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THE DRINFELD-GRINBERG-KAZHDAN THEOREM FOR FORMAL SCHEMES AND SINGULARITY THEORY

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Abstract. Let k be a field. In this article, we provide an extended version of the Drinfeld-Grinberg-Kazhdan Theorem in the context of formal geometry. We prove that, for every formal scheme V topologically of finite type over $\mathrm{Spf}(k[[T]])$, for every non-singular arc $\gamma \in \mathscr{L}_\infty(V)(k)$, there exists an affine noetherian adic formal k-scheme $\mathscr{S}$ and an isomorphism of formal k-schemes

$$\mathscr{L}_\infty(V)_\gamma \cong \mathscr{S} \times_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]).$$

We emphasize the fact that the proof is constructive and, when V is the completion of an affine algebraic k-variety, effectively implementable. Besides, we derive some properties of such an isomorphism in the direction of singularity theory.

1. Introduction

1.1. In [11, Theorem 0.1], V. Drinfeld proved the following statement (which was conjectured, under a weaker form, by V. Drinfeld himself in private communication; see [17, Introduction]):

Theorem 1.1. — *Let k be a field. Let V be a k-variety, with no connected component isomorphic to* $\mathrm{Spec}(k)$. *Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a rational point of the associated arc scheme, not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$. If $\mathscr{L}_\infty(V)_\gamma$ denotes the formal neighborhood of the k-scheme $\mathscr{L}_\infty(V)$ at the point γ, there exists an affine k-scheme S of finite type, with $s \in S(k)$, and an isomorphism of formal k-schemes:*

$$\mathscr{L}_\infty(V)_\gamma \cong S_s \times_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \tag{1.1}$$

This statement generalizes to every field k a previous version, due to M. Grinberg and D. Kazhdan, only proved for subfields of $\mathbf{C}$ (see [17]). See also [8] for an interesting point of view on this question and related problems. Very recently, this statement has been used in various directions, see [7, 6].

1.2. In this article we prove the following version of Theorem 1.1 (see Theorems 4.1 and 4.2 for a more precise statement):

Theorem 1.2. — *Let k be a field and $R \in \{k, k[[T]]\}$. Let V be either an R-scheme of finite type, or a formal $k[[T]]$-scheme topologically of finite type. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})(k)$. There exist an affine noetherian adic formal k-scheme $\mathscr{S}$ and an isomorphism of formal k-schemes:*

$$\theta_k(V)\colon \mathscr{L}_\infty(V)_\gamma \to \mathscr{S} \times_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \tag{1.2}$$

Moreover, when V is an R-scheme of finite type, there exist an affine k-scheme of finite type S and $s \in S(k)$ such that the formal k-scheme $\mathscr{S}$ is isomorphic to S_s. The isomorphism $\theta_k(V)$ has the following properties:

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1) *For every separable field extension K of k, it is compatible with base change to K.*
2) *For every integer $n \in \mathbf{N}$, if $\pi_n^\infty \colon \mathscr{L}_\infty(V) \to \mathscr{L}_n(V)$ is the truncation morphism of level n, then the induced morphism of formal k-schemes*

$$\hat{\pi}_n^\infty \colon \mathscr{L}_\infty(V)_\gamma \to \mathscr{L}_n(V)_{\pi_n^\infty(\gamma)}$$

factorizes through $S_s \times_k \mathrm{Spf}(k[[T_0, \ldots, T_n]])$.

Our proof of the first assertion of Theorem 1.2 follows the original strategy of V. Drinfeld's preprint, but the arguments we use are suited to our general context; they elaborate those used in [11] (see §3 and §4, in particular §4.5). This refinement allows us to complete the original statement by properties 1) and 2) of Theorem 1.2. Besides, when V is a k-scheme of finite type, our proof shows that the above procedure can be implemented as an effective algorithm taking as its input a suitable truncation of the arc γ and producing as its output a pointed affine k-scheme (S, s) such that $\mathscr{S} = S_s$ realizes isomorphism (1.2). A SAGE code of this algorithm can be found in [4] (unpublished). In §6, we provide various computations obtained thanks to this algorithmic approach.

1.3. Different works have proved that the arc scheme $\mathscr{L}_\infty(V)$ carries a part of the information on the singularities of the variety V. From our point of view, an important and natural question is to understand what amount of the information on singularities is contained in isomorphism (1.1). Motivated by this idea, we derive some results which provide the basic elements for an exegesis with respect to singularity theory of the Drinfeld-Grinberg-Kazhdan Theorem.

∘ If $\mathscr{N}$ is the nilradical of the ring $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$, Proposition 8.1 implies that there exists a smallest integer m such that $\mathscr{N}^m = 0$, that we denote by $m_\gamma(V)$. We introduce the *absolute nilpotency index* of (V, v) by the formula:

$$m_{\mathrm{abs}}(V, v) := \inf(m_\gamma(V)) \in \mathbf{N},$$

where the infimum is taken over the set of the arcs on V, with base-point v, which are not contained in $\mathscr{L}_\infty(V_{\mathrm{sing}})$ (see §8). Proposition 8.4 shows that this integer is a formal invariant of the singularity (V, v). Besides, it seems to us very plausible that this integer can be linked to "classical" numerical invariants of singularity theory.

∘ Thanks to O. Gabber's Cancellation Theorem (see §7, Theorem 7.1), which is, to the best of our knowledge, *new*, we show, in corollary 7.4, that the *minimal* part of the formal k-schemes $\mathscr{S}$ is independent (up to isomorphism) of the possible choice of isomorphism (1.2). We will call this minimal part the *minimal (finite dimensional) formal model* of the pair $(\mathscr{L}_\infty(V), \gamma)$.

1.4. In the end, this work addresses natural questions in the field. We prove that isomorphism (1.2) depends in general on the choice of the involved arc γ (see §6). We examinate whether isomorphism (1.2) may factorize through the formal neighborhood of a truncation of the involved arc (see remark 4.8). Besides, we use the general ideas of the proof of Theorem 1.2 to state a formal analog of the Denef-Loeser Fibration Theorem (see §5.1, Proposition 5.1), and a global version of Theorem 1.2 (see §5.2, Proposition 5.4).

1.5. **Conventions, notation.** ∘ In this article, k is a field of arbitrary characteristic (unless explicitly stated otherwise); $k[[T]]$ is the ring of power series over the field k. For the simplicity of the presentation, we use the notation $R \in \{k, k[[T]]\}$ to say $R = k$ or $R = k[[T]]$. For every integer $n \in \mathbf{N}$, we set $R_n := k[T]/\langle T^{n+1}\rangle$.

∘ If A, B are local k-algebras, the set of *local* morphisms from A to B is denoted by $\mathrm{Hom}_k^{\mathrm{loc}}(A,B)$. If A, B are two *admissible* local k-algebras (i.e., linearly topologized local rings which are separated and complete, and endowed with a structure of k-algebra), a morphism from A to B is defined as a morphism of local k-algebras supposed to be continuous for the involved topologies (see [18, 0/§7] for details). The set of morphisms between the admissible k-algebras A, B is denoted by $\mathrm{Hom}_k^{\mathrm{loct}}(A,B)$. An element of $\mathrm{Hom}_k^{\mathrm{loct}}(A,B)$ is called an *admissible morphism*. A *test-ring* A is a local k-algebra, with nilpotent maximal ideal $\mathfrak{M}_A$ and residue field k-isomorphic to k. A test-ring is in particular an admissible k-algebras, and the morphisms between test-rings are admissible morphisms. These data form a category that we denote by $\mathfrak{Tes}$. One says that a local noetherian complete k-algebra A is *cancellable* (as such a k-algebra) if there exists a local noetherian complete k-algebra B such that A is isomorphic to $B[[T]]$ (as such k-algebras). Thus, with every local noetherian complete k-algebra A, whose residue field is k-isomorphic to k, one can associate a local noetherian complete k-algebra $A_{\min}$, which is *non-cancellable*, unique (up to isomorphism) by Theorem 7.1, and satisfies $A \cong A_{\min}[[T_1, \ldots, T_n]]$ for some integer $n \in \mathbf{N}$.

∘ An *R-variety* is a scheme of finite type over $\mathrm{Spec}(R)$. A *formal tft $k[[T]]$-scheme* is a formal scheme topologically of finite type over $\mathrm{Spf}(k[[T]])$. For the simplicity of the presentation, we adopt the following convention in order to formally unify the treatment of various geometric contexts. We say that V is a *R-space* if V is either a R-variety (recall that $R = k$ or $R = k[[T]]$) according to the context) or a formal tft $k[[T]]$-scheme. If V is a R-variety, let us note that $\hat{V} := \varinjlim_n V \times_R \mathrm{Spec}(R_n)$ is a formal tft $k[[T]]$-scheme (recall that $R_n := k[T]/\langle T^{n+1}\rangle$); it is a formal tft $k[[T]]$-scheme called *the formal neighborhood of V along its special fiber*. We denote by $\mathbf{A}_R^n$ the R-space defined by $\mathbf{A}_R^n = \mathrm{Spec}(R[X_1, \ldots, X_n])$ in the algebraic case and $\mathbf{A}_{k[[T]]}^n = \mathrm{Spf}(k[[T]]\{X_1, \ldots, X_n\})$ in the formal case. If $V = \mathrm{Spec}(R[X_1, \ldots, X_n]/I)$, then we observe that $\hat{V} = \mathrm{Spf}(k[[T]]\{X_1, \ldots, X_n\}/I)$. In this article, we shall only consider those R-spaces having no connected components with relative dimension zero.

∘ If V is a R-space, we denote by $\mathscr{L}_n(V)$ the restriction *à la* Weil of the R_n-scheme $V \times_R \mathrm{Spec}(R_n)$ with respect to the morphism of k-algebras $k \hookrightarrow R_n$. This object exists, under this assumption, in the category of k-schemes; it is a k-scheme of finite type which we call the *n-jet scheme* of V. We introduce now the *arc scheme* associated with V by $\mathscr{L}_\infty(V) := \varprojlim_n (\mathscr{L}_n(V))$ which exists in the category of k-schemes. In the end, let us note that, if V is a R-variety, we have an isomorphism of k-varieties $\mathscr{L}_\infty(V) \cong \mathscr{L}_\infty(\hat{V})$. (See for example [29] for a unified treatment of the question in the context of algebraic and formal geometry). We always assume that $\mathscr{L}_\infty(V)(k) \neq \varnothing$. For every integer $n \in \mathbf{N}$, the canonical morphism of k-schemes $\pi_n^\infty : \mathscr{L}_\infty(V) \to \mathscr{L}_n(V)$ is called the *truncation morphism* of level n. Let A be a k-algebra. If A is assumed to be local, or the R-space V is assumed to be affine, there exists a natural bijection

$$\mathrm{Hom}_k(\mathrm{Spec}(A), \mathscr{L}_\infty(V)) \cong \mathrm{Hom}_R(\mathrm{Spec}(A[[T]]), V). \tag{1.3}$$

A point $\gamma \in \mathscr{L}_\infty(V)$ is called an *arc* on the variety V. We denote by $\mathscr{L}_\infty^\circ(V)$ the open subscheme of $\mathscr{L}_\infty(V)$ defined to be $\mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(V_{\mathrm{sing}})$.

$\circ$ Following the terminology introduced in [7], we call *a finite dimensional formal model* of the pair $(\mathscr{L}_\infty(V), \gamma)$, with $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$, every affine noetherian adic formal k-scheme realizing isomorphism (4.1).

$\circ$ If A is a ring and d is a non-negative integer, we denote by $A[T]_{\leqslant d}$ the set of polynomials in the indeterminate T with coefficients in A and with degree less than or equal to d.

2. General results

In this section, we establish some general results on formal neighborhoods of arc schemes.

2.1. **Functor of points.** We denote by $\mathfrak{Lacp}$ the category whose objects are the admissible local k-algebras, with residue field k-isomorphic to k, and which are isomorphic to $\mathfrak{M}$-adic completion of local k-algebras, and whose morphisms are admissible morphisms. We observe that the category $\mathfrak{Tes}$ also is a full subcategory of the category $\mathfrak{Lacp}$, since, for every test-ring $(A, \mathfrak{M}_A)$, we have $A = \varprojlim_n (A/\mathfrak{M}_A^n)$. Let us state a seemingly standard observation.

Let $\widehat{\mathcal{O}}$ be an object of the category $\mathfrak{Lacp}$. If $\widehat{\mathfrak{Tes}}$ is the category of pre-cosheaves on the category $\mathfrak{Tes}$ (i.e., covariant functors from the category $\mathfrak{Tes}$ to the category of sets), we construct a functor

$$F : \mathfrak{Lacp} \to \widehat{\mathfrak{Tes}}$$

by

$$\widehat{\mathcal{O}} \mapsto F_{\widehat{\mathcal{O}}} := \mathrm{Hom}_k^{\mathrm{loct}}(\widehat{\mathcal{O}}, \cdot).$$

By the Yoneda Lemma, the functor $\mathrm{Hom}_k^{\mathrm{loct}}(\widehat{\mathcal{O}}, \cdot)$ defined on the category $\mathfrak{Lacp}$ determines the ring $\widehat{\mathcal{O}}$ in $\mathfrak{Lacp}$. Let us emphasize that, here, $F_{\widehat{\mathcal{O}}}$ is the restriction of the latter functor to the category $\mathfrak{Tes}$.

Let $(A, \mathfrak{M}_A)$ be a test-ring (see definition in §1.5). By definition, the ring A is a $\mathfrak{M}_A$-adically complete local k-algebra. For every local k-algebra $\mathcal{O}$ such that $\widehat{\mathcal{O}} = \varprojlim_n \mathcal{O}/\mathfrak{M}_{\mathcal{O}}^n$, then we deduce:

$$\mathrm{Hom}_k^{\mathrm{loct}}(\widehat{\mathcal{O}}, A) \quad \cong \quad \mathrm{Hom}_k^{\mathrm{loc}}(\mathcal{O}, A). \tag{2.1}$$

Then, $F_{\widehat{\mathcal{O}}} \cong \mathrm{Hom}_k^{\mathrm{loc}}(\mathcal{O}, \cdot)$ on the category $\mathfrak{Tes}$. Furthermore, for every $\hat{B} = \varprojlim_n B/\mathfrak{M}_B^n \in \mathfrak{Lacp}$, we have by definition:

$$\begin{aligned} \mathrm{Hom}_k^{\mathrm{loct}}(\widehat{\mathcal{O}}, B) &= \mathrm{Hom}_k^{\mathrm{loc}}(\mathcal{O}, \hat{B}) \\ &\cong \varprojlim_n \mathrm{Hom}_k^{\mathrm{loc}}(\mathcal{O}, B/\mathfrak{M}_B^n). \end{aligned} \tag{2.2}$$

Let us note that, for every integer $n \in \mathbf{N}^*$, the local k-algebra $B/\mathfrak{M}_B^n$ is a test-ring. It follows from the Yoneda Lemma that the functor F is fully faithful. In the context of the present work, we will use this remark under the following formulation:

Observation 2.1. — Let $\widehat{\mathcal{O}} = \varprojlim_n \mathcal{O}/\mathfrak{M}_{\mathcal{O}}^n$ and $\widehat{\mathcal{O}'} = \varprojlim_n \mathcal{O}'/\mathfrak{M}_{\mathcal{O}'}^n$ be two admissible local k-algebras in the category $\mathfrak{Lacp}$. Then, we have the following property:

(1) A morphism of functors $F_{\widehat{\mathcal{O}'}} \to F_{\widehat{\mathcal{O}}}$ gives rise to a unique morphism of admissible local k-algebras $\widehat{\mathcal{O}'} \to \widehat{\mathcal{O}}$;

(2) If $\theta : \widehat{\mathcal{O}'} \to \widehat{\mathcal{O}}$ is a morphism of admissible local k-algebras, then the induced morphism of functors $F_{\widehat{\mathcal{O}'}} \to F_{\widehat{\mathcal{O}}}$ is an isomorphism if and only if θ is an isomorphism.

