functions (see [5]). The one given here is due to Kontsevich and Soibelman.

6.1. **Definition and main results.** We start by fixing a stacky motivic theory R satisfying $R(X/G) \cong R(X)^G$ for every quotient stack X/G with special group G. Moreover, let ϕ be a stacky vanishing cycle with values in R which is completely determined by its restriction to functions $f : X \to \mathbb{A}^1$ on smooth schemes X (cf. Lemma 5.31). As shown before, we could start with any vanishing cycle with values in a motivic theory and pass to the "stackification". Let us also assume that a square root $\mathbb{L}_R^{1/2}$ of $\mathbb{L}_R$ is contained in $R(\mathrm{Spec}\,\mathbb{C})$ such that $\sigma^n(\mathbb{L}_R^{1/2}) = 0$ for all $n \geqslant 2$.

Example 6.1. — Assume that $\sigma^n(a\mathbb{L}_R) = \sigma^n(a)\mathbb{L}_R^n$ holds for all $a \in R(X)$, all $n \in \mathbb{N}$ and all X. As shown in [5], Appendix B, one can extend the σ^n-operations to $R(X)[\mathbb{L}_R^{1/2}]$ such that $\sigma^n(-\mathbb{L}_R^{1/2}) = (-\mathbb{L}_R^{1/2})^n$ for all $n \in \mathbb{N}$ or equivalently $\sigma^n(\mathbb{L}_R^{1/2}) = 0$ for all $n \geqslant 2$. Thus, $R[\mathbb{L}_R^{1/2}]$ is a new motivic theory having the required square root of $\mathbb{L}_R$. We can apply the stackification to the canonical vanishing cycle $\phi_{can}^{R[\mathbb{L}_R^{1/2}]}$ of $R[\mathbb{L}_R^{1/2}]$ and obtain a pair $(R[\mathbb{L}_R^{1/2}]^{st}, \phi_{can}^{R[\mathbb{L}_R^{1/2}],st})$ satisfying our requirements. Note that the assumption on R is always fulfilled if we replace R with R^{gm} (cf. Exercise 4.9).

Example 6.2. — For every motivic theory R the element

$$[\mathbb{A}^1 \ni z \mapsto z^2 \in \mathbb{A}^1]_R \in R_{mon}^{gm}(\mathrm{Spec}\,\mathbb{C})$$

is the required square root of $\mathbb{L}_{R_{mon}^{gm}}$ as shown in Exercise 5.29. The stackification of ϕ^R with values in R_{mon}^{gm} will match our requirements. This applies in particular to ϕ^{mot} and ϕ^{mon}. Note that the stackification of $\mathrm{Con} = \mathrm{Con}_{mon}^{gm}$ and of $\underline{\mathrm{K}}_0(D_{con}^b(-, \mathbb{Q}))_{mon}^{gm}$ is zero as $[\mathrm{GL}(n)]_R = 0$ in both cases.

We fix a quiver Q with potential W, a stability condition ζ and use the notation introduced in Section 3.2. Recall that $\mathfrak{M}^{\zeta-ss}$ was the stack of ζ-semistable quiver representations with coarse moduli space $\mathcal{M}^{\zeta-ss}$ parameterizing polystable representations. Similarly $\mathfrak{M}$ was the stack of all quiver representations and $\mathcal{M}^{ssimp}$ its coarse moduli space parameterizing semisimple representations. There are various maps between these spaces as shown in the following diagram

$$
\begin{array}{ccc}
\mathfrak{M}^{\zeta-ss} \hookrightarrow & & \mathfrak{M} \\
\downarrow{\scriptstyle p^\zeta} & & \downarrow{\scriptstyle p} \\
\mathcal{M}^{\zeta-ss} \xrightarrow{q^\zeta} & \mathcal{M}^{ssimp} \xrightarrow{\dim \times \mathcal{T}r(W)} & \mathbb{N}^{Q_0} \times \mathbb{A}^1.
\end{array}
$$

Note that the maps in the lower horizontal row are homomorphisms of monoids with respect to (direct) sums. Moreover, q^ζ is proper. Denote the composition $\mathfrak{M}^{\zeta-ss} \hookrightarrow \mathfrak{M} \xrightarrow{\mathcal{T}r(W)\circ p} \mathbb{A}^1$ with $\mathfrak{Tr}(W)^\zeta$. For a fixed slope $\mu \in \mathbb{R}$, let us introduce the short hand $\phi_{\mathfrak{Tr}(W)^\zeta}(\mathcal{IC}_{\mathfrak{M}_\mu^{\zeta-ss}})$ for the object in $R(\mathfrak{M}^{\zeta-ss}) \cong \prod_{d \in \mathbb{N}^{Q_0}} R(\mathfrak{M}_d^{\zeta-ss})$ having components

$$
\mathbb{L}_R^{(d,d)/2} \phi_{\mathfrak{Tr}(W)^\zeta}|_{\mathfrak{M}_d^{\zeta-ss}} = \frac{\mathbb{L}_R^{(d,d)/2}}{[G_d]_R} \rho_{d!}\big(\phi_{\mathrm{Tr}(W)_d}|_{X_d^{\zeta-ss}}\big)
$$

if d has slope μ or $d = 0$ and 0 for the remaining dimension vectors d. Recall that $\rho_d : X_d^{\zeta-ss} \to X_d^{\zeta-ss}/G_d \cong \mathfrak{M}_d^{\zeta-ss}$ was the standard atlas of the quotient stack and (d,d) was defined in Definition 3.15. The idea behind the notation is the following. The vanishing cycle ϕ defines a map $\underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathfrak{M}^{\zeta-ss}})^{st} \to R(\mathfrak{M}^{\zeta-ss})$ mapping $\mathbb{1}_{\mathfrak{M}^{\zeta-ss}} = (\mathfrak{M}^{\zeta-ss} \to \mathrm{Spec}\,\mathbb{C})^*(1)$ to $\phi_{\mathfrak{Tr}(W)^\zeta}$. If we define $\mathcal{IC}_{\mathfrak{M}^{\zeta-ss}}$ to be the object in $\underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathfrak{M}^{\zeta-ss}})[\mathbb{L}^{1/2}]^{st}$ whose restriction to $\mathfrak{M}_d^{\zeta-ss}$ is $\mathbb{L}^{(d,d)/2} \mathbb{1}_{\mathfrak{M}_d^{\zeta-ss}}$ if d has slope μ or $d = 0$ and zero else, then $\phi_{\mathfrak{Tr}(W)^\zeta}(\mathcal{IC}_{\mathfrak{M}^{\zeta-ss}})$ is just the image of $\mathcal{IC}_{\mathfrak{M}_\mu^{\zeta-ss}}$ under the induced map $\underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathfrak{M}^{\zeta-ss}})[\mathbb{L}^{1/2}]^{st} \to R(\mathfrak{M}^{\zeta-ss})$. One should think of $\mathcal{IC}_{\mathfrak{M}_\mu^{\zeta-ss}}$ as (the class of) the motivic intersection complex of $\mathfrak{M}_\mu^{\zeta-ss} \subseteq \mathfrak{M}^{\zeta-ss}$.

Let us define the convolution product on $R(\mathcal{M}^{\zeta-ss})$ by means of

$$
R(\mathcal{M}^{\zeta-ss}) \otimes R(\mathcal{M}^{\zeta-ss}) \xrightarrow{\boxtimes} R(\mathcal{M}^{\zeta-ss} \times \mathcal{M}^{\zeta-ss}) \xrightarrow{\oplus_!} R(\mathcal{M}^{\zeta-ss})
$$

and operations $\mathrm{Sym}^n : R(\mathcal{M}^{\zeta-ss}) \to R(\mathcal{M}^{\zeta-ss})$ for $n \in \mathbb{N}$ via

$$
R(\mathcal{M}^{\zeta-ss}) \xrightarrow{\sigma^n} R(\mathrm{Sym}^n \mathcal{M}^{\zeta-ss}) \xrightarrow{\oplus_!} R(\mathcal{M}^{\zeta-ss}). \tag{6.1}
$$

LEMMA 6.3. — *For $a \in R(\mathcal{M}^{\zeta-ss})$ with $a_0 := a|_{\mathcal{M}_0^{\zeta-ss}} = 0$ the infinite sum $\mathrm{Sym}(a) := \sum_{n \in \mathbb{N}} \mathrm{Sym}^n(a)$ has only finitely many nonzero summands after restriction to $\mathcal{M}_d^{\zeta-ss}$ and, hence, defines a well defined element in*

$$
R(\mathcal{M}^{\zeta-ss}) \cong \prod_{d \in \mathbb{N}^{Q_0}} R(\mathcal{M}_d^{\zeta-ss}).
$$

Conversely, every element $b \in R(\mathcal{M}^{\zeta - ss})$ with $b_0 = 1 \in R(\mathcal{M}_0^{\zeta - ss}) = R(\operatorname{Spec} \mathbb{C})$ can be written uniquely as $\operatorname{Sym}(a)$. The map $\operatorname{Sym}(-)$ is a group homomorphism form the additive group $\{a \in R(\mathcal{M}^{\zeta - ss}) \mid a_0 = 0\}$ to the multiplicative group $\{b \in R(\mathcal{M}^{\zeta - ss}) \mid b_0 = 1\}$. The same holds true if we replace $\mathcal{M}^{\zeta - ss}$ with $\mathcal{M}^{ssimp}$ or with $\mathbb{N}^{Q_0}$.