2.2. **An important identification.** Let V be a R-space. Let $\gamma \in \mathscr{L}_\infty(V)(k)$. Let us denote by $\mathfrak{M}_\gamma$ the maximal ideal of the local k-algebra $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$. Then, in the sense of observation 2.1, the formal k-scheme $\mathscr{L}_\infty(V)_\gamma$ is uniquely determined by the functor $F_{\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}}$. Let A be a test-ring. Let $\gamma_A \in F_{\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}}(A) = \mathrm{Hom}_k^{\mathrm{loc}}(\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}, A)$. It corresponds to a commutative diagram of morphisms of local k-algebras:

$$\begin{tikzcd} \mathcal{O}_{\mathscr{L}_\infty(V),\gamma} \arrow[r, "\gamma_A"] \arrow[d] & A \arrow[d] \\ k \arrow[r, equal] & k \end{tikzcd} \tag{2.3}$$

Let U be an affine open subspace of V which contains $\gamma(0)$. Then, diagram (2.3) corresponds to a commutative diagram of morphisms of R-algebras:

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, "\gamma_A"] \arrow[d, "\gamma"'] & A[[T]] \arrow[d, "p"] \\ k[[T]] \arrow[r, equal] & k[[T]], \end{tikzcd} \tag{2.4}$$

(where $p\colon A[[T]] \to k[[T]]$ is the unique continuous morphism which extends the projection $A \to A/\mathfrak{M}_A \cong k$) or equivalently, to a commutative diagram of morphisms of R-schemes:

$$\begin{tikzcd} \mathrm{Spec}(A[[T]]) \arrow[r, "\gamma_A"] & V \arrow[d, equal] \\ \mathrm{Spec}(k[[T]]) \arrow[u] \arrow[r, "\gamma"] & V \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathrm{Spf}(A[[T]]) \arrow[r, "\gamma_A"] & V \arrow[d, equal] \\ \mathrm{Spf}(k[[T]]) \arrow[u] \arrow[r, "\gamma"] & V \end{tikzcd}$$

with respect to the algebraic or formal nature of the R-space V.

Definition 2.2. — Every morphism γ_A involved in the above commutative diagrams is called an *A-deformation of* γ.

Remark 2.3. — The following remark clarifies an important point in the Drinfeld-Grinberg-Kazhdan Theorem. For simplicity we specify it in case $V = \mathbf{A}_k^1$. Then, the k-scheme $\mathscr{L}_\infty(V)$ is isomorphic to $\mathrm{Spec}(k[(T_i)_{i\in\mathbf{N}}])$. So, for every (constant or non-constant) k-rational arc γ, we conclude that there exists a continuous isomorphism of admissible local k-algebras:

$$\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}} \cong k[[(T_i)_{i\in\mathbf{N}}]].$$

But, it is very important to keep in mind that this isomorphism of topological complete rings is not an isomorphism for the adic topologies. The topology on the

admissible local k-algebra $k[[(T_i)_{i\in\mathbf{N}}]]$ is the inverse limit topology induced by the isomorphism
$$k[[(T_i)_{i\in\mathbf{N}}]] = \varprojlim_N k[(T_i)_{i\in\mathbf{N}}]/\langle (T_i)_{i\in\mathbf{N}}\rangle^N$$
In constrast with the classical case of a power series in a finite number of indeterminate, the adic topology on $k[[(T_i)_{i\in\mathbf{N}}]]$ does not coincide with the inverse limit topology; in fact $k[[(T_i)_{i\in\mathbf{N}}]]$ is not even complete for its adic topology; see [34, Example 1.8]) (we thank A. Bouthier for pointing out this reference to us), [9] or [20].

2.3. **Reduction to branches.** Let k be a field. The following remark implies that the formal neighborhood of a given arc carries a part of the information on the mere singularities of the branch of the involved k-variety which contains this arc.

Lemma 2.4. — *Let V be an integral k-variety. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ be a non-constant arc on V. We assume that there exists a unique minimal prime ideal $\mathfrak{p}$ of the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ such that the induced morphism of admissible local k-algebras $\gamma\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to k[[T]]$ factorizes through $\widehat{\mathcal{O}_{V,\gamma(0)}} \to \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Then, for every test-ring $(A, \mathfrak{M}_A)$, for every A-deformation $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$ of γ, the induced morphism of admissible local k-algebras $\gamma_A\colon \widehat{\mathcal{O}_{V,\gamma(0)}} \to A[[T]]$ factorizes through $\widehat{\mathcal{O}_{V,\gamma(0)}} \to \widehat{\mathcal{O}_{V,\gamma(0)}}/\mathfrak{p}$. Besides, the ideal $\mathfrak{p}$ is the only minimal prime ideal with this property.*

In other words, if the arc γ is contained in a unique formal irreducible component at $\gamma(0)$, then every A-deformation of γ is contained in the same component (and only in this component).

Proof. — Let $(A, \mathfrak{M}_A)$ be a test-ring and $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$. By subsection 2.2, the datum of γ_A corresponds to that of a diagram of morphisms of complete local k-algebras:
$$\begin{tikzcd} \widehat{\mathcal{O}_{V,\gamma(0)}} \arrow[r, "\gamma_A"] \arrow[dr, "\gamma"'] & A[[T]] \arrow[d] \\ & k[[T]]. \end{tikzcd} \tag{2.5}$$
Then, we have $\ker(\gamma) = \gamma_A^{-1}(\mathfrak{M}_A[[T]])$. Let $\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the minimal prime ideals of the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$. We may assume that that $\ker(\gamma)$ contains $\mathfrak{p}$ and does not contain the $\mathfrak{q}_i$ for every $i \in \{1, \ldots, n\}$. Let us prove that $\mathfrak{p} \subset \ker(\gamma_A)$. Let $x \in \mathfrak{p}$. Since the ring $\widehat{\mathcal{O}_{V,\gamma(0)}}$ is reduced, we have $\mathfrak{p} \cap (\cap_{i=1}^n \mathfrak{q}_i) = (0)$. By assumption, for every integer $i \in \{1, \ldots, n\}$, there exists an element $y_i \in \mathfrak{q}_i$ such that $y_i \notin \ker(\gamma)$. Then, we deduce that $xy_1 \ldots y_n = 0$ and that
$$\begin{array}{rcl} \gamma_A(xy_1 \ldots y_n) & = & 0 \\ \gamma_A(x) \cdot \gamma_A(y_1) \ldots \gamma_A(y_n) & = & 0. \end{array} \tag{2.6}$$
Since, by construction, $y_i \notin \gamma_A^{-1}(\mathfrak{M}_A[[T]])$ for every integer $i \in \{1, \ldots, n\}$, we conclude that the element $\gamma_A(y_i)$ does not reduce to zero modulo $\mathfrak{M}_A[[T]]$. Then, by the Weierstrass Preparation Theorem, the element $\gamma_A(y_i)$ is not a zero divisor in the ring $A[[T]]$ (see §3.2); hence, by equation (2.6), we have $\gamma_A(x) = 0$, i.e., $x \in \mathrm{Ker}(\gamma_A)$.

In the end, if there exists $i \in \{1, \ldots, n\}$ such that $\mathfrak{q}_i \subset \gamma_A^{-1}(0)$, then we have $\mathfrak{q}_i \subset \gamma_A^{-1}(\mathfrak{M}_A[[T]]) = \mathrm{Ker}(\gamma)$, which contradicts our assumption. It concludes the proof of our statement. $\square$

Remark 2.5. — If one does not assume that the arc γ belongs to a unique formal irreducible component at $\gamma(0)$, the result does not hold anymore when $\dim(V) \geqslant 2$. Let us consider the example of the affiine k-variety $V = \mathrm{Spec}(k[X,Y,Z]/\langle Y^2 - X^3 - X\rangle)$. It is an integral k-variety and $\widehat{\mathcal{O}_{V,\mathfrak{o}}} \simeq k[[U,V,W]]/\langle UV\rangle$, where we denote by $\mathfrak{o}$ the origin of $\mathbf{A}^3_k$. Let $A = k[S]/\langle S^2\rangle$ and $s := \bar{S}$. Via this identification, we observe that the arc $\gamma(T) = (0,0,T)$ admits the A-deformation $\gamma_A(T) = (s,s,T)$, which is not contained in any formal irreducible component of the formal germ $(V, \mathfrak{o})$.

Corollary 2.6. — *Let $\mathscr{C}$ be an integral k-curve. Let $\gamma \in \mathscr{L}_\infty(\mathscr{C})(k)$ be a non-constant arc on $\mathscr{C}$. Then, for every test-ring $(A, \mathfrak{M}_A)$, for every A-deformation $\gamma_A \in \mathscr{L}_\infty(\mathscr{C})_\gamma(A)$ of γ, the A-deformation γ_A belongs to the unique formal irreducible component of the formal germ $(V, \gamma(0))$ which contains γ.*

2.4. **Invariance by étale morphisms.** We show that formal neighborhoods in arc scheme are invariant by étale morphisms.

Proposition 2.7. — *Let k be a field. Let V, V' be two R-varieties. Let $\gamma \in \mathscr{L}_\infty(V)(k)$ and $\gamma' \in \mathscr{L}_\infty(V')(k)$. Let $f\colon V' \to V$ be an étale morphism such that $\mathscr{L}_\infty(f)(\gamma') = \gamma$. The induced morphism of formal k-schemes*

$$\mathscr{L}_\infty(V')_{\gamma'} \to \mathscr{L}_\infty(V)_\gamma$$

is an isomorphism.

Proof. — By observation 2.1, it is enough to check that the natural map

$$\mathscr{L}_\infty(V')_{\gamma'}(A) := \mathrm{Hom}_k^{\mathrm{loct}}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(V'),\gamma'}}, A) \to \mathrm{Hom}_k^{\mathrm{loct}}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}, A) =: \mathscr{L}_\infty(V)_\gamma(A)$$

is bijective for every test-ring A. Let $(A, \mathfrak{M}_A)$ be a test-ring. Let $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$. By subsection 2.1, we know that the datum of γ_A corresponds to that of a commutative diagram of morphisms of R-schemes:

$$\begin{tikzcd}
\mathrm{Spec}(A[[T]]) \arrow[r, "\gamma_A"] & V \\
\mathrm{Spec}(k[[T]]) \arrow[u, "\mathfrak{M}_A=0"] \arrow[ur, "\gamma"] \arrow[r, dashed, "\gamma'"'] & V' \arrow[u, dashed, "f"']
\end{tikzcd} \tag{2.7}$$

We fix the structure of V-scheme on $\mathrm{Spec}(A[[T]])$ by considering γ_A and the nilpotent closed immersion $\iota : \mathrm{Spec}(k[[T]]) \to \mathrm{Spec}(A[[T]])$ in diagram (2.7) (which is nilpotent since the k-algebra A is a test-ring). From the definition of formal étaleness (see [19, 17.1.1]) applied to γ', we deduce that there exists a unique element

$$\gamma'_A \in \mathrm{Hom}_V(\mathrm{Spec}(A[[T]]), V')$$

such that $\iota \circ \gamma'_A = \gamma'$, and $f \circ \gamma'_A = \gamma_A$ because of the choice of the structure of V-scheme on $\mathrm{Spec}(A[[T]])$. Hence, γ'_A is the unique preimage of γ_A in $\mathscr{L}_\infty(V')_{\gamma'}(A)$. That concludes the proof. $\square$

2.5. **Behaviour under arc reparametrization.** In this subsection, we describe the behaviour of formal neighborhoods under the operation to reparametrize arc.

Proposition 2.8. — *Let V be a k-variety. Let $\gamma, \gamma' \in \mathscr{L}_\infty(V)(k)$. Assume that there exists a morphism of formal k-schemes $p : \mathrm{Spf}(k[[T]]) \to \mathrm{Spf}(k[[T]])$ such that $\gamma' = \gamma \circ p$. Then, there exists a unique morphism of formal k-schemes*

$$\theta_\gamma^{\gamma'} : \mathscr{L}_\infty(V)_\gamma \to \mathscr{L}_\infty(V)_{\gamma'}$$

which sends, for every test-ring A, the element $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$ to $\gamma'_A := \gamma_A \circ p \in \mathscr{L}_\infty(V)_{\gamma'}(A)$. If $p \neq 0$, this morphism is a closed immersion. Moreover, if p is an automorphism, then the morphism $\theta_\gamma^{\gamma'}$ is an isomorphism. The converse holds true when the field k is assumed to be of characteristic 0.

Proof. — ∘ By assumption, there exists $p \in T\,k[[T]]$ such that $\gamma' = \gamma \circ p$. We denote the common value of $\gamma(0)$ and $\gamma'(0)$ by v. By subsection 2.1, to prove the first part of the theorem, it suffices to observe that, for every test-ring A and every $\gamma_A \in \mathscr{L}_\infty(V)_\gamma(A)$, there exists a commutative diagram of morphism of local R-algebras:

$$\begin{tikzcd}
\mathcal{O}_{V,v} \arrow[r, "\gamma_A"] \arrow[dr, "\gamma"'] \arrow[drr, bend right=60, "\gamma'"'] & A[[T]] \arrow[r, "p^\#"] \arrow[d, "\mathfrak{m}_A=0"] & A[[T]] \arrow[d, "\mathfrak{m}_A=0"] \\
& k[[T]] \arrow[r, "p^\#"] & k[[T]].
\end{tikzcd} \tag{2.8}$$

∘ Let us prove that the morphism is a closed immersion when $p \neq 0$. Since the assertion is local, we may assume that the variety V is embedded in $\mathbf{A}^N_k$, say

$$V = \mathrm{Spec}(k[X_1, \ldots, X_N]/I).$$

and that $v := \gamma(0)$ is the origin of $\mathbf{A}^N_k$. Let B be a k-algebra and $(\varphi_i)_{i\in\mathbf{N}} \in B^{\mathbf{N}}$. In order to fix notations, let us consider the formal relation:

$$p^\sharp(\sum_{i\geqslant 0} \varphi_i T^i) = \sum_{i\geqslant 0} \varphi_i\,(p(T))^i = \sum_{i\geqslant 0} G_i((\varphi_j)_{j\leqslant i}) T^i$$

where the G_i are linear forms uniquely determined by the datum of p. In particular, we observe that $G_0(\varphi_0) = \varphi_0$. Besides, writing $p(T) = \alpha\, T^n(1 + Tq(T))$ with $\alpha \in k^\times$ and $n \geqslant 1$, for every integer $i \in \mathbf{N}^*$, there exists a unique linear form $\tilde{G}_i \in k[Y_1, \ldots, Y_{i-1}]$ which satisfies the following relation:

$$G_{n\,i}((\varphi_j)_{j\leqslant n\,i}) = \alpha\varphi_i + \tilde{G}_i((\varphi_j)_{j\leqslant i-1}). \tag{2.9}$$

Then, for every integer $i \in \{1, \ldots, N\}$, and every integer $j \in \mathbf{N}$, the associated morphism of k-algebras $(\theta_\gamma^{\gamma'})^\sharp$ is defined by $X_{i,j} \mapsto G_j((X_{i,\ell})_{\ell\leqslant j-1})$. Thanks to formulas (2.9), we conclude that it is surjective.

∘ Let us show the last part of the statement. If p is an automorphism (equivalently, if $p \in T\,(k[[T]])^\times$) the first part of the theorem applied to p^{-1} provides an inverse to $\theta_\gamma^{\gamma'}$, and shows that $\theta_\gamma^{\gamma'}$ is an isomorphism.

∘ Assume now that p is not an automorphism. We will show that there exist a test-ring A and an A-deformation γ'_A of γ_A wich can not be written as $\gamma_A \circ p$, where

γ_A is an A-deformation of A. Let us begin with the following remark : if A is a test-ring and $a \in \mathfrak{M}_A$, then there exists a unique local morphism of local k-algebra $\theta_a : k[[T]] \to A[[T]]$ such that $\theta_a(T) = T + a$, since a is nilpotent. Moreover the composition of $\gamma : \mathcal{O}_{V,v} \to k[[T]]$ with θ_a is an A-deformation of γ.

First assume that $p \neq 0$ (equivalently, there exists an integer $n \geqslant 2$ such that $p \in T^n(k[[T]])^\times$) and γ is non constant. Then, for every A-deformation γ_A of γ and every $x \in \mathfrak{M}_v$, the coefficients of $T, \ldots, T^{n-1}$ in $(\gamma_A \circ p)(x)$ vanish. Moreover, since γ is non-constant, there exist a positive integer $N \geqslant n$ and $x \in \mathfrak{M}_v$ such that $\gamma'(x) \in T^N\,(k[[T]])^\times$. Now let $A = k[S]/\langle S^{N-n+2}\rangle$, with $s := \bar{S}$ in the ring A. Consider the A-deformation of γ' given by $\gamma'_A = \gamma' \circ \theta_s$. Then one has

$$\gamma'_A(x) \in \binom{N}{n-1} s^{N-n+1}\, T^{n-1} + T^n A[[T]].$$

Thus, there is no A-deformation γ_A of γ such that $\gamma'_A = \gamma_A \circ p$.

Now assume that $p = 0$ (in particular γ' is the constant arc with $\gamma'(0) = v$). Then for every A-deformation γ_A of γ one has $(\gamma_A \circ p)(\mathfrak{M}_v) \subset A$. Thus it suffices to show that there exist a test-ring A and an A-deformation γ'_A of γ' such that $\gamma'_A(\mathfrak{M}_v) \subsetneq A$. Recall than an A-deformation of the constant arc γ' is a local morphism $\gamma'_A : \mathcal{O}_{V,v} \to A[[T]]$ such that $\gamma'_A(\mathfrak{M}_v) \subset \mathfrak{M}_A[[T]]$. Let $A = k[S]/\langle S^2\rangle$ with $s := \bar{S}$ in the ring A. Let $x_1, \ldots, x_r$ be a k-basis of $\mathfrak{M}_v/\mathfrak{M}_v^2$. Then there exists a unique local morphism of k-algebra $\mathcal{O}_{V,v}/\mathfrak{M}_v^2 \to A[[T]]$ sending x_i to $s\,T$. Composing with $\mathcal{O}_{V,v} \to \mathcal{O}_{V,v}/\mathfrak{M}_v^2$ one obtains an A-deformation γ'_A of γ' such that $\gamma'_A(\mathfrak{M}_v)$ contains non constant power series.

Finally, assume that γ (hence γ') is constant and that $p \neq 0$. Then there exists an integer $n \geqslant 2$ such that for every test-ring A , every A-deformation γ_A of γ and every $x \in \mathfrak{M}_v$, the coefficients of $T, \ldots, T^{n-1}$ in $(\gamma_A \circ p)(x)$ vanish. Using the same A-deformation of γ' as in the previous case, we conclude the proof.

□

Corollary 2.9. — *Let V, V' be two k-varieties. Let $\gamma \in \mathscr{L}_\infty(V)(k)$, $\gamma' \in \mathscr{L}_\infty(V')(k)$, $v = \gamma(0)$ and $v' = \gamma'(0)$. Assume that there exists isomorphisms $f : \widehat{\mathscr{O}_{V,v}} \to \widehat{\mathscr{O}_{V',v'}}$ and $p : k[[T]] \to k[[T]]$ such that $p \circ \gamma = \gamma' \circ f$. Then, there exists an isomorphism of formal k-schemes $\mathscr{L}_\infty(V)_\gamma \cong \mathscr{L}_\infty(V')_{\gamma'}$.*

Proof. — By Proposition 2.8, one may assume $\gamma = \gamma' \circ f$. Then, for every test-ring A, the composition by f induces the following diagram of maps, functorial in A:

$$\begin{tikzcd}
\mathrm{Hom}_k^{\mathrm{lcp}}(\widehat{\mathcal{O}_{V,v}}, A[[T]]) \arrow[r, "\circ f", "\sim"'] & \mathrm{Hom}_k^{\mathrm{lcp}}(\widehat{\mathcal{O}_{V',v'}}, A[[T]]) \\
\mathscr{L}_\infty(V)_\gamma(A) \arrow[u, hook] \arrow[r] & \mathscr{L}_\infty(V')_{\gamma'}(A) \arrow[u, hook]
\end{tikzcd}$$

where the map $\mathscr{L}_\infty(V)_\gamma(A) \to \mathscr{L}_\infty(V')_{\gamma'}(A)$, obtained by restriction, is also a bijection. Hence, the formal k-schemes $\mathscr{L}_\infty(V)_\gamma$ and $\mathscr{L}_\infty(V')_{\gamma'}$ are isomorphic. □

2.6. **The kernel of the completion.** The following statement mimicks, for non-singular arcs, [27, Theorem 3.13] which is obtained in *loc. cit.* for stable points of arc schemes. We would like to thank A. Reguera for pointing out this reference to us.

Proposition 2.10. — *Let k be a field of charcateristic zero. Let V be an integral hypersurface of $\mathbf{A}_k^N$. Let $v \in V_{\mathrm{sing}}(k)$ and $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$, with $\gamma(0) = v$. Then the canonical morphism of local k-algebras $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma} \to \mathcal{O}_{\mathscr{L}_\infty(V)_{\mathrm{red}},\gamma}$ induces an isomorphism of complete local k-algebras $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}} \to \widehat{\mathcal{O}_{\mathscr{L}_\infty(V)_{\mathrm{red}},\gamma}}$.*

Remark 2.11. — The nilpotent elements of the completion $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ should carry interesting informations about the singularity at the origin of the arc $\gamma(0)$ (see section 8 and [3]). It is interesting to note that Proposition 2.10 shows that such informations may not be visible at the level of the local ring $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$.

The proof of Proposition 2.10 is based on the following result:

Lemma 2.12 ([27, Lemma 2.7]). — *Let k be a field of characteristic zero. Let $F \in k[X_1, \ldots, X_N]$ be an irreducible polynomial. Then, for every partial derivative $\partial(F)$ of F, we have*

$$\{F\} := \sqrt{[F]} = ([F] \colon \partial(F)^\infty).$$

This statement can be interpreted as a simple formulation, in the hypersurface case, of the Rosenfeld Lemma ([22, IV/§9/Lemma 2]). Let us recall that, under notation of Lemma 2.12, and if we set

$$V = \mathrm{Spec}(k[X_1, \ldots, X_N]/(F)),$$

then we have $\mathscr{L}_\infty(V) = \mathrm{Spec}(k[(X_{i,j})_{i,j\in\mathbf{N}}]/[F])$ and that $[F]$ is the differential ideal of $(k[(X_{i,j})_{i,j\in\mathbf{N}}], \Delta)$ generated by F, where Δ is the k-derivation of the ring $k[(X_{i,j})_{i,j\in\mathbf{N}}]$ defined by the relations $\Delta(X_{i,j}) = X_{i,j+1}$ for every integer $i \in \{1, \ldots, N\}$ and every integer $j \in \mathbf{N}$.

Proof of prop 2.10. —

The surjective canonical morphism of local k-algebras

$$\mathcal{O}_{\mathscr{L}_\infty(V),\gamma} \to \mathcal{O}_{(\mathscr{L}_\infty(V))_{\mathrm{red}},\gamma}$$

has a kernel equal to $\{F\}/[F]$. Then, it induces a surjective morphism of complete local k-algebras:

$$\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}} \to \widehat{\mathcal{O}_{(\mathscr{L}_\infty(V))_{\mathrm{red}},\gamma}}$$

whose kernel is the $\mathfrak{M}_\gamma$-completion of the differential ideal $\{F\}/[F]$, i.e.,

$$\mathrm{Ker}(\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}} \to \widehat{\mathcal{O}_{(\mathscr{L}_\infty(V))_{\mathrm{red}},\gamma}}) = \varprojlim_n((\mathfrak{M}_\gamma^n + \{F\})/(\mathfrak{M}_\gamma^n + [F])).$$

(See [25, Theorem 8.1].) We have to prove that this kernel equals 0.

We show the following formula:

$$\{F\} \subset \cap_{n\in\mathbf{N}^*}(\mathfrak{M}_\gamma^n + [F]) \tag{2.10}$$

in the local ring $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$. Let $P \in \{F\}$. By Lemma 2.12, there exists an integer $a \in \mathbf{N}^*$ such that

$$\partial(F)^a \cdot P \in [F]. \tag{2.11}$$

Let us note that $[F] \subset \mathfrak{M}_\gamma$ and $\{F\} \subset \mathfrak{M}_\gamma$. We claim that there exist a partial derivative $\partial(F)$ and a smallest integer i_0 such that $\Delta^{(i_0)}(\partial(F)) \notin \mathfrak{M}_\gamma$. Indeed, let us assume that $\Delta^{(i)}(\partial(F)) \in [F]$ for every integer $i \in \mathbf{N}$ and every $\partial(F)$. In that case, the image of $\partial(F)$ by the morphism of k-algebras $\mathcal{O}(\mathscr{L}_\infty(V)) \to k$, corresponding to the datum of $\mathfrak{M}_\gamma$, would equal 0; hence, by adjunction,

$$\partial(F)(\gamma(T)) = 0$$

in the ring $k[[T]]$. That contradicts our assumption on the arc γ, since F is assumed to be irreducible.

Then, the integer ai_0 is the smallest integer such that:

$$\Delta^{(ai_0)}(\partial(F)^a) \notin \mathfrak{M}_\gamma. \tag{2.12}$$

By deriving ai_0 times equation (2.11), we conclude that

$$\Delta^{(ai_0)}(\partial(F)^a) \cdot P + \sum_{i=1}^{ai_0} \binom{ai_0}{i} \Delta^{(ai_0-i)}((\partial(F))^a)\Delta^{(i)}(P) \in [F] \tag{2.13}$$

So,

$$P \in \mathfrak{M}_\gamma^2 + [F]$$

in the local ring $\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}$. Note that we have shown the inclusion $\{F\} \subset \mathfrak{M}_\gamma^2 + [F]$. In particular, for every integer $n \in \mathbf{N}^*$, one has

$$\Delta^{(n)}(P) \in \mathfrak{M}_\gamma^2 + [F];$$

hence, by equation (2.13),

$$P \in \mathfrak{M}_\gamma^3 + [F].$$

By iterating this process, we prove formula (2.10) and conclude the proof. □

Remark 2.13. — Let k be a field of characteristic 0. In [12] or [23, 32, 30], it is shown in particular that the integral plane k-curve X is smooth if and only $\mathscr{L}_\infty(X)$ is reduced. Let $\gamma \in \mathscr{L}_\infty(X)(k)$ be a primitive k-parametrization. Thus, the morphism of k-algebras:

$$\mathcal{O}(\mathscr{L}_\infty(X)) \to \widehat{\mathcal{O}_{\mathscr{L}_\infty(X),\gamma}} \tag{2.14}$$

is injective if and only if the k-curve X is smooth. Indeed, Proposition 2.10 and [23, 30] obviously imply that, if X is not smooth, morphism (2.14) is not injective. Conversely, if X is smooth over k, there is an open subscheme $U = \operatorname{Spec}(A)$ of X, which contains $\gamma(0)$, endowed with an étale morphism $U \to \mathbf{A}^1_k$. So, we deduce that $\mathscr{L}_\infty(U) = \operatorname{Spec}(A[(T_i)_{i\geqslant 1}])$. Now, morphism (2.14) factorizes through the morphism

$$A[(T_i)_{i\geqslant 1}] \to k[[T_0]][[(T_i)_{i\geqslant 1}]],$$

which is given by $A \to k[[T_0]]$ (the completion morphism) and $T_i \mapsto T_i$ for every integer $i \in \mathbf{N}^*$. Since the morphism $\mathcal{O}(\mathscr{L}_\infty(X)) \to \mathcal{O}(\mathscr{L}_\infty(U))$ is injective, we conclude that morphism (2.14) also is injective.

3. Preliminaries to the proof of Theorem 4.1

To state our generalization of the Drinfeld-Grinberg-Kazhdan Theorem, we need to fix the notation which will be used in its proof. In this section, we also recall some classical results and state different technical statements. Let k be a field. Recall that $R = k$ or $R = k[[T]]$.