Proof. — Fix $d \neq 0$. Since $\oplus$ maps $\operatorname{Sym}^n \mathcal{M}_e^{\zeta - ss}$ to $\mathcal{M}_{ne}^{\zeta - ss}$, we get

$$\operatorname{Sym}^n(a)|_{\mathcal{M}_d^{\zeta - ss}} = 0$$

for all $n > |d| = \sum_{i \in Q_0} d_i$, and the infinite sum is finite after restriction to $\mathcal{M}_d^{\zeta - ss}$. Conversely, given b we set $a_0 = 0$. Suppose $a_e \in R(\mathcal{M}_e^{\zeta - ss})$ has been constructed for all dimension vectors[19] $e < d$. We put

$$a_d := b_d - \sum_{\substack{n_1 e_1 + \ldots n_r e_r = d \\ 0 \neq n_j \in \mathbb{N}, 0 \neq e_i \neq e_j \in \mathbb{N}^{Q_0}}} \prod_{j=1}^{r} \operatorname{Sym}^{n_j}(a_{e_j}).$$

Then $b = \operatorname{Sym}(a)$ for $a \in R(\mathcal{M}^{\zeta - ss})$ with $a|_{\mathcal{M}_d^{\zeta - ss}} = a_d$. Using the properties of $\sigma^n : R(\mathcal{M}^{\zeta - ss}) \to R(\operatorname{Sym}^n \mathcal{M}^{\zeta - ss})$ we get $\operatorname{Sym}(0) = 1$ and $\operatorname{Sym}(a + b) = \operatorname{Sym}(a) \operatorname{Sym}(b)$. In particular,

$$\{b \in R(\mathcal{M}^{\zeta - ss}) \mid b_0 = 1\} \cong \{a \in R(\mathcal{M}^{\zeta - ss}) \mid a_0 = 0\}$$

as groups. The proof for $\mathcal{M}^{ssimp}$ and $\mathbb{N}^{Q_0}$ is similar. $\square$

Exercise 6.4. — Let $\iota : \mathcal{M}^{\zeta - st} \hookrightarrow \mathcal{M}^{\zeta - ss}$ be the inclusion of the moduli space of ζ-stable representations. Show that $\mathbb{1}_{\mathcal{M}^{\zeta - ss}} = \operatorname{Sym}\left(\iota_! \mathbb{1}_{\mathcal{M}^{\zeta - st}}\right)$ holds in $\underline{\mathrm{K}}_0(\operatorname{Sch}_{\mathcal{M}^{\zeta - ss}})$. Hint: Proof that the strata of the Luna stratification of $\mathcal{M}^{\zeta - ss}$ and the strata of the natural stratification of $\bigsqcup_{\substack{n_1, \ldots, n_r \\ d_i \neq d_j \forall i \neq j}} \prod_{i=1}^{r} (\mathcal{M}_{d_i}^{\zeta - ss})^{n_i} /\!\!/ S_{n_i}$ given by the conjugacy types of the S_{n_i}-stabilizers coincide. In other words, the canonical map $\oplus : \operatorname{Sym}(\mathcal{M}^{\zeta - st}) \to \mathcal{M}^{\zeta - ss}$ is a "constructible" isomorphism.

Exercise 6.5. — Show that the restriction of $p_!^{\zeta}\left(\phi_{\mathfrak{Tr}(W)^{\zeta}}(\mathcal{IC}_{\mathfrak{M}_{\mu}^{\zeta - ss}})\right)$ to $\mathcal{M}_0^{\zeta - ss}$ is 1.

Using the last exercise and the previous lemma, the following definition makes sense.

DEFINITION 6.6. — The Donaldson-Thomas function $\mathcal{DT}(Q, W)^{\zeta} \in R(\mathcal{M}^{\zeta - ss})$ is the unique element with $\mathcal{DT}(Q, W)^{\zeta}|_{\mathcal{M}_0^{\zeta - ss}} = 0$ such that the restriction

$$\mathcal{DT}(Q, W)_{\mu}^{\zeta} := \mathcal{DT}(Q, W)|_{\mathcal{M}_{\mu}^{\zeta - ss}}$$

solves the equation

$$p_!^{\zeta}\left(\phi_{\mathfrak{Tr}(W)^{\zeta}}(\mathcal{IC}_{\mathfrak{M}_{\mu}^{\zeta - ss}})\right) = \operatorname{Sym}\left(\frac{\mathcal{DT}(Q, W)_{\mu}^{\zeta}}{\mathbb{L}_R^{1/2} - \mathbb{L}_R^{-1/2}}\right)$$

---

[19]We write $e < d$ if $d = e + e'$ with $0 \neq e' \in \mathbb{N}^{Q_0}$.

in $R(\mathcal{M}^{\zeta-ss})$ for all $\mu \in (-\infty, +\infty]$. We also use the notation $\mathcal{DT}(Q,W)^{\zeta}_d :=$ $\mathcal{DT}(Q,W)^{\zeta}|_{\mathcal{M}^{\zeta-ss}_d}$. The element

$$\int_{\mathcal{M}^{\zeta-ss}_d} \mathcal{DT}(Q,W)^{\zeta}_d =: \mathrm{DT}(Q,W)^{\zeta}_d \in R(\mathrm{Spec}\,\mathbb{C})$$

is called the Donaldson-Thomas invariant of (Q,W) with respect to ζ for dimension vector d. If $W = 0$, we simply write $\mathcal{DT}(Q)^{\zeta}_d$ and $\mathrm{DT}(Q)^{\zeta}_d$.

In view of Exercise 6.4, one might hope that $\mathcal{DT}(Q,W)^{\zeta}_d/(\mathbb{L}^{1/2}_R - \mathbb{L}^{-1/2}_R)$ is something like $p^{\zeta}_! \phi_{\mathfrak{Tr}(W)^{\zeta}}(j_! \mathcal{IC}_{\mathfrak{M}^{\zeta-st}_{\mu}})$, where $j : \mathfrak{M}^{\zeta-st}_{\mu} \hookrightarrow \mathfrak{M}^{\zeta-ss}_{\mu}$ denotes the inclusion, and $\mathcal{IC}_{\mathfrak{M}^{\zeta-st}_{\mu}}$ is defined similarly to $\mathcal{IC}_{\mathfrak{M}^{\zeta-ss}_{\mu}}$. Let us assume, we were allowed to commute $p^{\zeta}_!$ with $\phi_{\mathfrak{Tr}(W)^{\zeta}}$ which is a priori not clear as p^{ζ} is not proper. Then

$$p^{\zeta}_! \phi_{\mathfrak{Tr}(W)^{\zeta}}(j_! \mathcal{IC}_{\mathfrak{M}^{\zeta-st}_{\mu}}) = \phi_{\mathcal{Tr}(W)\circ q^{\zeta}}\left(\frac{\iota_! \mathcal{IC}_{\mathcal{M}^{\zeta-ss}_{\mu}}}{\mathbb{L}^{1/2}_R - \mathbb{L}^{-1/2}_R}\right).$$

for $\mathcal{IC}_{\mathcal{M}^{\zeta-st}_d} = \mathbb{L}^{((d,d)-1)/2}_R \mathbb{1}_{\mathcal{M}^{\zeta-st}_d}$ and the inclusion $\iota : \mathcal{M}^{\zeta-st} \hookrightarrow \mathcal{M}^{\zeta-ss}$ (cf. Exercise 6.4). We conclude $\mathcal{DT}(Q,W)^{\zeta}_d = \phi_{\mathcal{Tr}(W)\circ q^{\zeta}_d}(\iota_! \mathcal{IC}_{\mathcal{M}^{\zeta-ss}_d})$. This is not quite true. It turns out that the extension $\iota_! \mathcal{IC}_{\mathcal{M}^{\zeta-st}_{\mu}}$ of $\mathcal{IC}_{\mathcal{M}^{\zeta-st}_d}$ by zero has to be replaced with the "correct" extension $\mathcal{IC}_{\overline{\mathcal{M}^{\zeta-st}_d}}$ which restricts to $\mathcal{IC}_{\mathcal{M}^{\zeta-st}_d}$, but might also be nonzero on the boundary of $\mathcal{M}^{\zeta-st}_d$ inside $\mathcal{M}^{\zeta-ss}_d$. However, we have not defined $\mathcal{IC}_{\overline{\mathcal{M}^{\zeta-st}_d}}$ yet.

DEFINITION 6.7. — We denote by $\mathcal{IC}^{mot}_{\overline{\mathcal{M}^{\zeta-st}}} \in \underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathcal{M}^{\zeta-ss}})[\mathbb{L}^{1/2}]^{st}$ the Donaldson-Thomas function $\mathcal{DT}(Q)^{\zeta} = \mathcal{DT}(Q,0)^{\zeta}$ computed for $W = 0$ with the stackification of the canonical vanishing cycle $\phi^{\underline{\mathrm{K}}_0(\mathrm{Sch})[\mathbb{L}^{1/2}]}_{can}$. Note that $\overline{\mathcal{M}^{\zeta-st}_d} = \mathcal{M}^{\zeta-ss}_d$ if $\mathcal{M}^{\zeta-st}_d \neq \emptyset$ and $\overline{\mathcal{M}^{\zeta-st}_d} = \emptyset$ else.

The following result justifies the definition.

THEOREM 6.8 ([26]). — *If ζ is generic (see Definition 3.15), the element $\mathcal{IC}^{mot}_{\overline{\mathcal{M}^{\zeta-st}}}$ maps to the classical intersection complex[20] $\mathcal{IC}_{\overline{\mathcal{M}^{\zeta-st}}}$ of the closure of $\mathfrak{M}^{\zeta-st}$ inside $\mathcal{M}^{\zeta-ss}$ under the map*

$$\underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathcal{M}^{\zeta-ss}})[\mathbb{L}^{1/2}]^{st} \to \underline{\mathrm{K}}_0(\mathrm{MHM}(\mathcal{M}^{\zeta-ss}))[\mathbb{L}^{1/2}]^{st}$$

constructed in Lemma 4.7.

Example 6.9. — Assuming equation (4.1) for all $a \in R(X)$ and all $n \in \mathbb{N}$ so that ϕ^R_{can} has a stacky extension $\phi^{R[\mathbb{L}^{1/2}_R]}_{can}$. In this case, $\mathcal{DT}(Q,0)^{\zeta} = \mathcal{DT}(Q)^{\zeta}$ is just the image of $\mathcal{IC}^{mot}_{\overline{\mathcal{M}^{\zeta-st}}}$ under the canonical map

$$\underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathcal{M}^{\zeta-ss}})[\mathbb{L}^{1/2}]^{st} \to R(\mathcal{M}^{\zeta-ss})[\mathbb{L}^{1/2}]^{st}$$

---

[20]Strictly speaking one has to normalize the class of the classical (shifted) intersection complex of $\mathcal{M}^{\zeta-ss}_d$ by multiplication with $(-\mathbb{L}^{1/2})^{(d,d)-1}$ which does not change the underlying perverse sheaf.

of Lemma 4.7, and we may define

$$\mathcal{IC}_{\overline{\mathcal{M}^{\zeta-st}}} := \mathcal{DT}(Q)^\zeta \in R(\mathcal{M}^{\zeta-ss})[\mathbb{L}^{1/2}]^{st}.$$

As for mixed Hodge modules one can show that $\mathcal{IC}_{\overline{\mathcal{M}^{\zeta-st}}}$ has a lift in

$$R(\mathcal{M}^{\zeta-ss})[\mathbb{L}^{-1/2}]$$

if the canonical map $\underline{K}_0(\mathrm{Sch}) \to R$ factorizes trough $\underline{K}_0(\mathrm{MHM})$ or if R is a sheaf in the big étale topology with $[\mathrm{GL}(r)]_R$ acting as a nonzero divisor in each group $R(X)$ for all $r \geqslant 1$. The first case follows from Theorem 6.8 and for the second case we refer to Corollary 6.14 in [26] where the Donaldson-Thomas function is constructed étale locally (over strata of a stratification). The nonzero divisor condition is need to glue these étale local functions to a global function as the cocycle condition on overlaps holds only up to elements annihilated by certain $[\mathrm{GL}(r)]_R$. Using Corollary 6.6 in [26] one can relax the nonzero divisor condition by replacing $[\mathrm{GL}(r)]_R$ with $[\mathbb{P}^{r-1}]_R$ which is a factor of $[\mathrm{GL}(r)]_R$ according to Exercise 4.6(ii). Note that we can replace R with R^{gm} to ensure equation (4.1).

One can also prove the following result which should be seen as the analogue of Exercise 6.4.

THEOREM 6.10 ([5]). — *Recall that every stacky vanishing cycle with values in R satisfying our assumptions defines a map*

$$\phi_{\mathcal{T}r(W) \circ q^\zeta} : \underline{K}_0(\mathrm{Sch}_{\mathcal{M}^{\zeta-ss}})^{st} \to R(\mathcal{M}^{\zeta-ss}).$$

If this map commutes with the Sym^n-operations of equation (6.1) for every $n \in \mathbb{N}$, and if ζ is generic, then $\mathcal{DT}(Q,W)^\zeta = \phi_{\mathcal{T}r(W) \circ q^\zeta}\big(\mathcal{IC}^{mot}_{\mathcal{M}^{\zeta-st}}\big)$.

Example 6.11. — The assumption on $\phi_{\mathcal{T}r(W)q^\zeta}$ is true for $\phi = \phi^{mon}$. Hence, if ζ is generic, $\mathcal{DT}(Q,W)^\zeta_d = \phi^{mon}_{\mathcal{T}r(W) \circ q^\zeta_d}\big(\mathcal{IC}_{\mathcal{M}^{\zeta-st}_d}\big)$ if $\mathcal{M}^{\zeta-st}_d \neq \emptyset$ and zero else. It is not known yet whether or not $\phi^{mot}_{\mathcal{T}r(W) \circ q^\zeta}$ commutes with the Sym^n-operations, and we cannot apply the theorem. However, a counterexample is also not known, and conjecturally the theorem also holds for ϕ^{mot}.

Exercise 6.12. — Use the support property of ϕ^R (see Corollary 5.27) to show that $\mathcal{DT}(Q,W)^\zeta_d$ is supported on $\mathcal{M}^{W,\zeta-ss}_d$, i.e. is an element in $R(\mathcal{M}^{W,\zeta-ss}_d)$, where $\mathcal{M}^{W,\zeta-ss}_d$ the moduli space parameterizing ζ-polystable $\mathbb{C}Q$-representations V of dimension d such that $\partial W/\partial \alpha = 0$ on V for all $\alpha \in Q_1$.

6.2. **Examples.** The aim of this section is to provide some examples of (motivic) Donaldson-Thomas invariants.

6.2.1. *The m-loop quiver.* Let us consider the quiver $Q^{(m)}$ with one vertex and m loops. The choice of a stability condition is irrelevant as $\mathfrak{M}^{\zeta-ss} = \mathfrak{M}$. We take the canonical vanishing cycle of $\underline{K}_0(\mathrm{Sch})[\mathbb{L}^{1/2}]$ and are only interested in Donaldson-Thomas invariants. As

$$\underline{K}(\mathrm{Sch}_\mathbb{N})[\mathbb{L}^{1/2}]^{st} \cong K(\mathrm{Sch}_\mathbb{C})[\mathbb{L}^{-1/2}, (\mathbb{L}^n-1)^{-1} : n \in \mathbb{N}_*][[t]],$$

we end up with the following power series

$$A^{(m)}(t) := (\dim \circ p)_! \big(\phi_{\mathfrak{T}r(W)}(\mathcal{IC}_\mathfrak{M})\big) = \sum_{d \geqslant 0} \frac{\mathbb{L}^{(m+1)d^2/2}}{[\mathrm{GL}(d)]} t^d = \sum_{d \geqslant 0} \frac{\mathbb{L}^{(md^2+d)/2}}{\prod_{i=1}^d (\mathbb{L}^i - 1)} t^d.$$

Note that the series is also well-defined for $m \in \mathbb{Z}$.