3.1. One of the crucial ingredients in the following presentation is the Taylor expansion for polynomials or restricted power series. So we recall this formula under a form appropriate for our purposes: let N and M be positive integers, $X = (X_1, \ldots, X_N)$ and $Y = (Y_1, \ldots, Y_M)$ be unknows and F be an element of $R[X, Y]$ (or more generally, in case $R = k[[T]]$, of $R\{X, Y\}$). Then, if $U = (U_1, \ldots, U_{N+M})$, there exist polynomials $G_{j,k} \in R[X, Y, U]$ (or restricted power series $G_{j,k} \in R\{X, Y, U\}$) such that the element $F(X_1 + U_1, \ldots, X_N + U_N, Y_1 + U_{N+1}, \ldots, Y_M + U_{N+M})$ equals:

$$\begin{aligned} F(X,Y) + \textstyle\sum_{i=1}^{N} U_i \partial_{X_i}(F)(X,Y) + \sum_{i=1}^{M} U_{i+N}\partial_{Y_i}(F)(X,Y) \\ + \textstyle\sum_{j,k\in\{1,\ldots,N+M\}} U_j U_k G_{j,k}(X,Y,U). \end{aligned} \tag{3.1}$$

We refer to [1, Chapitre III/§4/5] for details in the formal context.

3.2. For every complete local ring A and non-negative integer d, an element $q(T) \in A[T]$ with degree d is said to be a *Weierstrass (or distinguished) polynomial* if $q(T)$ is monic and $q(T) - T^d \in \mathfrak{M}_A[T]$. We denote by $\mathscr{W}(A, d)$ the set of Weierstrass polynomials of degree d with coefficients in A. By uniqueness in the Weierstrass Division Theorem ([24, Theorem 9.1]), a Weierstrass polynomial is not a zero divisor in $A[[T]]$. More generally, any $r_A(T) \in A[[T]]$ whose reduction modulo $\mathfrak{M}_A[[T]]$ is a non zero element of $k[[T]]$ is not a zero divisor in $A[[T]]$; indeed, by the Weierstrass Preparation Theorem (see [24, Theorem 9.2]), one may write $r_A(T) = q_A(T)u_A(T)$ where $q_A(T)$ is a Weierstrass polynomial and $u_A(T)$ is invertible, and is in particular not a zero divisor. Besides, if $r_A(T) = q_A(T)u_A(T)$ is the Weierstrass decomposition of $r_A(T)$ and if $f\colon A \to A'$ is a morphism of local rings, the uniqueness also implies that $f_T(q_A(T))$ is the Weierstrass polynomial in the Weierstrass decomposition of $f_T(r_A(T))$. Here, we denote by $f_T : A[[T]] \to A'[[T]]$ the morphism of local k-algebras induced by the action of f on coefficients.

3.3. Let us note an important fact.

Lemma 3.1. — *Let $d \geqslant 1$ and $N \geqslant 0$ be integers, and let $X = (X_0, \ldots, X_{d-1})$, $Y = (Y_1, \ldots, Y_N)$ be tuples of indeterminates. Let $p = \sum_{i\in\mathbf{N}} p_i(Y)T^i$ be an element of $k[Y][[T]]$. For every integer $i \in \{0, \ldots, d-1\}$, there exist power series $\mathscr{D}_{d,N,p,i} \in k[[X, Y]]$, such that the following property holds: for every complete local k-algebra $(A, \mathfrak{M}_A)$, for every pair $(a, b) \in \mathfrak{M}_A^{d+N}$, the polynomial $T^d + \sum_{i=0}^{d-1} a_i T^i$ divides $\sum_{i\in\mathbf{N}} p_i(b)T^i$ in $R[[T]]$ if and only if, for every integer $i \in \{0, \ldots, d-1\}$, we have*

$$\mathscr{D}_{d,N,p,i}(a, b) = 0.$$

Proof. — Let $\mathfrak{N} = \langle X, Y\rangle$ be the maximal ideal of the complete local ring $k[[X, Y]]$. Set $q(T) := T^d + \sum_{i=0}^{d-1} X_i T^i$. Let $n \in \mathbf{N}$. Since for every integer $m \geqslant 1$, one has

$$T^{md} = \Big(-\sum_{i=0}^{d-1} X_i T^i\Big)^m \mod \langle q(T)\rangle,$$

we deduce that

$$\sum_{i\in\mathbf{N}} p_i(Y)T^i = \sum_{i\leqslant nd-1} p_i(Y)T^i \mod \langle \mathfrak{N}^n, q(T)\rangle.$$

This remark implies the existence of power series $\mathscr{D}_i^{(n)} \in k[[X,Y]]$, for every integer $i \in \{0,\dots,n-1\}$, such that

$$\sum_{i\in\mathbf{N}} p_i(Y)T^i = \sum_{i=0}^{d-1} \mathscr{D}_i^{(n)}(X,Y)T^i \mod \langle \mathfrak{N}^n, q(T)\rangle,$$

where we observe that $\sum_{i=0}^{d-1} \mathscr{D}_i^{(n)}(X,Y)T^i \bmod \mathfrak{N}^n[[T]]$ is the remainder of the Weierstrass division in $(k[[X,Y]]/\mathfrak{N}^n)[[T]]$ of $p(T) \bmod \mathfrak{N}^n[[T]]$ by $q(T) \bmod \mathfrak{N}^n[[T]]$. By the uniqueness of the Weierstrass division, we deduce that, for every integer $i \in \{0,\dots,d-1\}$,

$$\mathscr{D}_i^{(n+1)}(X,Y) = \mathscr{D}_i^{(n)}(X,Y) \mod \mathfrak{N}^n.$$

The datum of the family $(\mathscr{D}_i^{(n)})_n$ gives rise to an element $\mathscr{D}_{d,N,p,i} \in k[[X,Y]]$ with the required properties. □

Let $e \geqslant 1$ be an integer and $Z = (Z_1,\dots,Z_e)$ be a tuple of indeterminates. If $p(T) \in k[Y][[T]]$ and $q(T)$ is an element of $T^d + (k[Z])[T]_{\leqslant d-1}$ whose reduction modulo $\langle Z\rangle$ is T^d, we denote by $\mathscr{D}_{d,N,i}(q(T),p(T))$ the element of $k[[Y,Z]]$ obtained by substituting the coefficients of q to the X_i's in $\mathscr{D}_{d,N,p,i}$.

3.4. When $p \in k[Y,T]$ is a polynomial, Lemma 3.1 can be specialized and precised. Following the previous ideas, it is easy to see that there exist polynomials $D_{d,N,p,i} \in k[X,Y]$ for $i \in \{0,\dots,d-1\}$ satisfying the following property: for every ring A, for every $(a,b) \in A^{d+N}$, we have $D_{d,N,p,i}(a,b) = 0$ for every integer $i \in \{0,\dots,d-1\}$ if and only if the polynomial $T^d + \sum_{i=0}^{d-1} a_iT^i$ divides $p(b,T)$ in the ring $A[T]$. In the same way, we adopt the notation $D_{d,N,i}(q(T),p(T)) \in k[Y,Z]$ for the specialization of $D_{d,N,p,i}$ at the coefficients of $q(T)$.

4. The Drinfeld-Grinberg-Kazhdan Theorem for formal schemes

This section is the core of the article. It is devoted to the statement and the proof of our main result. The last subsections present different important remarks deduced from Theorem 4.1 and (global) generalizations.

4.1. **Statements and remarks.** Let k be a field. Recall that we denote by $R = k$ or $R = k[[T]]$. If K is a field extension of the field k, then we set $R_K := K$ if $R = k$, and $R_K := K[[T]]$ if $R = k[[T]]$. Let V be a R-space, i.e., either an R-scheme of finite type, or, in case $R = k[[T]]$, a formal tft R-scheme. Let us denote by $\mathscr{L}_\infty^\circ(V)$ the open subscheme of $\mathscr{L}_\infty(V)$ defined by $\mathscr{L}_\infty^\circ(V) = \mathscr{L}_\infty(V) \setminus \mathscr{L}_\infty(V_{\mathrm{sing}})$. If K is a field extension of the field k, we set $V_K = V \times_R \operatorname{Spec}(R_K)$ if V is an R-scheme of finite type and $V_K = V \times_R \operatorname{Spf}(R_K)$ if $R = k[[T]]$ and V is a formal tft R-scheme.

Theorem 4.1. — *Let V be a R-space. Let $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$. There exists an affine noetherian formal k-scheme $\mathscr{S}$ and an isomorphism of formal k-schemes:*

$$\theta_k(V)\colon \mathscr{L}_\infty(V)_\gamma \to \mathscr{S} \times_k \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \tag{4.1}$$

Moreover, in case V is an R-scheme of finite type, there exist an affine k-scheme of finite type S and $s \in S(k)$ such that $\mathscr{S}$ is isomorphic to S_s.

The isomorphism $\theta_k(V)$ also have the following properties:

Theorem 4.2. — *With the notation as in Theorem 4.1, we have the following properties:*

(1) *The isomorphism $\theta_k(V)$ constructed in this way satisfies the following assertions:*

(a) *For every separable field extension K of k, we have the following commutative diagram of maps:*

$$\begin{tikzcd} \mathscr{L}_\infty(V_K)_\gamma \arrow[r, "\theta_K(V_K)"] \arrow[d] & \mathscr{S} \times_k \mathrm{Spf}(K[[(T_i)_{i\in\mathbf{N}}]]) \arrow[d] \\ \mathscr{L}_\infty(V)_\gamma \arrow[r, "\theta_k(V)"'] & S_s \times_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \end{tikzcd}$$

(b) *For every integer $n \in \mathbf{N}$, there exists a morphism of formal k-schemes $\mathscr{S} \times_k \mathrm{Spf}(k[[T_0,\dots,T_n]]) \to \mathscr{L}_n(V)_{\pi_n^\infty(\gamma)}$ making the following diagram of morphisms of formal k-schemes commute:*

$$\begin{tikzcd} \mathscr{L}_\infty(V)_\gamma \arrow[r, "\hat{\pi}_n^\infty"] \arrow[d, "\theta_k(V)"'] & \mathscr{L}_n(V)_{\pi_n^\infty(\gamma)} \\ \mathscr{S} \times_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]) \arrow[r] & \mathscr{S} \times_k \mathrm{Spf}(k[[T_0,\dots,T_n]]), \arrow[u, dashed] \end{tikzcd}$$

where $\pi_n^\infty \colon \mathscr{L}_\infty(V) \to \mathscr{L}_n(V)$ is the truncation morphism of level n.

(2) *Once having chosen an affine neighborhood of $\gamma(0)$ in V and an embedding of this affine neighborhood in an explicit complete intersection in an affine space, there exists a uniquely determined and completely explicit procedure allowing to construct a suitable formal scheme $\mathscr{S}$ and a suitable isomorphism $\theta_k(V)$.*

(3) *When V is a k-scheme of finite type, the above procedure can be implemented as an effective algorithm taking as its input a suitable truncation of the arc γ and producing as its output a pointed affine k-scheme (S,s) such that $\mathscr{S} = S_s$ realizes isomorphism* (4.1).

Remark 4.3. — If there is no confusion, we omit to write V in the notation $\theta_k(V)$.

Remark 4.4. — The statement of Theorem 4.1 in particular extends Theorem 1.1 to the case of a family of varieties parametrized by $\mathrm{Spec}(k[[T]])$ and to the case of formal schemes topologically of finite type over $\mathrm{Spf}(k[[T]])$. Our proof follows the strategy of Drinfeld's original proof in the preprint [11], but adapts the crucial arguments in our general setting. We give in particular a detailed account of these arguments. Assertion (b) of the statement of Theorem 4.2 had been stated in the first available version of the preprint [7] (in the framework of algebraic varieties over a field and without any proof).

Remark 4.5. — In [4], we provide a SAGE code which, following the algorithm alluded to in the statement of Theorem 4.1, computes a presentation of a pointed k-scheme (S,s) realizing isomorphism (1.1) in case V is an affine plane curve. We note that the algorithm is not computationally very efficient. It seems to us an interesting algorithmic question to find a modified or an alternative version of Drinfeld's arguments providing a more computationally effective way to obtain a presentation of a pointed k-scheme (S, s) realizing isomorphism (1.1).

Remark 4.6. — Theorem 4.1 and results such as Greenberg's Theorem [16] and Denef-Loeser's Fibration Theorem ([10, Lemma 4.1]) deeply differ in nature. In particular, Theorem 4.1 establishes the existence of an isomorphism in the category of formal schemes, which is *a priori* not a completion of an algebraic morphism (this point is crucial, see remark 2.3).

Remark 4.7. — As it will clearly transpire from the explicit description of $\theta_k(V)$ given below, being given an arc $\gamma \in \mathscr{L}_\infty^\circ(V)$, there exists an integer n such that for every $\eta \in \mathscr{L}_\infty^\circ(V)$ such that $\pi_n^\infty(\gamma) = \pi_n^\infty(\eta)$ one has $\mathscr{L}_\infty(V)_\gamma \cong \mathscr{L}_\infty(V)_\eta$. More precisely the dependance in γ of the constructed formal scheme $\mathscr{S}$ is only through $\pi_n^\infty(\gamma)$.

Remark 4.8. — Let V be a R-scheme of finite type. Let $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$. A *DGK-isomorphism* for the arc γ is a triple (S, s, θ) where S is an affine k-scheme of finite type, $s \in S(k)$, and θ is an isormophism of formal k-schemes

$$\theta\colon \mathscr{L}_\infty(V)_\gamma \xrightarrow{\sim} S_s \times_k \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]).$$

We say that such a DGK-isomorphism *factorizes through a finite level* if there exists an integer n such that the morphism $\mathscr{L}_\infty(V)_\gamma \to S_s$ obtained by composing θ with the projection onto S_s factors through the morphism $\mathscr{L}_\infty(V)_\gamma \to \mathscr{L}_n(V)_{\pi_n^\infty(\gamma)}$ induced by the truncation morphism π_n^∞. It is not hard to prove that the DGK-isomorphism $\theta_k(V)$ constructed in the proof of Theorem 4.1 does not factorize through a finite level. Let us briefly indicate how this is done; see the notation of subsection 4.2. Assume that $\theta_k(V)$ factorizes at level $n-1$. Let $N \in \mathbf{N}$, A be the test ring $k[S]/\langle S^N\rangle$ and $s := \bar{S}$ be the image of S in A. Now consider the two A-deformations $\gamma(T+s)$ and $\gamma(T+s(1+T^n))$. From the explicit description of the morphism θ_k (see in particular equations (4.2) and (4.3)) and the fact that $\theta_k(V)$ factors at level $n-1$, we easily deduce that $\det(\mathscr{J}_V)(\gamma(T+s))$ and $\det(\mathscr{J}_V)(\gamma(T+s(1+T^n)))$ have the same Weierstrass polynomial in their Weierstrass decomposition, say $q_A(T)$. Now using equation (4.4) to estimate $q_A(T)$ in both cases, we find that $(T+s)^d$ must divide $(T+s(1+T^n))^d$ in $A[[T]]$. Specializing in $T = -s$ gives a contradiction for $N \geqslant (n+1)d+1$.

As a matter of fact, the existence of DGK-isomorphisms which do not factorize through a finite level is not very surprising, since one easily sees that it holds even for arcs γ with a smooth origin. In [3], the authors prove the following stronger fact: in general, it may happen for a given arc γ that *no* DGK-isomorphism factorizes through a finite level.

Remark 4.9. — A natural question is to ask whether the finite dimensional models have an interpretation in terms of the geometry of the involved R-space. We strongly believe that this interpretation is linked to singularity theory. To justify this affirmation let us mention the following statement due to the authors (see [5]): Let k be a field, V be a k-variety and $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$. Assume that the k-variety V is unibranch at $\gamma(0)$. Then, there exists an isomorphism of formal k-schemes $\mathscr{L}_\infty(V)_\gamma \cong \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$ if and only if the point $\gamma(0)$ is a smooth point of V.

4.2. **Reduction to the case of a complete intersection.** By considering an affine neighborhood of $\gamma(0)$ in V, one reduces immediatly the proof of Theorems 4.1 and 4.2 to the case where V is affine.

∘ Assume that V is an affine scheme of finite type over $\mathrm{Spec}(R)$, defined by the datum of an ideal I_V of the polynomial ring $R[X_1,\dots,X_N]$. Then we denote by J_V the ideal generated by the elements $h\delta\in R[X_1,\dots,X_N]$, where δ is a minor of the Jacobian matrix of a r-tuple $(F_1,\dots,F_r)$ of elements of I_V, for some integer $r\in\mathbf{N}$, and $h\in((F_1,\dots,F_r):I_V)$. Using the Jacobian criterion, one may show (see [14, §0.2],[33, §4]) that the *singular locus* V_{sing} of V, i.e., the reduced closed subscheme associated with the non-smooth locus, can be described as the support of the closed subscheme of V associated with the datum of the ideal I_V+J_V.

∘ Similarly, when $R=k[[T]]$ and V is a formal affine tft R-scheme, defined by the datum of an ideal I_V of the ring of restricted power series $R\{X_1,\dots,X_N\}$, one defines J_V as in the previous case and one shows that the reduced formal R-scheme V_{sing} is associated with the support of the closed formal R-subscheme of V defined by I_V+J_V (see [29, Lemme 2.3.10]).

Using the above description of the non-smooth locus, we will now reduce the proof of Theorem 4.1 to the case of a complete intersection. This kind of reduction is a classical "trick" in the construction of motivic measures (see [10] or, e.g., [29]). We thank O. Gabber for pointing out to us that adapting such an idea to the context of Theorem 4.1 and arc deformations needs some extra argument.

Lemma 4.10. — *Let V be an affine R-space and $\gamma\in\mathscr{L}^{\circ}_{\infty}(V)(k)$. Then, there exists an R-space V' defined in the affine R-space $\mathbf{A}_R^{N+M}$ by M elements $F_1(X,Y)\in\mathcal{O}(\mathbf{A}_R^{N+M})$,..., $F_M(X,Y)\in\mathcal{O}(\mathbf{A}_R^{N+M})$, where we denote by $X=(X_1,\dots,X_N)$, $Y=(Y_1,\dots,Y_M)$ the coordinates on $\mathbf{A}_R^{N+M}$, and a closed immersion $V\hookrightarrow V'$, such that the determinant of the matrix $(\partial_{Y_j}(F_i(X,Y)))_{i,j}$ does not vanish at γ. Besides, the induced morphism of formal k-schemes $\mathscr{L}_\infty(V)_\gamma\cong\mathscr{L}_\infty(V')_\gamma$ is an isomorphism.*

Proof. — Since $\gamma\in\mathscr{L}^{\circ}_{\infty}(V)(k)$, up to shrinking V, we may assume that V is embedded in $\mathbf{A}_R^{N+M}$ defined by the ideal I_V, and that there exist a minor of the Jacobian matrix of the M-tuple $F=(F_1,\dots,F_M)\in I_V^M$ and $h\in((F_1,\dots,F_M):I_V)$ which do not vanish at γ thanks to the above description of the non-smooth locus. Let V' be the R-space defined by the datum of F. The last property of the statement is then a consequence of Lemma 4.11 below. □

Lemma 4.11. — *Let V' be an affine R-space, V be a closed R-subspace of V' and $h\in(0:I_V)\subset\mathcal{O}(V')$. Let $\gamma\in\mathscr{L}_\infty(V)(k)$ be such that $h(\gamma)\neq0$. Then, still denoting by γ the image of γ in V', the natural morphism of formal schemes $\mathscr{L}_\infty(V)_\gamma\to\mathscr{L}_\infty(V')_\gamma$ is an isomorphism of formal k-schemes.*

Proof. — It suffices to show that the induced map $\mathscr{L}_\infty(V)_\gamma(A)\to\mathscr{L}_\infty(V')_\gamma(A)$ is bijective for every test-ring A. Injectivity is clear; so let us show surjectivity. We pick out $\gamma_A\in\mathscr{L}_\infty(V')_\gamma(A)$ and $G\in I_V$. We have to show that $G(\gamma_A)=0$. By hypothesis, one has $h(\gamma_A)G(\gamma_A)=0$. Since $h(\gamma)\neq0$, the reduction of $h(\gamma_A)$ modulo $\mathfrak{M}_A$ is not zero. By §3, one infers that $G(\gamma_A)=0$. □

With the notation $X=(X_1,\dots,X_N)$ and $Y=(Y_1,\dots,Y_M)$, Lemma 4.10 allows us to assume *from now on* that:

(1) The involved R-space V is either of the form $\operatorname{Spec}(R[X,Y]/\langle (F_i)_{1\leqslant i\leqslant M}\rangle)$ or $\operatorname{Spf}(R\{X,Y\}/\langle (F_i)_{1\leqslant i\leqslant M}\rangle)$.
(2) If $\mathscr{J}_V$ denotes the Jacobian matrix $[\partial_{Y_j}F_i]_{1\leqslant i,j\leqslant M}$, the k-rational arc γ satisfies the relation $\det(\mathscr{J}_V)(\gamma(T))\neq 0$.

We set $d := \operatorname{ord}_T[\det(\mathscr{J}_V)(\gamma)]$. We shall also designate by $\underline{F}$ the column vector ${}^t(F_1\dots F_M)$ and by $\operatorname{ad}(\mathscr{J}_V)$ the adjugate of the square matrix $\mathscr{J}_V$, which is defined to be the transpose of the cofactor matrix of $\mathscr{J}_V$.

4.3. **A candidate for the formal k-scheme $\mathscr{S}$ in isomorphism** (4.1)**.** Set

$$\gamma(T) =: (\sum_{j\geqslant 0}\gamma_{i,j}T^j)_{1\leqslant i\leqslant N+M}, \quad \gamma_{i,j}\in k.$$

Let $U=(U_{i,j})_{\substack{1\leqslant i\leqslant N,\\ 0\leqslant j\leqslant 2d-1}}$, $V=(V_{\ell,j})_{\substack{1\leqslant \ell\leqslant M,\\ 0\leqslant j\leqslant d-1}}$, and $W=(W_r)_{0\leqslant r\leqslant d-1}$ be indeterminates. We define the following elements of $k[U,V,W,T]$:

$$\begin{cases} \tilde{x}_i(T) & := & \sum_{j=0}^{2d-1}(U_{i,j}+\gamma_{i,j})T^j, \quad i\in\{1,\dots,N\},\\ \tilde{x}(T) & := & (\tilde{x}_1(T),\dots,\tilde{x}_N(T)),\\ \tilde{y}_i(T) & := & \sum_{j=0}^{d-1}(V_{\ell,j}+\gamma_{\ell+N,j})T^j, \quad \ell\in\{1,\dots,M\},\\ \tilde{y}(T) & := & (\tilde{y}_1(T),\dots,\tilde{y}_M(T)),\\ q(T) & := & T^d+\sum_{j=0}^{d-1}W_jT^j. \end{cases}$$

For every polynomial $P\in k[U,V,W,T]$, for every k-algebra C, and for every element $c=((u_{i,j}),(v_{\ell,j}),(w_r))\in C^{2dN+dM+d}$, we denote by $P_c(T)\in C[T]$ the evaluation of P at c.

◦ *First case: we assume that V is an R-scheme of finite type.* Let $\mathcal{I}$ be the ideal of $k[U,V,W]$ generated by the polynomials, for every integer $\alpha\in\{0,\dots,d-1\}$,

$$D_{d,2dN+dM,\alpha}(q(T),\det(\mathscr{J}_V)(\tilde{x}(T),\tilde{y}(T)))$$

and, for every integer $\alpha\in\{0,\dots,2d-1\}$,

$$D_{2d,2dN+dM,\alpha}(q(T)^2,[\operatorname{ad}(\mathscr{J}_V)\cdot\underline{F}](\tilde{x}(T),\tilde{y}(T))).$$

We set

$$S := \operatorname{Spec}(k[U,V,W]/\mathcal{I}).$$

Thus, by the construction of the involved polynomials and subsection 3.4, for every k-algebra C, an element $c\in C^{2dN+dM+d}$ lies in $S(C)$ if and only if the polynomial $q_c(T)$ divides the polynomial $\det(\mathscr{J}_V)(\tilde{x}_c(T),\tilde{y}_c(T)))$ and the polynomial $q_c(T)^2$ divides the polynomial $[\operatorname{ad}(\mathscr{J}_V)\cdot\underline{F}](\tilde{x}_c(T),\tilde{y}_c(T))$.

Remark 4.12. — Let $\mathfrak{o}\in\mathbf{A}_k^{2dN+dN+d}(k)$ be the origin. Note that

$$\gamma(T)\in(\tilde{x}_{\mathfrak{o}}(T)+T^{2d}k[[T]],\tilde{y}_{\mathfrak{o}}(T)+T^dk[[T]]).$$

Thus, using the Taylor expansion, one has

$$\det(\mathscr{J}_V)(\gamma(T)) = \det(\mathscr{J}_V)(\tilde{x}_{\mathfrak{o}}(T),\tilde{y}_{\mathfrak{o}}(T)) \mod \langle T^d\rangle.$$

As $d=\operatorname{ord}_T[\det(\mathscr{J}_V)(\gamma(T))]$, clearly $T^d=q_{\mathfrak{o}}(T)$ divides $\det(\mathscr{J}_V)(\tilde{x}_{\mathfrak{o}}(T),\tilde{y}_{\mathfrak{o}}(T))$.

Moreover since $F_i(\gamma(T))=0$ for every integer $i\in\{1,\dots,M\}$, using again the Taylor expansion, one has

$$\underline{F}(\tilde{x}_{\mathfrak{o}}(T),\tilde{y}_{\mathfrak{o}}(T))\in T^d\mathscr{J}_V(\tilde{x}_{\mathfrak{o}}(T),\tilde{y}_{\mathfrak{o}}(T))^t(r_1(T),\dots,r_M(T))k[[T]].$$

where $r_1(T), dots, r_M(T)$ are elements of $k[[T]]$.Multiplying the previous relation by $\mathrm{ad}(\mathscr{J}_V)(\tilde{x}_{\mathfrak{o}}(T), \tilde{y}_{\mathfrak{o}}(T))$ and using the previous observation, one sees that that $T^{2\,d} = q_{\mathfrak{o}}(T)^2$ divides $[\mathrm{ad}(\mathscr{J}_V) \cdot \underline{F}](\tilde{x}_{\mathfrak{o}}(T), \tilde{y}_{\mathfrak{o}}(T))$.

Thanks to this remark, the point $\mathfrak{o}$ defines an element of $S(k)$ which we will denote by s. We set $\mathscr{S} := S_s$. Thus, for every complete local k-algebra $(C, \mathfrak{M}_C)$, there is a natural bijection from $\mathscr{S}(C)$ onto the set of elements $c = ((u_{i,j}), (v_{\ell,j}), w_r) \in \mathfrak{M}_C^{2\,d\,N+d\,M+d}$ such that $q_c(T)$ divides the polynomial $\det(\mathscr{J}_V)(\tilde{x}_c(T), \tilde{y}_c(T))$ and the polynomial $q_c(T)^2$ divides the polynomial $[\mathrm{ad}(\mathscr{J}_V) \cdot \underline{F}](\tilde{x}_c(T), \tilde{y}_c(T))$.

◦ *Second case: we assume that $R = k[[T]]$ and V is a tft formal R-scheme.* Let $\mathcal{I}$ be the ideal of $k[[U, V, W]]$ generated by the power series, for every integer $\alpha \in \{0, \dots, d-1\}$,

$$\mathscr{D}_{d,2\,d\,N+d\,M,\alpha}(q(T), \det(\mathscr{J}_V)(\tilde{x}(T), \tilde{y}(T)))$$

and, for every integer $\alpha \in \{0, \dots, 2d-1\}$,

$$\mathscr{D}_{2\,d,2\,d\,N+d\,M,\alpha}(q(T)^2, [\mathrm{ad}(\mathscr{J}_V) \cdot \underline{F}](\tilde{x}(T), \tilde{y}(T))).$$

Arguing as in remark 4.12, one sees that $\mathcal{I}$ is contained in the maximal ideal of $k[[U, V, W]]$. Thus, one may set

$$\mathscr{S} := \mathrm{Spf}(k[[U, V, W]]/\mathcal{I}).$$

Thus, for every complete local k-algebra $(C, \mathfrak{M}_C)$, there is a natural bijection from $\mathscr{S}(C)$ onto the set of elements $c = ((u_{i,j}), (v_{\ell,j}), w_r) \in \mathfrak{M}_C^{2\,d\,N+d\,M+d}$ such that the polynomial $q_c(T)$ divides the formal series $\det(\mathscr{J}_V)(\tilde{x}_c(T), \tilde{y}_c(T))$ and the polynomial $q_c(T)^2$ divides the formal series $[\mathrm{ad}(\mathscr{J}_V) \cdot \underline{F}](\tilde{x}_c(T), \tilde{y}_c(T))$.