Exercise 6.13. —   Prove the identity $A^{(m)}(\mathbb{L}t) - A^{(m)}(t) = \mathbb{L}^{(m+1)/2} t \, A^{(m)}(\mathbb{L}^m t)$ for all $m \in \mathbb{Z}$.

For $m \in \mathbb{N}$ we introduce the series
$$B^{(m)}(t) := A^{(m)}(\mathbb{L}t)/A^{(m)}(t) = \mathrm{Sym}\left(\sum_{d \geqslant 1} \mathbb{L}^{1/2}[\mathbb{P}^{d-1}] \, \mathrm{DT}(Q^{(m)})_d \, t^d\right),$$
where we used the properties of Sym and the fact that $\dim_!$ commutes with Sym. Moreover, due to the previous exercise
$$B^{(m)}(t) = 1 + \mathbb{L}^{(m+1)/2} t \prod_{i=0}^{m-1} B^{(m)}(\mathbb{L}^i t).$$

For $m = 0$ the empty product on the right hand side is 1, and we obtain $B^{(0)}(t) = 1 + \mathbb{L}^{1/2} t$ as well as
$$\mathrm{DT}(Q^{(0)})_d = \begin{cases} 1 & \text{for } d = 1, \\ 0 & \text{else.} \end{cases}$$
This is in fully agreement with Theorem 6.8 as $\mathcal{M}_d^{simp} = \mathrm{Spec}\,\mathbb{C}$ for $d = 1$ and $\mathcal{M}_d^{simp} = \emptyset$ else.

For $m = 1$, we get $B^{(1)}(t) = 1 + \mathbb{L}t B^{(1)}(t)$, and $B^{(1)}(t) = 1/(1 - \mathbb{L}t) = \sum_{d \in \mathbb{N}} \mathbb{L}^d t^d$ follows. Hence,
$$\mathrm{DT}(Q^{(1)})_d = \begin{cases} \mathbb{L}^{1/2} & \text{for } d = 1, \\ 0 & \text{else.} \end{cases}$$
Again, this is in fully agreement with Theorem 6.8 as $\mathcal{M}_d^{simp} = \mathbb{A}^1$ for $d = 1$ and $\mathcal{M}_d^{simp} = \emptyset$ else.

Solving the pseudo-algebraic equation for $m \geqslant 2$ is much more complicated, but the answer is given as follows. Note that $\mathbb{Z}/(d) =: C_d$ acts on the set
$$U_d := \{(a_1, \ldots, a_d) \in \mathbb{N}^d \mid a_1 + \ldots + a_d = (m-1)d\}$$
by cyclic permutation. We call $a = (a_1, \ldots, a_d)$ primitive if $\mathrm{Stab}_{C_d}(a) = \{0\}$, and almost primitive if a is primitive or $m \equiv 0(2), d \equiv 2(4)$ and
$$a = (b_1, \ldots, b_{d/2}, b_1, \ldots, b_{d/2})$$
for some primitive $(b_1, \ldots, b_{d/2})$. The subset
$$U_d^{ap} := \{a \in U_d \mid a \text{ is almost primitive}\}$$
is obviously stable under the C_d-action. Define $\deg(a) = \sum_{i=1}^d (d-i)a_i$ and
$$\deg(C_d \cdot a) = \min\{\deg(a') \mid a' \in C_d \cdot a\}.$$

THEOREM 6.14 ([32]). —   *Let $d \geqslant 1$ and $m \geqslant 2$. Then*
$$\dim(\mathcal{M}_d^{simp}) = (m-1)d^2 + 1$$
and
$$\mathrm{DT}(Q^{(m)})_d = \mathbb{L}^{\frac{(m-1)d^2+1}{2}} \frac{1 - \mathbb{L}^{-1}}{1 - \mathbb{L}^{-d}} \sum_{C_d \cdot a \in U_d^{ap}/C_d} \mathbb{L}^{-\deg(C_d \cdot a)}.$$

In particular, $\chi_c(\mathrm{DT}(Q^{(m)})_d) = \frac{(-1)^{(m-1)d^2+1}}{d} |U_d^{as}/C_d|$.

Exercise 6.15. — Taking the Euler characteristic of every coefficient of $B^{(m)}(t)$, we get a series $\beta^{(m)}(t) \in \mathbb{Z}[[t]]$ satisfying $\beta^{(m)}(t) = 1 + (-1)^{m+1}\beta^{(m)}(t)^m$, but also $\beta^{(m)}(t) = \prod_{d \geqslant 1}(1 - t^d)^{d\Omega_d^{(m)}}$ using the shorthand $\Omega_d^{(m)} := \chi_c(\mathrm{DT}(Q^{(m)})_d)$. Prove that

$$\Omega_d^{(2)} = \chi_c(\mathrm{DT}(Q^{(2)})_d) = \frac{1}{2d^2}\sum_{n|d}(-1)^{n+1}\mu(d/n)\binom{2n}{n}$$

for $d \geqslant 1$, where $\mu(-)$ denotes the Möbius function. Hint: Solve the quadratic equation for $\beta^{(2)}(t)$ and use the logarithmic derivative to prove the formula

$$\sum_{d \geqslant 1} 2d^2\Omega_d^{(2)}\frac{t^d}{1-t^d} = \sum_{n \geqslant 1}\Big(\sum_{d|n} 2d^2\Omega_d^{(2)}\Big)t^n = 1 - \frac{1}{\sqrt{1+4t}}.$$

The Taylor expansion of $1/\sqrt{1+4t}$ is $\sum_{n\in\mathbb{N}}(-1)^n\binom{2n}{n}t^n$. Finally, use the Möbius function to solve for $\Omega_d^{(2)}$.

One can show $\Omega_d^{(2)} = F(d)$ for $F(d)$ being the coefficients introduced in [15], equation (53).

6.2.2. *Dimension reduction.* Let Q be a quiver with potential W. Let ζ be a stability condition and $\mu \in [-\infty, \infty)$. Assume $\mathfrak{M}_d^{\zeta-ss} = \mathfrak{M}_d$ for all $d \in \Lambda_\mu$. As an example, we may take the King stability condition $\theta = 0$ and $\mu = 0$. Given a motivic theory R, the following arguments apply to $\phi^{R,st}$ with values in $R_{mon}^{gm,st}$. Without loss of generality, we consider the case $R = \underline{\mathrm{K}}_0(\mathrm{Sch})$, i.e. $\phi = \phi^{mot,st}$. As in the previous example, we are only interested in Donaldson-Thomas invariants. A subset $C \subset Q_1$ such that every cycle in W contains exactly one arrow in C is called a cut of W. Let $\mathbb{G}_m$ act on $X_d = \prod_{\alpha:i\to j}\mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ by multiplying a linear map corresponding to $\alpha \in C$ with $g \in \mathbb{G}_m$. By assumption, $\mathrm{Tr}(W)_d$ is homogeneous of degree one, and $\int_{X_d}\phi_{\mathrm{Tr}(W)_d}^{mot}$ is the class of $[X_d \xrightarrow{\mathrm{Tr}(W)_d} \mathbb{A}^1]$ in $\mathrm{K}_0(\mathrm{Sch}_\mathbb{C})_{mon}$ according to Theorem 5.28. Consider the projection $\tau_d : X_d \to Y_d$ with $Y_d = \prod_{C\not\ni\alpha:i\to j}\mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ which is a trivial vector bundle with fiber $F = \prod_{C\ni\alpha:i\to j}\mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$. As $\mathrm{Tr}(W)_d$ is linear along the fibers, we can think of it as being a section σ_d^W of the dual bundle with fiber $F^\vee = \prod_{C\ni\alpha:i\to j}\mathrm{Hom}(\mathbb{C}^{d_j}, \mathbb{C}^{d_i})$ using the trace pairing. Indeed, it maps a point $M = (M_\alpha)_{\alpha\notin C}$ to $(\frac{\partial W}{\partial \alpha}(M))_{\alpha\in C} \in F^\vee$.

Exercise 6.16. — Show that

$$[\tau_d^{-1}\{\sigma_d^W \neq 0\} \xrightarrow{\mathrm{Tr}(W)_d} \mathbb{A}^1] \in \mathrm{pr}_\mathbb{C}^*\mathrm{K}_0(\mathrm{Sch}_\mathbb{C}) \subset \mathrm{K}_{0,\mathbb{G}_m}(\mathrm{Sch}_{\mathbb{A}^1}).$$

Hint: Choose an open cover $\bigcup_{i\in I}U_i = \{\sigma_d^W \neq 0\} \subseteq Y_d$ such that $\tau_d : \tau_d^{-1}U_i \to U_i$ splits into the kernel of $\mathrm{Tr}(W)_d$ and a complement of rank one.

Using the exercise, we obtain

$$[X_d \xrightarrow{\mathrm{Tr}(W)_d} \mathbb{A}^1] = [\tau_d^{-1}\{\sigma_d^W = 0\} \xrightarrow{0} \mathbb{A}^1] = [F][\{\sigma_d^W = 0\}]$$
$$= \mathbb{L}^{\sum_{C\ni\alpha:i\to j}d_id_j}\Big[\Big\{ M \in Y_d \mid \frac{\partial W}{\partial \alpha}(M) = 0 \,\forall\, \alpha \in C\Big\}\Big].$$

in $K_0(\text{Sch}_{\mathbb{C}})_{mon}$. Therefore,

$$\text{Sym}\left(\frac{\text{DT}(Q,W)_\mu^\zeta}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}\right) = (\dim \circ p)_! \big(\phi_{\mathfrak{Tr}(W)}(\mathcal{IC}_{\mathfrak{M}_\mu})\big)$$

$$= \sum_{d \in \Lambda_\mu} \mathbb{L}^{(d,d)/2 + \sum_{C \ni \alpha : i \to j} d_i d_j} \frac{\left[\left\{M \in Y_d \mid \frac{\partial W}{\partial \alpha}(M) = 0 \,\forall\, \alpha \in C\right\}\right]}{[G_d]} t^d,$$

is actually in $K_0(\text{Sch}_{\mathbb{C}})[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}_*][[t_i \mid i \in Q_0]]$, where we consider

$$K_0(\text{Sch}_{\mathbb{C}})[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}_*][[t_i : i \in Q_0]] = \underline{K}(\text{Sch}_{\mathbb{N}^{Q_0}})[\mathbb{L}^{1/2}]^{st}$$

as a subgroup inside $\underline{K}(\text{Sch}_{\mathbb{N}^{Q_0}})_{mon}^{st}$. This reduction procedure is usually called dimension reduction, and $\left\{M \in Y_d \mid \frac{\partial W}{\partial \alpha}(M) = 0 \,\forall\, \alpha \in C\right\}/G_d$ is the stack of d-dimensional representations of the algebra $\mathbb{C}Q/(\alpha, \partial W/\partial \alpha \mid \alpha \in C)$.

6.2.3. *0-dimensional sheaves on a Calabi-Yau 3-fold.* Let us illustrate the concept of dimension reduction using the quiver $Q^{(3)}$ with one vertex and three loops x, y, z. The choice of the stability condition does not matter. We take the potential $W = [x,y]z = xyz - yxz$, and $\mathbb{C}Q/(\partial W/\partial \alpha \mid \alpha \in Q_1^{(3)}) = \mathbb{C}[x,y,z]$ follows. Hence, representations of this algebra are just 0-dimensional sheaves of finite length d on the Calabi-Yau 3-fold $\mathbb{A}^3$. We can take $C = \{z\}$ and obtain $\mathbb{C}Q/(z, \partial W/\partial z) = \mathbb{C}[x,y]$, and representations of this algebra are 0-dimensional sheaves of finite length d on the Calabi-Yau 2-fold $\mathbb{A}^2$. Using $(d,d)^2 = -2d^2$, we have to compute

$$\sum_{d \in \mathbb{N}} \frac{\left[\left\{M \in Y_d \mid \frac{\partial W}{\partial \alpha}(M) = 0 \,\forall\, \alpha \in C\right\}\right]}{[\text{GL}(d)]} t^d$$

which has already been done by Feit and Fine half a century ago in [9]. The answer is

$$\text{Sym}\left(\frac{1}{\mathbb{L} - 1} \sum_{d \geqslant 1} [\mathbb{A}^2] t^d\right),$$

and $DT(Q^{(3)}, W)_d = \mathbb{L}^{3/2} = \int_{\mathbb{A}^3} \mathcal{IC}_{\mathbb{A}^3}^{mot}$ follows for all $d \geqslant 1$ if we define $\mathcal{IC}_X^{mot} = \mathbb{L}^{-\dim(X)/2} \mathbb{1}_X \in \underline{K}_0(\text{Sch}_X)_{mon}^{st}$ for every smooth equidimensional variety X. This example has been generalized to arbitrary Calabi-Yau 3-folds by Behrend, Bryan, Szendrői.

THEOREM 6.17 ([2]). — *The motivic Donaldson-Thomas invariant for 0-dimensional sheaves of length $d \geqslant 1$ on a Calabi-Yau 3-fold X is given by*

$$\int_X \mathcal{IC}_X^{mot} = \mathbb{L}^{-3/2}[X] \in K_0(\text{Sch}_{\mathbb{C}})_{mon}^{st}.$$

Note that the Donaldson-Thomas function $\mathcal{DT}(Q^{(3)}, W)_d$ is supported on the moduli space $\text{Sym}^d(\mathbb{A}^3)$ of semisimple $\mathbb{C}[x,y,z]$ representations but the sublocus of simple representations is empty for $d > 1$. However, the space of simple $\mathbb{C}Q^{(3)}$-representations is nonempty even for $d > 1$ which is the reason why $DT(Q^{(3)}, W)_d \neq 0$. It seems plausible that $\mathcal{DT}(Q^{(3)}, W)_d$ is $\Delta_{d!}(\mathcal{IC}_{\mathbb{A}^3}^{mot})$, where $\Delta_d : \mathbb{A}^3 \hookrightarrow \text{Sym}^d(\mathbb{A}^3)$ is the diagonal embedding.

6.2.4. *The 1-loop quiver with potential.* Let us come back to the 1-loop quiver with $\mathbb{C}Q^{(1)} = \mathbb{C}[x]$ and choose an arbitrary potential $W \in \mathbb{C}[x]$. Representations of the algebra $\mathbb{C}Q/(dW/dx) = \mathbb{C}[x]/(dW/dx)$ can be interpreted as (0-dimensional) sheaves of length d on $\mathrm{Crit}(W) \subseteq \mathbb{A}^1$. Form the prime decomposition $dW/dx = c\prod_{i=1}^r (x - a_i)^{d_i - 1}$ for some $1 < d_i \in \mathbb{N}, c \in \mathbb{C}^\times, a_i \in \mathbb{C}$ satisfying $a_i \neq a_j$ for $i \neq j$. As before, the choice of a stability condition does not effect the Donaldson-Thomas function.

THEOREM 6.18 ([6]). — *The Donaldson-Thomas function $\mathcal{DT}(Q^{(1)}, W)_1$ computed using ϕ^{mot} is supported on $\mathrm{Crit}(W)$, i.e. is contained in*

$$\underline{\mathrm{K}}_0(\mathrm{Sch}_{\mathrm{Crit}(W)})^{st}_{mon} \cong \prod_{i=1}^r \mathrm{K}_0(\mathrm{Sch}_\mathbb{C})^{st}_{mon}.$$

Its "value" at a_i is given by

$$\mathbb{L}^{-1/2}[\mathbb{A}^1 \xrightarrow{z^{d_i}} \mathbb{A}^1] \in \mathrm{K}_0(\mathrm{Sch}_\mathbb{C})^{st}_{mon}.$$

Moreover, $\mathcal{DT}(Q^{(1)}, W)_d = 0$ for $d \neq 1$.

Exercise 6.19. — Prove this result using Theorem 6.10 and the explicit form of the vanishing cycle given by embedded resolutions as in Theorem 5.25.

The formula remains true if we replace ϕ^{mot} with ϕ^R due to Exercise 5.24.