4.4. **Notation.** Let us begin by introducing the notation which will be used until the end of this section. Let us write $\gamma(T) = (x(T), y(T))$, where $x(T) \in k[[T]]^N$ and $y(T) \in k[[T]]^M$. We also write $x(T) = T^{2\,d} z(T) + \tilde{x}(T)$ where $\tilde{x}(T) \in (k[T]_{\leqslant 2\,d-1})^N$ and $z(T) \in k[[T]]^N$. Then, for every test-ring $(A, \mathfrak{M}_A)$ and every integer $n \geqslant 0$, we define:

- the set $\mathcal{A}(A) := \mathscr{L}_\infty(V)_\gamma(A)$ whose elements

$$(x_A(T), y_A(T)) \in A[[T]]^N \times A[[T]]^M$$

satisfy the following equations:

$$(x_A(T), y_A(T)) = \gamma(T) \mod \mathfrak{M}_A[[T]] \quad \text{and} \quad F_i(x_A(T), y_A(T)) = 0, \quad 1 \leqslant i \leqslant M;$$

- the set $\mathcal{A}_n(A) := \mathscr{L}_n(V)_{\gamma_n}(A)$ whose elements

$$(x_A(T), y_A(T)) \in (A[T]/\langle T^{n+1}\rangle)^N \times (A[T]/\langle T^{n+1}\rangle)^M$$

satisfy the following equations:

$$(x_A(T), y_A(T)) = \gamma(T) \mod \langle T^{n+1}, \mathfrak{M}_A\rangle$$

$$\text{and} \quad F_i(x_A(T), y_A(T)) = 0 \mod \langle T^{n+1}\rangle, \quad 1 \leqslant i \leqslant M;$$

- the set $\mathcal{B}(A)$ whose elements are of the form

$$(z_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T)) \in A[[T]]^N \times A[T]^N_{\leqslant 2d-1} \times A[T]^M_{\leqslant d-1} \times \mathscr{W}(A, d)$$

and satisfy the relations:

$$\left\{\begin{array}{l} z_A(T) = z(T) \mod \mathfrak{M}_A[[T]]; \\ \tilde{x}_A(T) = \tilde{x}(T) \mod \langle T^{2\,d}, \mathfrak{M}_A\rangle; \\ \tilde{y}_A(T) = y(T) \mod \langle T^d, \mathfrak{M}_A\rangle; \\ q_A(T) \text{ divides } \det(\mathscr{J}_V)(\tilde{x}_A(T), \tilde{y}_A(T)); \\ q_A(T)^2 \text{ divides } \operatorname{ad}(\mathscr{J}_V)\cdot \underline{F}(\tilde{x}_A(T), \tilde{y}_A(T)) \end{array}\right. .$$

(We recall that $\operatorname{ad}(\mathscr{J}_V)$ designates the adjugate of the square matrix $\mathscr{J}_V$, and $\underline{F} = {}^t(F_1 \ldots F_M)$.);

- the set $\mathcal{B}_n(A)$ whose elements are of the form

$$(z_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T)) \in (A[T]/\langle T^{n+1}\rangle)^N \times A[T]^N_{\leqslant 2d-1} \times A[T]^M_{\leqslant d-1} \times \mathscr{W}(A,d)$$

and satisfy the relations:

$$\left\{\begin{array}{l} z_A(T) = z(T) \mod \langle T^{n+1}, \mathfrak{M}_A\rangle; \\ \tilde{x}_A(T) = \tilde{x}(T) \mod \langle T^{2\,d}, \mathfrak{M}_A\rangle; \\ \tilde{y}_A(T) = y(T) \mod \langle T^d, \mathfrak{M}_A\rangle; \\ q_A(T) \text{ divides } \det(\mathscr{J}_V)(\tilde{x}_A(T), \tilde{y}_A(T)); \\ q_A(T)^2 \text{ divides } \operatorname{ad}(\mathscr{J}_V)\cdot \underline{F}(\tilde{x}_A(T), \tilde{y}_A(T)) \end{array}\right.$$

Remark 4.13. — The definitions of $\mathcal{A}(A)$, $\mathcal{A}_n(A)$, $\mathcal{B}(A)$ and $\mathcal{B}_n(A)$ are meaningful for every local k-algebra $(A, \mathfrak{M}_A)$.

4.5. **Proof of Theorems 4.1 and 4.2.** We now show that the formal k-scheme $\mathscr{S}$ described in subsection 4.3 realizes isomorphism (4.1). By subsection 2.1, we know that it is sufficient to construct a suitable natural bijection at the level of the A-points for every test-ring A, functorial in A.

The obvious map $\mathcal{A}(A) \to \mathcal{A}_n(A)$ corresponds to the morphism of formal schemes $\hat{\pi}_n^\infty\colon \mathscr{L}_\infty(V)_\gamma \to \mathscr{L}_n(V)_{\pi_n^\infty(\gamma)}$ deduced from the truncation morphism π_n^∞ at level n. Furthermore,we deduce from the very definitions, that, if $\mathscr{S}$ is the affine formal k-scheme constructed in subsection 4.3, there is a natural bijective map

$$\mathcal{B}(A) \to (\mathscr{S} \times_k \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]))(A).$$

The key point of the proof is to construct a natural map $\theta_A : \mathcal{A}(A) \to \mathcal{B}(A)$. We describe it from now on. For every $(x_A(T), y_A(T)) \in \mathcal{A}(A)$, by the Weierstrass Preparation Theorem, there is a unique decomposition

$$\det(\mathscr{J}_V)(x_A(T), y_A(T)) = q_A(T)u_A(T), \quad q_A(T) \in \mathscr{W}(A,d),\ u_A(T) \in A[[T]]^\times. \tag{4.2}$$

Let us write

$$x_A(T) = z_A(T)\, q_A(T)^2 + \tilde{x}_A(T), \quad \text{where } \tilde{x}_A(T) \in A[T]^N_{\leqslant 2d-1} \text{ and } z_A(T) \in A[[T]]^N,$$

and

$$y_A(T) = r_A(T)\, q_A(T) + \tilde{y}_A(T), \quad \text{where } \tilde{y}_A(T) \in A[T]^M_{\leqslant d-1} \text{ and } r_A(T) \in A[[T]]^M.$$

○ *Claim 1.* The tuple $(z_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T))$ belongs to $\mathcal{B}(A)$. The first three relations follow from the very definitions and the fact that $q_A(T) \in \mathscr{W}(A,d)$.

The first divisibility condition comes directly from equation (4.2) and the Taylor formula (see §3.1). As for the second divisibility condition, applying the Taylor formula to each of the F_i's yields the relation

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} F_1(x_A(T), y_A(T)) \\ \vdots \\ F_M(x_A(T), y_A(T)) \end{pmatrix}$$

$$= \begin{pmatrix} F_1(\tilde{x}_A(T), \tilde{y}_A(T)) \\ \vdots \\ F_M(\tilde{x}_A(T), \tilde{y}_A(T)) \end{pmatrix} + q_A(T)\mathscr{J}_V(\tilde{x}_A(T), \tilde{y}_A(T)) \begin{pmatrix} r_{1,A}(T) \\ \vdots \\ r_{M,A}(T) \end{pmatrix} \bmod \langle q_A(T)^2 \rangle.$$

Multiplying by $\mathrm{ad}(\mathscr{J}_V)(\tilde{x}_A(T), \tilde{y}_A(T))$, we obtain modulo $\langle q_A(T)^2 \rangle$:

$$\mathrm{ad}(\mathscr{J}_V)\cdot \begin{pmatrix} F_1 \\ \vdots \\ F_M \end{pmatrix} (\tilde{x}_A(T), \tilde{y}_A(T)) = -q_A(T)\cdot \underbrace{\det(\mathscr{J}_V)(\tilde{x}_A(T), \tilde{y}_A(T))}_{\text{divisible by } q_A(T)} \cdot \begin{pmatrix} r_{1,A}(T) \\ \vdots \\ r_{M,A}(T) \end{pmatrix}.$$

Let us note that the same kind of argument also shows that the two divisibility conditions in the definition of $\mathcal{B}(A)$ only depends on the classes in $A[[T]]$ of $\tilde{x}_A(T)$ modulo $q_A(T)^2$ and $\tilde{y}_A(T)$ modulo $q_A(T)$.

◦ *Definition of θ_k.* Then, we set

$$\theta_A((x_A(T), y_A(T))) = (z_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T)). \tag{4.3}$$

This definition is functorial in A in an obvious way, since for every test-ring A' and every local morphism $f\colon A \to A'$ between test-rings, the relation

$$\det(\mathscr{J}_V)(f_T(x_A(T)), f_T(y_A(T))) = f_T(q_A(T)) f_T(u_A(T))$$

is the Weierstrass decomposition of $\det(\mathscr{J}_V)(f_T(x_A(T)), f_T(y_A(T)))$, as we have seen in §3. Here, we denote by $f_T : A[[T]] \to A'[[T]]$ the morphism of local k-algebras induced by the action of f on coefficients. Then, by the Yoneda Lemma, the datum of $(\theta_A)_A$ for A running over the collection of test-rings corresponds to the datum of a morphism of formal k-schemes:

$$\theta_k(V)\colon \mathscr{L}_\infty(V)_\gamma \to \mathscr{S} \times_k \mathrm{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]).$$

Remark 4.14. — The construction of θ_A still makes sense if A is only assumed to be a complete local k-algebra.

◦ *Claim 2.* The morphism $\theta_k := \theta_k(V)$ is an isomorphism. By observation 2.1, we have to prove that, for every test-ring A, the map $\theta_A : \mathcal{A}(A) \to \mathcal{B}(A)$ is bijective.

Let $u(T) \in k[[T]]^\times$ such that

$$\det(\mathscr{J}_V)(x(T), y(T)) = T^d\, u(T). \tag{4.4}$$

We write

$$y(T) = w(T)\, T^d + \tilde{y}(T), \quad \tilde{y}(T) \in (k[T]_{\leqslant d-1})^M,\ w(T) \in k[[T]]^M.$$

Let $(z_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T)) \in \mathcal{B}(A)$. We set $x_A(T) = \tilde{x}_A(T) + q_A(T)^2 z_A(T)$ and $\hat{y}_A(T) = \tilde{y}_A(T) + q_A(T) w(T)$.

◦ Let us show that there exists a unique $w_A(T) \in \mathfrak{M}_A[[T]]^M$ such that, for every integer $i \in \{1, \ldots, M\}$,

$$F_i(x_A(T), \hat{y}_A(T) + w_A(T) q_A(T)) = 0.$$

First note that the latter requirement is equivalent to the condition

$$\mathrm{ad}(\mathscr{J}_V)(x_A(T), \hat{y}_A(T)) \begin{pmatrix} F_1(x_A(T), \hat{y}_A(T) + w_A(T)\, q_A(T)) \\ \vdots \\ F_M(x_A(T), \hat{y}_A(T) + w_A(T)\, q_A(T)) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Indeed, if the latter is satisfied, multiplying by $\mathscr{J}_V(x_A(T), \hat{y}_A(T))$, we obtain

$$\det(\mathscr{J}_V)(x_A(T), \hat{y}_A(T))\ F_i(x_A(T), \hat{y}_A(T) + w_A(T)\, q_A(T)) = 0$$

for every integer $i \in \{1, \dots, M\}$. But, $\det(\mathscr{J}_V)(x_A(T), \hat{y}_A(T))$ is not a zero divisor in $A[[T]]$, since its reduction modulo $\mathfrak{M}_A[[T]]$ is $\det(\mathscr{J}_V)(x(T), y(T)) \neq 0$ (see §3.2).

∘ There exists $u_A(T) \in A[[T]]^\times$ such that

$$\det(\mathscr{J}_V)(x_A(T), \hat{y}_A(T)) = q_A(T)\, u_A(T) \tag{4.5}$$

and $v_A(T) \in \mathfrak{M}_A[[T]]^M$ such that

$$\mathrm{ad}(\mathscr{J}_V) \cdot \begin{pmatrix} F_1 \\ \vdots \\ F_M \end{pmatrix} (x_A(T), \hat{y}_A(T)) = q_A(T)^2\, v_A(T). \tag{4.6}$$

By the Taylor formula (see §3.1), there exist polynomials or restricted power series $\{H_{i,j,k}\}_{1 \leqslant i,j,k \leqslant M}$ such that

$$\begin{aligned} &\begin{pmatrix} F_1 \\ \vdots \\ F_M \end{pmatrix} (x_A(T), \hat{y}_A(T) + w_A(T)\, q_A(T)) \\ &\quad = \begin{pmatrix} F_1 \\ \vdots \\ F_M \end{pmatrix} (x_A(T), \hat{y}_A(T)) + q_A(T) \cdot \mathscr{J}_V(x_A(T), \hat{y}_A(T)) \begin{pmatrix} w_{A,1}(T) \\ \vdots \\ w_{A,M}(T) \end{pmatrix} \\ &\qquad + q_A(T)^2 \sum_{1 \leqslant j,k \leqslant M} w_{A,j}(T) w_{A,k}(T) \begin{pmatrix} H_{1,j,k}(w_A(T)) \\ \vdots \\ H_{M,j,k}(w_A(T)) \end{pmatrix}. \end{aligned} \tag{4.7}$$

Note that the $\{H_{i,j,k}\}$ depend only on the F_i's, $x_A(T)$, $\hat{y}_A(T)$ and $q_A(T)$, and not on $w_A(T)$. Indeed, using the notation of §3.1, for every $i, j, k \in \{1, \dots, M\}^3$, one has

$$H_{i,j,k}(w_A(T)) = G_{i,j,k}(x_A(T), \hat{y}_A(T), q_A(T) w_A(T))$$

and the $\{G_{i,j,k}\}$ depend only on the F_i's.

Multiplying the above relation by $\mathrm{ad}(\mathscr{J}_V)(x_A(T), \hat{y}_A(T))$, we obtain that

$$\mathrm{ad}(\mathscr{J}_V)(x_A(T), \hat{y}_A(T)) \cdot \begin{pmatrix} F_1 \\ \vdots \\ F_M \end{pmatrix} (x_A(T), \hat{y}_A(T) + w_A(T)\, q_A(T)) \tag{4.8}$$

equals, up to modifying the $H_{i,j,k}$,

$$q_A(T)^2 \left[v_A(T) + u_A(T) \begin{pmatrix} w_{A,1}(T) \\ \vdots \\ w_{A,M}(T) \end{pmatrix} + \sum_{1\leqslant j,k\leqslant M} w_{A,j}(T) w_{A,k}(T) \begin{pmatrix} H_{1,j,k}(w_A(T)) \\ \vdots \\ H_{M,j,k}(w_A(T)) \end{pmatrix} \right].$$

Thus, thanks to the first remark and §3.2, we are reduced to prove that there exists a unique tuple $(w_{A,i}(T)) \in \mathfrak{M}_A[[T]]^M$ such that

$$v_A(T) + u_A(T)\, w_{A,i}(T) + \sum_{1\leqslant j,k\leqslant M} w_{A,j}(T) w_{A,k}(T) H_{i,j,k}(w_A(T)) = 0 \tag{4.9}$$

for every integer $i \in \{1,\dots,M\}$. This assertion directly follows from Lemma 4.15. Thus we have proved Claim 2 and the first assertion of the theorem.