6.2.5. *Sheaves on (-2)-curves.* Consider the following quiver Q
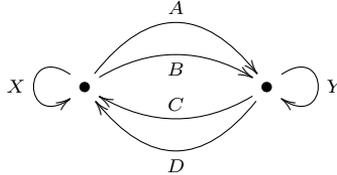
with potential

$$W_n = \frac{1}{n+1}\left(X^{n+1} - Y^{n+1}\right) - XCA + XDB + YAC - YBD \text{ for some } 0 < n \in \mathbb{N}.$$

The bounded derived category $D^b \mathrm{Jac}(Q, W_n)$ has also a geometric interpretation. For this consider the singular affine variety

$$X_n = \{x^2 + y^2 + (z + w^n)(z - w^n) = 0\} \subset \mathbb{A}^4$$

which is a local model for a 3-fold with an A_n-singularity. By blowing-up $\{x = z \pm w^d = 0\}$ in X_n we get two minimal resolutions $Y_n^\pm$ with an exceptional locus $C \cong \mathbb{P}^1$. The normal bundle of C inside $Y_n^\pm$ is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ for $n = 1$ and $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ for $n > 1$. In particular, $Y_n^\pm$ is a Calabi-Yau 3-fold which has actually a locally trivial fibration over C with fiber $\mathbb{A}^2$. For $d = 1$, $Y_1^\pm$ is isomorphic to the normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and known as the conifold resolution. For $d > 1$ this fibration is not a vector bundle as the transition functions are not linear. The resolutions Y_n^+ and Y_n^- are related via a flop over X_n, and also isomorphic to each other. Moreover,

$$D^b \mathrm{Jac}(Q, W_n) \cong D^b \mathrm{Coh}(Y_n^\pm)$$

and (complexes of) nilpotent representations on the left hand side correspond to (complexes of) sheaves supported on $C \subseteq Y_n^{\pm}$. We are only interested in these objects and choose any stability condition $\zeta = (\zeta_1, \zeta_2)$ with $\zeta_1 \nparallel \zeta_2$ in $\mathbb{R}^2 \cong \mathbb{C}$.

THEOREM 6.20 ([4]). — *The Donaldson-Thomas invariant*

$$\mathrm{DT}(Q, W_n)_{(d_1, d_2)}^{nilp} \in \mathrm{K}_0(\mathrm{Sch}_{\mathbb{C}})_{mon}^{st}$$

of nilpotent representations computed with respect to ϕ^{mot} is given by

$$\mathrm{DT}(Q, W_n)_{(d_1, d_2)}^{\mathrm{nilp}} = \begin{cases} \mathbb{L}^{-3/2}[C] = \mathbb{L}^{-3/2}(\mathbb{L}+1) & \text{if } 0 \neq d_1 = d_2, \\ \mathbb{L}^{-1/2}[\mathbb{A}^1 \xrightarrow{z^{n+1}} \mathbb{A}^1] & \text{if } |d_1 - d_2| = 1, \\ 0 & \text{else.} \end{cases}$$

Of course, the formula remains true if we replace ϕ^{mot} with ϕ^R due to Exercise 5.24. Note that $\mathrm{DT}(Q, W_n)_{(d_1, d_1)}^{nilp}$ is just "counting" 0-dimensional sheaves on $Y_n^{\pm}$ supported on C which explains the answer in view of Theorem 6.17. For $|d_1 - d_2| = 1$, there is just one simple nilpotent $\mathrm{Jac}(Q, W_n)$-representation V with $\mathrm{Ext}^1(V, V)$ being of dimension one. However, the obstruction of deforming V as a representation of $\mathrm{Jac}(Q, W_n)$ is controlled by some potential of the form z^{n+1} induced by W_n. Hence, we are back in the context of the previous example. The case of $n = 1$ has been studied earlier by Morrison, Mozgovoy, Nagao, Szendrői in [27].

6.3. **The Ringel-Hall algebra.** In the previous section we have seen some examples of Donaldson-Thomas invariants and functions. In all of these cases the choice of the stability condition did not play a crucial role. However, for a generic quiver this is not the case and the Donaldson-Thomas functions and invariants change as we vary the stability condition. There is a wall and chamber structure on the moduli space of stability conditions and these changes will only happen if we jump over a wall into a different chamber. There is, however, a formula — the wall-crossing formula — relating the Donaldson-Thomas functions and invariants for various stability conditions. Before we state and prove the formula, let us introduce some fundamental objects in Donaldson-Thomas theory.

Fix two dimension vectors d, d' and recall the following commutative diagram using the notation of section 2

$$\begin{array}{ccccc} \mathfrak{M}_d \times \mathfrak{M}_{d'} & \xleftarrow{\pi_1 \times \pi_3} & \mathfrak{M}_{d,d'} & \xrightarrow{\pi_2} & \mathfrak{M}_{d+d'} \\ {\scriptstyle p_d \times p_{d'}} \downarrow & & & & \downarrow {\scriptstyle p_{d+d'}} \\ \mathcal{M}_d^{ssimp} \times \mathcal{M}_{d'}^{ssimp} & & \xrightarrow{\quad \oplus \quad} & & \mathcal{M}_{d+d'}^{ssimp} \end{array}$$

with $\mathfrak{M}_{d,d'}$ being the stack of short exact sequences $0 \to V^{(1)} \to V^{(2)} \to V^{(3)} \to 0$ such that $\dim V^{(1)} = d, \dim V^{(3)} = d'$. The morphism π_i maps a sequence to its i-th entry. By taking the disjoint union over all dimension vectors, we end up with

$$\mathfrak{M} \times \mathfrak{M} \xleftarrow{\pi_1 \times \pi_3} \mathfrak{Exact} \xrightarrow{\pi_2} \mathfrak{M},$$

where $\mathfrak{Exact}$ denotes the stack of all short exact sequences.

DEFINITION 6.21. — For a given motivic theory R, we call the $R(\operatorname{Spec}\mathbb{C})$-module $R(\mathfrak{M})$ with the Ringel-Hall product

$$* : R(\mathfrak{M}) \otimes R(\mathfrak{M}) \xrightarrow{\boxtimes} R(\mathfrak{M}) \xrightarrow{(\pi_1 \times \pi_3)^*} R(\mathfrak{Exact}) \xrightarrow{\pi_2 !} R(\mathfrak{M})$$

the Ringel-Hall algebra of the quiver Q with respect to R.

LEMMA 6.22. — *The Ringel-Hall algebra is an associative algebra with unit.*

The proof is not very difficult but a nice exercise in dealing with successive extensions.

Exercise 6.23. — Let us fix three dimension vectors d, d', d''.

(1) Consider the following diagram, where the maps are given by mapping (successive) extensions to its subquotients or intermediate extensions and also by the identity on the factors $\mathfrak{M}$ not being part of an extension.

$$
\begin{array}{ccccc}
\mathfrak{M}_d \times \mathfrak{M}_{d'} \times \mathfrak{M}_{d''} & \longleftarrow & \mathfrak{M}_d \times \mathfrak{M}_{d',d''} & \longrightarrow & \mathfrak{M}_d \times \mathfrak{M}_{d'+d''} \\
\uparrow & & \uparrow & & \uparrow \\
\mathfrak{M}_{d,d'} \times \mathfrak{M}_{d''} & \longleftarrow & \mathfrak{M}_{d,d',d''} & \longrightarrow & \mathfrak{M}_{d,d'+d''} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}_{d+d'} \times \mathfrak{M}_{d''} & \longleftarrow & \mathfrak{M}_{d+d',d''} & \longrightarrow & \mathfrak{M}_{d+d'+d''}
\end{array}
$$

Show that the diagram commutes and that every square is cartesian.

(2) Use this diagram and the base change property of a motivic theory to prove associativity of the Ringel-Hall product.

(3) The zero representation induces a map $\operatorname{Spec}\mathbb{C} \xrightarrow{0} \mathfrak{M}$. Show that $1_0 := 0_!(1) \in R(\mathfrak{M})$ is a unit for the Ringel-Hall product.

Let us form the following cartesian product

$$
\begin{array}{ccc}
\mathfrak{M}^{\zeta-ss}_{\mu,\mu'} & \lhook\joinrel\longrightarrow \mathfrak{Exact} \xrightarrow{\pi_2} \mathfrak{M} \\
\downarrow & \qquad\quad \downarrow {\scriptstyle \pi_1 \times \pi_3} \\
\mathfrak{M}^{\zeta-ss}_{\mu} \times \mathfrak{M}^{\zeta-ss}_{\mu'} & \lhook\joinrel\longrightarrow \mathfrak{M} \times \mathfrak{M}.
\end{array}
$$

If $\mu > \mu'$, the composition $\mathfrak{M}^{\zeta-ss}_{\mu,\mu'} \to \mathfrak{M}$ is an isomorphism onto the image which is the substack of $\mathfrak{M}$ consisting of all representations whose Harder-Narasimhan filtration has only one subquotient in $\mathfrak{M}^{\zeta-ss}_{\mu}$ and another one in $\mathfrak{M}^{\zeta-ss}_{\mu'}$. Indeed, the functoriality of the Harder-Narasimhan filtration ensures that taking the Harder-Narasimhan filtration provides an inverse morphism to $\mathfrak{M}^{\zeta-ss}_{\mu,\mu'} \to \mathfrak{M}$. Fixing another slope μ'' with $\mu' > \mu''$ we continue this way and take the fiber product

$$
\begin{array}{ccc}
\mathfrak{M}^{\zeta-ss}_{\mu,\mu',\mu''} & \lhook\joinrel\longrightarrow \mathfrak{Exact} \xrightarrow{\pi_2} \mathfrak{M} \\
\downarrow & \qquad\quad \downarrow {\scriptstyle \pi_1 \times \pi_3} \\
\mathfrak{M}^{\zeta-ss}_{\mu,\mu'} \times \mathfrak{M}^{\zeta-ss}_{\mu''} & \lhook\joinrel\longrightarrow \mathfrak{M} \times \mathfrak{M}.
\end{array}
$$

which can be identified with the substack of representations having a Harder-Narasimhan filtration with subquotients in $\mathfrak{M}_\mu^{\zeta-ss}, \mathfrak{M}_{\mu'}^{\zeta-ss}, \mathfrak{M}_{\mu''}^{\zeta-ss}$. As every quiver representation has a unique Harder-Narasimhan filtration, we obtain a locally finite stratification of $\mathfrak{M}\backslash\{0\}$ with strata $\mathfrak{M}_{\mu_1,\ldots,\mu_r}^{\zeta-ss}\backslash\{0\} \hookrightarrow \mathfrak{M}\backslash\{0\}$, where $\mathfrak{M}_{\mu_1,\ldots,\mu_r}^{\zeta-ss}$ is defined as above by means of $r-1$ fiber products for every strictly decreasing sequence $\mu_1 > \ldots > \mu_r$ in $(-\infty, +\infty]$ of length r. Using the notation

$$\delta_{\mu_1,\ldots,\mu_r}^\zeta := (\mathfrak{M}_{\mu_1,\ldots,\mu_r}^{\zeta-ss} \hookrightarrow \mathfrak{M})_!(\mathbb{1}_{\mathfrak{M}_{\mu_1,\ldots,\mu_r}^{\zeta-ss}}) \in R(\mathfrak{M}),$$

this stratification can be written as

$$\mathbb{1}_{\mathfrak{M}} = 1_0 + \sum_{\substack{0 < r \in \mathbb{N} \\ \mu_1 > \ldots > \mu_r}} (\delta_{\mu_1,\ldots,\mu_r}^\zeta - 1_0).$$

Exercise 6.24. — Prove the formula $\delta_{\mu_1,\ldots,\mu_r}^\zeta = \delta_{\mu_1}^\zeta * \ldots * \delta_{\mu_r}^\zeta$ for all sequences $\mu_1 > \ldots > \mu_r$ of real numbers.

Applying the formula proven in the exercise we can rewrite the infinite sum as an infinite product

$$\mathbb{1}_{\mathfrak{M}} = \prod_{\mu \searrow}^* \delta_\mu^\zeta \tag{6.2}$$

which is well-defined as for every dimension vector only finitely many factors contribute. Note that the (infinite) Ringel-Hall product has to be taken in decreasing order of the slopes. If ζ' is another stability condition, we conclude the formula

$$\prod_{\mu \searrow}^* \delta_\mu^\zeta = \prod_{\mu \searrow}^* \delta_\mu^{\zeta'} \tag{6.3}$$

which relates elements in $R(\mathfrak{M})$ defined by means of two different stability conditions ζ and ζ'. In order to obtain a similar formula for Donaldson-Thomas functions, we need to related the Ringel-Hall algebra with corresponding objects on the coarse moduli space which will be the topic of the next subsection.

6.4. Integration map.

As the Donaldson-Thomas function was an object defined on $\mathcal{M}^{\zeta-ss}$, we cannot compare Donaldson-Thomas functions taken with respect to different stability conditions since $\mathcal{M}^{\zeta-ss}$ might change. To make them comparable, we need to push them down along $q^\zeta : \mathcal{M}^{\zeta-ss} \to \mathcal{M}^{ssimp}$. The map q^ζ can be understood in many ways. Firstly, q^ζ maps a ζ-polystable representation to the direct sum of its simple factors taken in the category of all quiver representations. Notice that a ζ-stable representation of slope μ is only simple in the subcategory of ζ-semistable representations of slope μ but it might have subrepresentations of smaller slopes. Secondly, the composition $\mathfrak{M}^{\zeta-ss} \hookrightarrow \mathfrak{M} \to \mathcal{M}^{ssimp}$ must factorize (uniquely) as $\mathfrak{M}^{\zeta-ss} \to \mathcal{M}^{\zeta-ss} \xrightarrow{q^\zeta} \mathcal{M}^{ssimp}$ because the coarse moduli space of $\mathfrak{M}^{\zeta-ss}$ is (per definition) an initial object in the category of all morphisms $\mathfrak{M}^{\zeta-ss} \to X$ to schemes X. Thirdly, due to Prop. 3.5 in [29], the morphism $q_d^\zeta : \mathcal{M}_d^{\zeta-ss} \to \mathcal{M}_d^{ssimp}$ is projective and surjective, at least for ζ of the special form $\zeta_i = -\theta_i + \sqrt{-1}$ with $\theta_i \in \mathbb{Z}$ for all $i \in Q_0$. This statement can be generalized to arbitrary ζ since due to [25] we can replace ζ with another ζ' (depending on d) of this special form

without changing $\mathcal{M}_d^{\zeta-ss}$. As $\mathcal{M}_d^{ssimp}$ is by construction the spectrum of the G_d-invariant functions on X_d, hence affine, we conclude that $q_d^\zeta : \mathcal{M}_d^{\zeta-ss} \to \mathcal{M}_d^{ssimp}$ is just the affinization of $\mathcal{M}_d^{\zeta-ss}$, i.e. the map to the spectrum of the ring of global regular functions on $\mathcal{M}_d^{\zeta-ss}$.

Fix a stacky vanishing cycle ϕ, and use the fact that $q_!^\zeta$ commutes with Sym, that the open embedding $j : \mathfrak{M}_\mu^{\zeta-ss} \hookrightarrow \mathfrak{M}$ is smooth, and the projection formula to conclude

$$\mathrm{Sym}\left(\frac{q_!^\zeta \mathcal{DT}(Q,W)_\mu^\zeta}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}\right) = q_!^\zeta \mathrm{Sym}\left(\frac{\mathcal{DT}(Q,W)_\mu^\zeta}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}\right)$$
$$= (q^\zeta \circ p^\zeta)_! \left(\phi_{\mathfrak{Tr}(W)^\zeta}(\mathcal{IC}_{\mathfrak{M}_\mu^{\zeta-ss}})\right)$$
$$= (p \circ j)_! \left(\phi_{\mathfrak{Tr}(W)^\zeta}(j^* \mathcal{IC}_{\mathfrak{M}})\right)$$
$$= p_! j_! \left(j^* \phi_{\mathfrak{Tr}(W)}(\mathcal{IC}_{\mathfrak{M}})\right)$$
$$= p_! \left(\delta_\mu^\zeta \cap \phi_{\mathfrak{Tr}(W)}(\mathcal{IC}_{\mathfrak{M}})\right).$$

DEFINITION 6.25. — The $R(\mathrm{Spec}\,\mathbb{C})$-linear map

$$I^W : R(\mathfrak{M}) \ni a \mapsto p_! \left(a \cap \phi_{\mathfrak{Tr}(W)}(\mathcal{IC}_{\mathfrak{M}})\right) \in R(\mathcal{M}^{ssimp})$$

is called integration map.

Hence, we have proven

$$I^W(\delta_\mu^\zeta) = \mathrm{Sym}\left(\frac{q_!^\zeta \mathcal{DT}(Q,W)_\mu^\zeta}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}\right).$$

We wish to apply I^W to equation (6.2) or (6.3) to obtain a wall-crossing formula for the Donaldson-Thomas functions $\underline{\mathcal{DT}}(Q,W)_\mu^\zeta := q_!^\zeta \mathcal{DT}(Q,W)_\mu^\zeta$. Unfortunately, I^W will not be an $R(\mathrm{Spec}\,\mathbb{C})$-algebra homomorphism from the Ringel-Hall algebra $(R(\mathfrak{M}), *)$ to $R(\mathcal{M}^{ssimp})$ with the convolution product.