Lemma 4.15. — *Let A be a test-ring. Let M be a positive integer, $u_A(T) \in A[[T]]^{\times}$ and $(v_{A,i}(T))_{i\in\{1,\dots,M\}} \in \mathfrak{M}_A[[T]]^M$. Let $\{H_{i,j,k}\}_{1\leqslant i,j,k\leqslant M}$ be polynomials or (restricted) power series in M variables with coefficients in $A[[T]]$. Then, there exists a unique M-tuple $(w_{A,i}(T))_{i\in\{1,\dots,M\}} \in \mathfrak{M}_A[[T]]^M$ which satisfies, for every integer $i \in \{1,\dots,M\}$, the equations*

$$v_{A,i}(T) + u_A(T)\, w_{A,i}(T) + \sum_{1\leqslant j,k\leqslant M} w_{A,j}(T) w_{A,k}(T) H_{i,j,k}(w_A(T)) = 0.$$

Remark 4.16. — Lemma 4.15 still holds if A is only assumed to be a local henselian k-algebra. Indeed, by [26, 28], one knows that the ring $A[[T]]$ is henselian, and we may solve the equation by standard characterizations of henselian rings.

Proof of Lemma 4.15. — Multiplying by $u_A(T)^{-1}$, we may assume $u_A(T) = 1$. We show the existence and uniqueness of the element $(w_{A,i}(T))$ by induction on the smallest integer ν such that $\mathfrak{M}_A^{\nu} = 0$. If $\nu = 1$, the assertion is obvious. Let $\nu \geqslant 2$. The assertion then holds for the test-ring $A/\mathfrak{M}_A^{\nu-1}$ by induction hypothesis. Thus, we may assume that there exists $(w_{A,i}(T)) \in \mathfrak{M}_A[[T]]^M$, unique up to the addition of an element of $\mathfrak{M}_A^{\nu-1}[[T]]^M$, such that, for every integer $i \in \{1,\dots,M\}$, we have:

$$v_{A,i}(T) + w_{A,i}(T) + \sum_{1\leqslant j,k\leqslant M} w_{A,j}(T) w_{A,k}(T) H_{i,j,k}(w_A(T)) =: \tilde{v}_{A,i}(T)$$

in $\mathfrak{M}_A^{\nu-1}[[T]]$. We have to show that there exists a unique $(\tilde{w}_{A,i}) \in \mathfrak{M}_A^{\nu-1}[[T]]^M$ such that, for every integer $i \in \{1,\dots,M\}$, the expression:

$$\begin{aligned} &v_{A,i}(T) + w_{A,i}(T) + \tilde{w}_{A,i}(T) \\ &\quad + \sum_{1\leqslant j,k\leqslant M} (w_{A,j}(T) + \tilde{w}_{A,j}(T))\,(w_{A,k}(T) + \tilde{w}_{A,k}(T)) H_{i,j,k}(w_A(T) + \tilde{w}_A(T)) \end{aligned} \tag{4.10}$$

vanishes. But, since for every integer $i \in \{1,\dots,M\}$ we have $w_{A,i}(T) \in \mathfrak{M}_A[[T]]$ and $\tilde{w}_{A,i}(T) \in \mathfrak{M}_A^{\nu-1}[[T]]$, relations (4.10) may be rewritten in the following form:

$$\tilde{v}_{A,i}(T) + \tilde{w}_{A,i}(T) = 0$$

for every integer $i \in \{1,\dots,M\}$, since $\mathfrak{M}_A^{\nu} = 0$. That concludes the proof. □

Remark 4.17. — By remark 4.16, the above arguments show in fact that, for every local admissible k-algebra A, the map θ_A is a bijection. Moreover, by the same remark, they show the existence of a family of functorial maps $\phi_A : \mathcal{B}(A) \to \mathcal{A}(A)$ for A running over the collection of local henselian k-algebras; if A is admissible, ϕ_A is a bijection and $\phi_A^{-1} = \theta_A$.

∘ *Claim 3.* The datum of the morphism θ_k is functorial in k. Let K be a field extension of the field k. Recall that we set $V_K = V \times_R \operatorname{Spec}(R_K)$ if V is an R-scheme of finite type and $V_K = V \times_R \operatorname{Spec}(R_K)$ if $R = k[[T]]$ and V is a formal tft R-scheme. It follows from the very definitions and scalar extension that the K-scheme $\mathscr{L}_\infty(V_K)$ is canonically isomorphic to $\mathscr{L}_\infty(V) \times_k \operatorname{Spec}(K)$. In particular, if the extension K of k is separable, it corresponds to $\gamma \in \mathscr{L}_\infty^\circ(V)$ a unique element of $\mathscr{L}_\infty^\circ(V_K)$ which also is denoted by γ. Let A be a test-ring over K. Then, again by scalar extention,

$$\mathscr{L}_\infty(V)_\gamma(A) \cong \mathscr{L}_\infty(V_K)_\gamma(A)$$

is a natural bijection. Thus, $\mathscr{L}_\infty(V)_\gamma \times_k \operatorname{Spec}(K) \cong \mathscr{L}_\infty(V_K)_\gamma$. From this, we easily conclude that $\theta_K(V_K) \cong \theta_k(V) \times_k \operatorname{Spec}(K)$ by construction. That concludes the proof of Claim 3.

∘ *Claim 4.* For every integer $n \geqslant 0$, the morphism of formal k-schemes $\hat{\pi}_n^\infty$ factorizes through $\mathscr{S} \times_k \operatorname{Spf}(k[[T_0, \ldots, T_n]])$. By observation 2.1, it is equivalent to prove, that for every integer $n \geqslant 0$, the natural map $\mathcal{A}(A) \to \mathcal{A}_n(A)$ factorizes through $\mathcal{B}_n(A)$. Let us begin by showing the following lemma:

Lemma 4.18. — *Let A be a test-ring and M be a positive integer. Suppose $u_A(T), \bar{u}_A(T) \in A[[T]]^\times$ and $(v_{A,i}(T)), (\bar{v}_{A,i}(T)) \in \mathfrak{M}_A[[T]]^M$. Let $\{H_{i,j,k}\}_{1 \leqslant i,j,k \leqslant M}$ and $\{\bar{H}_{i,j,k}\}_{1 \leqslant i,j,k \leqslant M}$ be polynomials or (restricted) power series in M variables with coefficients in $A[[T]]$. Let $n \geqslant 0$ be an integer such that, for every integer $i, j, k \in \{1, \ldots, M\}$, the following equalities hold:*

$$\begin{cases} v_{A,i}(T) &= \bar{v}_{A,i}(T) & \mod \langle T^{n+1} \rangle, \\ u_A(T) &= \bar{u}_A(T) & \mod \langle T^{n+1} \rangle, \\ H_{i,j,k} &= \bar{H}_{i,j,k} & \mod \langle T^{n+1} \rangle. \end{cases}$$

Let $(w_{A,i}(T))$ (respectively $(\bar{w}_{A,i}(T))$) be the unique element of $\mathfrak{M}_A[[T]]^M$ satisfying, for every integer $i \in \{1, \ldots, M\}$, the equations:

$$v_{A,i}(T) + u_A(T)\, w_{A,i}(T) + \sum_{1 \leqslant j,k \leqslant M} w_{A,j}(T) w_{A,k}(T) H_{i,j,k}(w_A(T)) = 0$$

respectively

$$\bar{v}_{A,i}(T) + \bar{u}_A(T)\, \bar{w}_{A,i}(T) + \sum_{1 \leqslant j,k \leqslant M} \bar{w}_{A,j}(T) \bar{w}_{A,k}(T) \bar{H}_{i,j,k}(\bar{w}_A(T)) = 0$$

(which exists by Lemma 4.15). Then, for every integer $i \in \{1, \ldots, M\}$ we have:

$$w_{A,i}(T) = \bar{w}_{A,i}(T) \mod \langle T^{n+1} \rangle.$$

Proof. — We prove this by induction on the smallest integer ν such that $\mathfrak{M}_A^\nu = 0$. If $\nu = 1$, the assertion is obvious. Let $\nu \geqslant 2$. By Lemma 4.15 and the induction hypothesis, there exists $(w_{A,i,\nu-1}(T)), (\bar{w}_{A,i,\nu-1}(T)) \in \mathfrak{M}_A[[T]]^M$ such that, for every integer $i \in \{1,\dots,M\}$, the elements $\tilde{v}_{A,i}(T)$ and $\bar{\tilde{v}}_{A,i}(T)$, respectively defined by:

$$v_{A,i}(T) + u_A(T)\, w_{A,i,\nu-1}(T) + \sum_{1\leqslant j,k\leqslant M} w_{A,j,\nu-1}(T) w_{A,k,\nu-1}(T) H_{i,j,k}(w_{A,\nu-1}(T))$$

$$\bar{v}_{A,i}(T) + \bar{u}_A(T)\, \bar{w}_{A,i,\nu-1}(T) + \sum_{1\leqslant j,k\leqslant M} \bar{w}_{A,j,\nu-1}(T) \bar{w}_{A,k,\nu-1}(T) \bar{H}_{i,j,k}(\bar{w}_{A,\nu-1}(T)), \tag{4.11}$$

belong to $\in \mathfrak{M}_A^{\nu-1}[[T]]$ and, for every integer $i \in \{1,\dots,M\}$,

$$w_{A,i,\nu-1}(T) = \bar{w}_{A,i,\nu-1}(T) \qquad \mathrm{mod}\ \langle \mathfrak{M}_A^{\nu-1}[[T]], T^{n+1}\rangle.$$

By adding to the $\bar{w}_{A,i,\nu-1}(T)$ suitable elements of $\mathfrak{M}_A^{\nu-1}[[T]]$, we may assume that, for every integer $i \in \{1,\dots,M\}$,

$$w_{A,i,\nu-1}(T) = \bar{w}_{A,i,\nu-1}(T) \qquad \mathrm{mod}\ \langle T^{n+1}\rangle,$$

and (4.11) still holds. Then, by the assumptions of the lemma, for every integer $i \in \{1,\dots,M\}$, we have

$$\tilde{v}_{A,i}(T) = \bar{\tilde{v}}_{A,i}(T) \qquad \mathrm{mod}\ \langle T^{n+1}\rangle. \tag{4.12}$$

Moreover, for every integer $i \in \{1,\dots,M\}$, thanks to the uniqueness of the involved objects, we have

$$\begin{array}{rcl} w_{A,i}(T) & = & w_{A,i,\nu-1}(T) - u_A(T)^{-1}\tilde{v}_{A,i}(T), \\ \bar{w}_{A,i}(T) & = & \bar{w}_{A,i,\nu-1}(T) - \bar{u}_A(T)^{-1}\bar{\tilde{v}}_{A,i}(T). \end{array} \tag{4.13}$$

Hence, by the assumptions of the lemma, we deduce, from equations (4.12) and (4.13), that

$$w_{A,i}(T) = \bar{w}_{A,i}(T) \qquad \mathrm{mod}\ \langle T^{n+1}\rangle.$$

□

Then, Proposition 4.19 directly implies Claim 4.

Proposition 4.19. — *Let A be a test-ring. Let $(x_A(T), y_A(T)) \in \mathcal{A}(A)$ and set $\theta_A(x_A(T), y_A(T)) := (z_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T))$. Let $n \geqslant 0$ be an integer and $\bar{z}_A(T) \in A[[T]]^N$ such that $\bar{z}_A(T) = z_A(T) \mod \langle T^{n+1}\rangle$. Let*

$$(\bar{x}_A(T), \bar{y}_A(T)) := \theta_A^{-1}(\bar{z}_A(T), \tilde{x}_A(T), \tilde{y}_A(T), q_A(T))).$$

Then, we have the following relations:

$$\left\{\begin{array}{rcll} \bar{x}_A(T) & = & x_A(T) & \mathrm{mod}\ \langle T^{n+1}\rangle \\ \bar{y}_A(T) & = & y_A(T) & \mathrm{mod}\ \langle T^{n+1}\rangle. \end{array}\right.$$

Proof. — Let us recall some notation used in the proof of Claim 2: let $y(T) = w(T)\,T^d + \tilde{y}(T)$ be the division of $y(T)$ by T^d, and $\hat{y}_A(T) = \tilde{y}_A(T) + q_A(T)w(T)$. By the description of the map θ_A, we also have $x_A(T) = \tilde{x}_A(T) + q_A(T)^2 z_A(T)$ and $\bar{x}_A(T) = \tilde{x}_A(T) + q_A(T)^2 \bar{z}_A(T)$. Since, by assumption, one has $\bar{z}_A(T) = z_A(T)$ mod $\langle T^{n+1}\rangle$, we obtain in particular that

$$\bar{x}_A(T) = x_A(T) \quad \mathrm{mod}\ \langle T^{n+1}\rangle. \tag{4.14}$$

Recall from equations (4.5) and (4.6) the definitions of $u_A(T)$ and $v_A(T)$, and let us write

$$\det(\mathscr{J}_V)(\bar{x}_A(T), \hat{y}_A(T)) = q_A(T)\, \bar{u}_A(T)$$

and

$$\operatorname{ad}(\mathscr{J}_V)\cdot\begin{pmatrix}F_1\\ \vdots\\ F_M\end{pmatrix}(\bar{x}_A(T),\hat{y}_A(T)) = q_A(T)^2\,\bar{v}_A(T).$$

By the Taylor formula (see §3.1) and the assumptions of the proposition, we see that $\bar{u}_A(T)$ and the components of $\bar{v}_A(T)$ are polynomials (or restricted power series) with coefficients in $A[[T]]$ (depending only on the F_i's, $\tilde{x}_A(T)$, $\tilde{y}_A(T)$, $q_A(T)$ and $w(T)$) in the components of $\bar{z}_A(T)$. We deduce that

$$\begin{cases}\bar{u}_A(T) &= & u_A(T) \mod \langle T^{n+1}\rangle\\ \bar{v}_A(T) &= & v_A(T) \mod \langle T^{n+1}\rangle.\end{cases}$$

Moreover, we have by the proof of Claim 2

$$\bar{y}_A(T) = \hat{y}_A(T) + \bar{w}_A(T)q_A(T),$$

where $\bar{w}_A(T)$ is the unique element of $\mathfrak{M}_A[[T]]^M$ satisfying, for every integer $i \in \{1,\dots,M\}$,

$$\bar{v}_{A,i}(T) + \bar{u}_A(T)\,\bar{w}_{A,i}(T) + \sum_{1\leqslant j,k\leqslant M}\bar{w}_{A,j}(T)\bar{w}_{A,k}(T)\bar{H}_{i,j,k}(\bar{w}_A(T)) = 0,$$

for suitable polynomials (or restricted power series) $\bar{H}_{i,j,k}$. Moreover, it transpires from the proof of Claim 2 that equation (4.14) implies the relations

$$\bar{H}_{i,j,k} = H_{i,j,k} \mod \langle T^{n+1}\rangle$$

for every integer $i,j,k \in \{1,\dots,M\}$. From Lemma 4.18, we deduce that $\bar{w}_A(T) = w_A(T) \mod \langle T^{n+1}\rangle$. Since $y_A(T) = \hat{y}_A(T) + w_A(T)q_A(T)$, we have $\bar{y}_A(T) = y_A(T) \mod \langle T^{n+1}\rangle$. □

5. Further results

In this section, we complete the analysis of Theorem 4.1, in the direction of its relations with the Denef-Loeser Fibration Theorem, and formulate a global version of this statement. The proofs use arguments strongly similar to those used in the previous section and are sketched or omitted.