DEFINITION 6.26. — Define the "quantum" or deformed convolution product on the abelian group $R(\mathcal{M}^{ssimp})$ by means of

$$(a_d)_{d \in \mathbb{N}^{Q_0}} * (b_{d'})_{d' \in \mathbb{N}^{Q_0}} := \left(\sum_{d+d'=d''} \mathbb{L}^{\langle d,d' \rangle/2} a_d b_{d'}\right)_{d'' \in \mathbb{N}^{Q_0}}$$

and similarly on $R(\mathbb{N}^{Q_0})$.

As $\dim : \mathcal{M}^{ssimp} \to \mathbb{N}^{Q_0}$ is a monoid homomorphisms, it will preserve the convolution and, hence, also the deformed convolution product. The main result about the integration map was essentially proven by Reineke in [28] for $W = 0$ and sketched in a much broader context by Kontsevich and Soibelman in [19] for general 3-Calabi-Yau categories.[21] Following the arguments of Kontsevich and Soibelman given in [21], a rigorous proof for representations of a quiver with potential can also be found in [5].

THEOREM 6.27. — The map $I^W : (R(\mathfrak{M}), *) \to (R(\mathcal{M}^{ssimp}), *)$ is a homomorphism of $R(\mathrm{Spec}\,\mathbb{C})$-algebras.

---

[21]Notice that the abelian category of finite dimensional representations of $\mathrm{Jac}(Q,W)$ considered in this paper is a heart of a bounded t-structure on a 3-Calabi-Yau category. Kontsevich and Soibelman consider the more general situation of all objects in this 3-Calabi-Yau category.

6.5. **The wall-crossing identity.** Let us assume the conditions of Theorem 6.27. By applying the integration map I^W to the equations (6.2) and (6.3), we finally get the wall-crossing identity

$$I^W(\mathbb{1}_{\mathfrak{M}}) = \prod_{\mu \searrow}^{*} \mathrm{Sym}\left(\frac{\mathcal{DT}(Q,W)_\mu^\zeta}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right) = \prod_{\mu \searrow}^{*} \mathrm{Sym}\left(\frac{\mathcal{DT}(Q,W)_\mu^{\zeta'}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right)$$

relating the Donaldson-Thomas functions $\mathcal{DT}(Q,W)^\zeta$ and $\mathcal{DT}(Q,W)^{\zeta'}$ of two stability conditions ζ and ζ'. Since $\dim_!$ commutes with the deformed convolution product, we obtain the same formula for the Donaldson-Thomas invariants

$$\dim_! I^W(\mathbb{1}_{\mathfrak{M}}) = \prod_{\mu \searrow}^{*} \mathrm{Sym}\left(\frac{\mathrm{DT}(Q,W)_\mu^\zeta}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right) = \prod_{\mu \searrow}^{*} \mathrm{Sym}\left(\frac{\mathrm{DT}(Q,W)_\mu^{\zeta'}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right).$$

Let us illustrate this formula with an example.

Example 6.28 (cf. Example 2.13). — Consider the A_2-quiver $Q : \bullet_1 \to \bullet_2$ with potential $W = 0$ and the canonical vanishing cycle of $\underline{K}_0(\mathrm{Sch})[\mathbb{L}^{1/2}]^{st}$.

Exercise 6.28.4. — Show that every representation $V_1 \xrightarrow{M} V_2$ of Q is a direct sum of copies of $S_1 = (\mathbb{C} \xrightarrow{0} 0)$, $S_2 = (0 \xrightarrow{0} \mathbb{C})$ and $S_{12} = (\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C})$.

Fix a stability condition $\zeta = (\zeta_1, \zeta_2)$ satisfying $\arg(\zeta_1) < \arg(\zeta_2)$. Given a ζ-semistable representation $V \cong S_1^{m_1} \oplus S_{12}^{m_{12}} \oplus S_2^{m_2}$ with $d_1 = m_1 + m_{12}$ and $d_2 = m_2 + m_{12}$, two of the multiplicities m_1, m_{12}, m_2 must be zero. Moreover, m_{12} must be zero, too, since $S_2 \hookrightarrow S_{12}$ destabilizes S_{12} and the latter cannot be (semi)stable. Thus, $V \cong S_1^{d_1}$ or $V \cong S_2^{d_2}$. In particular, the category of representations of dimension vector $(d_1, 0)$ respectively $(0, d_2)$ is isomorphic to the category of representations of the quiver $Q^{(0)}$ with one vertex and no loop. Using $R(\mathbb{N}^{Q_0}) \cong R[[t_1, t_2]]$ we, therefore, obtain

$$\dim_! I^W(\mathbb{1}_{\mathfrak{M}}) = A^{(0)}(t_2) * A^{(0)}(t_1) = \mathrm{Sym}\left(\frac{t_2}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right) * \mathrm{Sym}\left(\frac{t_1}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right).$$

On the other hand, if we assume $\arg(\zeta_1) > \arg(\zeta_2)$, the representation S_{12} is ζ-stable, and $V \cong S_{12}^{d_1}$ is another class of semistable objects. Thus,

$$\dim_! I^W(\mathbb{1}_{\mathfrak{M}}) = A^{(0)}(t_1) * A^{(0)}(t_1 t_2) * A^{(0)}(t_2)$$

$$= \mathrm{Sym}\left(\frac{t_1}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right) * \mathrm{Sym}\left(\frac{t_1 t_2}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right) * \mathrm{Sym}\left(\frac{t_2}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right),$$

and the so-called quantum dilogarithm identity

$$A^{(0)}(t_2) * A^{(0)}(t_1) = A^{(0)}(t_1) * A^{(0)}(t_1 t_2) * A^{(0)}(t_2)$$

follows. Let us also consider the case $\arg(\zeta_1) = \arg(\zeta_2)$. Then, all representations are semistable with S_1, S_2 being the only stable representations, and

$$\dim_! I_W(\mathbb{1}_{\mathfrak{M}}) = \mathrm{Sym}\left(\frac{\sum_{(d_1,d_2)\neq(0,0)} \mathrm{DT}(Q)_{(d_1,d_2)}^\zeta \, t_1^{d_1} t_2^{d_2}}{\mathbb{L}^{1/2}-\mathbb{L}^{-1/2}}\right)$$

allows the computation of the Donaldson-Thomas invariants as the left hand side of the equation is already known by the previous two cases. For example, comparing

the coefficients of $t_1 t_2$ yields

$$\frac{\mathbb{L}^{1/2} - 1}{(\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})^2} = \frac{\mathrm{DT}(Q)^\zeta_{(1,1)}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}},$$

and $\mathrm{DT}(Q)^\zeta_{(1,1)} = 1/(1 + \mathbb{L}^{-1/2})$ follows. Note that $\mathrm{DT}(Q)^\zeta_{(1,1)} \in \mathrm{K}_0(\mathrm{Sch}_\mathbb{C})[\mathbb{L}^{1/2}]^{st}$ cannot be lifted under the map $\mathrm{K}_0(\mathrm{Sch}_\mathbb{C})[\mathbb{L}^{-1/2}] \to \mathrm{K}_0(\mathrm{Sch}_\mathbb{C})[\mathbb{L}^{1/2}]^{st}$ which does not contradict Theorem 6.8 as ζ is not generic. In particular, the assumption of being generic cannot be dropped in Theorem 6.8 and 6.10.

Let us finally say some words about the Donaldson-Thomas functions $\mathcal{DT}(Q)^\zeta_d$. For an arbitrary stability condition ζ there is at most one polystable object of given dimension vector d as the previous discussion shows. In particular, $\mathcal{M}^{\zeta-ss}_d = \mathcal{M}^{ssimp}_d = \mathrm{Spec}\,\mathbb{C}$ or $\mathcal{M}^{\zeta-ss}_d = \emptyset$, and $\mathcal{DT}(Q)^\zeta_d = \mathrm{DT}(Q)^\zeta_d$ follows for all $d \in \mathbb{N}^{Q_0}$.

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Sven MEINHARDT
School of Mathematics and Statistics, University of Sheffield, Western Bank, Sheffield S10
2TN, United Kingdom
s.meinhardt@sheffield.ac.uk