5.1. **Analogue of the Denef-Loeser Fibration Theorem.** We explain how a formal analogue of the Denef-Loeser Fibration Theorem can be obtained by following the spirit of Theorem 4.1.

Proposition 5.1. — *Let k be a field and $R \in \{k, k[[T]]\}$. Let V be a R-space. Let $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$. Let n be a nonnegative integer and set $\mathscr{L}_\infty(V)^{(\gamma,n)} := (\pi_n^\infty)^{-1}(\pi_n^\infty(\gamma))$. Then there exists an affine noetherian adic formal k-scheme $\mathscr{S}$ and an isomorphism of formal k-schemes:*

$$\theta_k^{(n)}(V)\colon \mathscr{L}_\infty(V)_\gamma^{(\gamma,n)} \xrightarrow{\sim} \mathscr{S}\times_k \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]]). \tag{5.1}$$

Remark 5.2. — For n large enough, one has $\mathscr{L}_\infty(V)_\gamma^{(\gamma,n)} \cong \operatorname{Spf}(k[[(T_i)_{i\in\mathbf{N}}]])$. This may be shown using arguments very similar to those needed in the proof of the Denef-Loeser Fibration Theorem (see [10, Lemma 4.1] and [29, Lemme 4.5.4] in the formal case).

Proof (sketch). — Owing to §4.2, we reduce to the case where the R-space V is defined by M equations $F_1,\dots,F_M$ in the variables $X_1,\dots,X_N,Y_1,\dots,Y_M$, and $\gamma\in\mathscr{L}_\infty(V)(k)$ has the property that $\det(\mathscr{J}_V)$ does not vanish at γ, where $\mathscr{J}_V:=[\partial_{Y_j}F_i]_{1\leqslant i,j\leqslant M}$.

Let $d:=\mathrm{ord}_T\,[\det(\mathscr{J}_V)(\gamma)]$. Write $\gamma(T)=(x(T),y(T))$, where $x(T)\in k[[T]]^N$ and $y(T)\in k[[T]]^M$. Let n be an nonnegative integer. Let us write

$$x(T)=x^{(0)}(T)+T^{n+1}x^{(1)}(T)\text{ and }y(T)=y^{(0)}(T)+T^{n+1}y^{(1)}(T),$$

where $x^{(0)}(T),y^{(0)}(T)\in k[T]_{\leqslant n}$. We also write $x^{(1)}(T)=T^{2\,d}z(T)+\tilde{x}(T)$, where $\tilde{x}(T)\in k[T]^N_{\leqslant 2\,d-1}$ and $z(T)\in k[[T]]^N$.
Then, for every test-ring A, we define:

- the set $\mathcal{A}(A):=\mathscr{L}_\infty(V)^{(\gamma,n)}_\gamma(A)$ whose elements
$$(x_A(T),y_A(T))\in A[[T]]^N\times A[[T]]^M$$
satisfy the following equations:
$$(x_A(T),y_A(T))=(x^{(1)}(T),y^{(1)}(T))\mod \mathfrak{M}_A[[T]]\quad\text{and}$$
$$F_i(x^{(0)}(T)+T^{n+1}x_A(T),y^{(0)}(T)+T^{n+1}y_A(T))=0,\quad 1\leqslant i\leqslant M;$$
- the set $\mathcal{B}(A)$ whose elements are of the form
$$(z_A(T),\tilde{x}_A(T),\tilde{y}_A(T),q_A(T))\in A[[T]]^N\times A[T]^N_{\leqslant 2d-1}\times A[T]^M_{\leqslant d-1}\times\mathscr{W}(A,d)$$
and satisfy the relations:

$$\begin{cases} z_A(T)=z(T) & \mod \mathfrak{M}_A[[T]];\\ \tilde{x}_A(T)=\tilde{x}(T) & \mod \langle T^{2d},\mathfrak{M}_A\rangle;\\ \tilde{y}_A(T)=y^{(1)}(T) & \mod \langle T^{d},\mathfrak{M}_A\rangle;\\ q_A(T)\mid \det(\mathscr{J}_V)(x^{(0)}(T)+T^{n+1}\tilde{x}_A(T),y^{(0)}(T)+T^{n+1}\tilde{y}_A(T)); & \\ T^{n+1}\,q_A(T)^2\mid \mathrm{ad}(\mathscr{J}_V)\cdot\underline{F}(x^{(0)}(T)+T^{n+1}\tilde{x}_A(T),y^{(0)}(T)+T^{n+1}\tilde{y}_A(T)). & \end{cases}$$

Now one has to construct a natural bijection from $\mathcal{A}(A)$ to $\mathcal{B}(A)$, functorial in A. One does this with essentially the same kind of arguments as in §4.5, still using as a crucial ingredient the Weierstrass Preparation Theorem. One replaces the use of Lemma 4.15 by that of the more elementary version given by Lemma 5.3 below ☐

Lemma 5.3. — *Let A be a k-algebra and let M be a positive integer. Suppose $(w_{A,i}(T))_{i\in\{1,\dots,M\}}\in A[[T]]^M$, $u_A(T)\in A[[T]]^\times$ and $(v_{A,i}(T))_{i\in\{1,\dots,M\}}\in A[[T]]^M$. For every integer $i\in\{1,\dots,M\}$ let H_i be a polynomial or a (restricted) power series in M variables with coefficients in $A[[T]]$. Then there exists a unique element $(w_{A,i}(T))_{i\in\{1,\dots,M\}}\in A[[T]]^M$ which satisfies, for every integer $i\in\{1,\dots,M\}$, the equations:*

$$v_{A,i}(T)+u_A(T)\,w_{A,i}(T)+T\,H_i(w_A(T))=0. \tag{5.2}$$

The proof of Lemma 5.3 easily follows from coefficient identification.

5.2. **A global version of Theorem 4.1.** Lemma 5.3 also allows to establish more global versions of Theorem 4.1. Let us now describe a result which may be obtained in this way. This is a somewhat expanded version of a result appearing in [6]. Let d be a nonnegative integer. Let $\mathscr{L}_\infty(\mathbf{A}^1)^{=d}$ be the locally closed subscheme of $\mathscr{L}_\infty(\mathbf{A}^1)$ defined, for every k-algebra A, by

$$\mathscr{L}_\infty(\mathbf{A}^1)^{=d}(A) = \{\gamma \in A[[T]],\, \gamma_0 = \cdots = \gamma_{d-1} = 0,\, \gamma_d \in A^\times\}.$$

We consider an R-scheme of finite type V and $\gamma \in \mathscr{L}^\circ_\infty(V)(k)$ such that locally aroung γ one may write

$$\mathscr{L}_\infty(V) = \mathscr{L}_\infty(\operatorname{Spec}(R[X_1,\dots,X_N,Y_1,\dots,Y_M]/\langle F_1,\dots,F_M\rangle))$$

where $F_1,\dots,F_M \in R[\underline{X},\underline{Y}]$ and $\det(\mathscr{J}_V)(\gamma(T)) \neq 0$, where $\mathscr{J}_V = [\partial_{Y_j} F_i]_{1\leqslant i,j\leqslant M}$. This property holds for example in case V is locally a complete intersection.

Let us consider the morphism $\delta\colon \mathscr{L}_\infty(V) \to \mathscr{L}_\infty(\mathbf{A}^1)$ mapping γ to $\det(\mathscr{J}_V)(\gamma)$. For every non-negative integer d, we set $\mathscr{L}_\infty(V)^{=d}$ to be the locally closed subscheme of $\mathscr{L}_\infty(V)$ defined by $\delta^{-1}(\mathscr{L}_\infty(\mathbf{A}^1)^{=d})$. Similarly, we define the locally closed subscheme $\mathscr{L}_n(V)^{=d}$ of $\mathscr{L}_n(V)$.

Let r and d be positive integers. We define the locally closed subscheme $Y^{=d,r}$ of $(\mathbf{A}_k^{(r+1)\,d})^N \times (\mathbf{A}_k^{r\,d})^M$ as follows: for every k-algebra A, let $Y^{=d,r}(A)$ be the set of elements

$$(\tilde{x}_A(T),\tilde{y}_A(T)) \in A[T]^N_{\leqslant (r+1)\,d-1} \times A[T]^M_{\leqslant r\,d-1}$$

such that $\det(\mathscr{J}_V)(\tilde{x}_A(T),\tilde{y}_A(T)) = T^d u_A(T)$ with $u_A(T) \in A[[T]]^\times$ (in other words $u_A(0) \in A^\times$) and

$$\operatorname{ord}_T\left(\operatorname{ad}(\mathscr{J}_V)\cdot\begin{pmatrix}F_1\\ \vdots\\ F_M\end{pmatrix}(\tilde{x}_A(T),\tilde{y}_A(T))\right) \geqslant (r+1)\,d.$$

If $r' \geqslant r$, we note using the Taylor formula (see §3.1) and the very definitions that maping $(\tilde{x}(T),\tilde{y}(T))$ to

$$\left(\tilde{x}(T) \mod T^{(r+1)d}, \tilde{y}(T) \mod T^{r\,d}, \frac{1}{T^{(r+1)d}}(\tilde{x}(T) - (\tilde{x}(T) \mod T^{(r+1)d}))\right)$$

defines a morphism $\iota_r^{r'} : Y^{=d,r'} \to Y^{=d,r} \times (\mathbf{A}_k^{(r-r')d})^N$.

Now consider $r \geqslant 2$ and n such that $d \leqslant n \leqslant r\,d-1$. Then if $(\tilde{x}_A(T),\tilde{y}_A(T)) \in Y^{=d,r}(A)$ we have $\operatorname{ord}_T(F_i(\tilde{x}_A(T),\tilde{y}_A(T))) \geqslant r\,d$ for every integer $i \in \{1,\dots,M\}$. This shows that maping $(\tilde{x}(T),\tilde{y}(T))$ to $(\tilde{x}(T) \mod T^{n+1}, \tilde{y}(T) \mod T^{n+1})$ defines a morphism

$$\alpha_n^{=d,r} : Y^{=d,r} \to \mathscr{L}_n(V)^{=d}.$$

Proposition 5.4. — *Let $d > 0$. For every $r' \geqslant r \geqslant 2$, $\iota_r^{r'}$ is an isomorphism. Moreover, for every $r \geqslant 2$, there is an isomorphism of schemes*

$$\theta^{=d,r} : Y^{=d,r} \times \mathscr{L}_\infty(\mathbf{A}_k^N) \xrightarrow{\sim} \mathscr{L}_\infty(V)^{=d}$$

with the following properties:

(1) *for every $r' \geqslant r \geqslant 2$, we have a commutative diagram*

$$\begin{tikzcd}
Y^{=d,r'} \times \mathscr{L}_\infty(\mathbf{A}_k^N) \arrow[r, "\theta^{=d,r'}"] \arrow[d, "\iota_r^{r'} \times \mathrm{Id}"'] & \mathscr{L}_\infty(V)^{=d} \\
Y^{=d,r} \times (\mathbf{A}_k^{(r'-r)d})^N \times \mathscr{L}_\infty(\mathbf{A}_k^N) \arrow[d] & \\
Y^{=d,r} \times \mathscr{L}_\infty(\mathbf{A}_k^N) \arrow[uur, bend right, "\theta^{=d,r}"'] &
\end{tikzcd}$$

where the lower vertical arrow is induced by the isomorphism

$$\mathbf{A}_k^{(r'-r)d} \times \mathscr{L}_\infty(\mathbf{A}_k^1) \xrightarrow{\sim} \mathscr{L}_\infty(\mathbf{A}_k^1) \quad \textit{given by} \quad (p(T), \gamma(T)) \mapsto p(T) + T^{(r'-r)d}\gamma(T);$$

(2) *for every $r \geqslant 2$ and any $d \leqslant n \leqslant n' \leqslant r\,d - 1$, the diagram*

$$\begin{tikzcd}
\mathscr{L}_\infty(V)^{=d} \arrow[r, "\pi^\infty_{n'}"] \arrow[d, "\theta^{=d,r}"'] & \mathscr{L}_{n'}(V)^{=d} \arrow[r, "\pi^{n'}_n"] & \mathscr{L}_n(V)^{=d} \\
Y^{=d,r} \times \mathscr{L}_\infty(\mathbf{A}_k^N) \arrow[r, "\mathrm{pr}_1"'] & Y^{=d,r} \arrow[u, "\alpha_{n'}^{=d,r}"] \arrow[ur, "\alpha_n^{=d,r}"'] &
\end{tikzcd}$$

is commutative ; moreover, $\alpha_n^{=d,r}$ is a piecewise trival fibration with fiber $\mathbf{A}_k^{(r+1)(d-n)}$ over its image (see [13, §4] *for a precise definition).*

6. Examples

Though the procedure described previously to construct a formal scheme $\mathscr{S}$ realizing isomorphism (4.1) is completely explicit, obtaining a sensible and useful description of $\mathscr{S}$ seems to remain a difficult problem, both from the computational and theoretical point of view. We give below some examples where a simple description may be achieved. Example 6.3 shows in particular that the finite dimensional models, in general, depend on the choice of the involved arc.

6.1. **Example.** Let k be a field whose characteristic does not divide 2 or 3. Let $\mathscr{C}$ be the affine plane k-curve defined by the polynomial $X^3 - Y^2 \in k[X, Y]$ and let $\gamma(T) = (T^2, T^3)$. Applying the algorithm described in section 4 shows that equations for a formal k-scheme realizing isomorphism (4.1) may be derived from the relations

$$q_A(T) \text{ divides } \tilde{y}_A(T), \quad q_A(T)^2 \text{ divides } \tilde{x}_A(T)^3 - \tilde{y}_A(T)^2$$

where A is a test-ring, $q_A(T)$ is a Weierstrass polynomial of degree 3, $\tilde{x}_A(T)$ (resp. $\tilde{y}_A(T)$) is a polynomial of degree $\leqslant 5$ (resp. $\leqslant 2$); note that in particular one must have $\tilde{y}_A(T) = 0$. Subsequent easy simplifications show that

$$\mathscr{S} = \mathrm{Spf}(k[[X_0]]/\langle X_0^2 \rangle)$$

realizes isomorphism (4.1). Note that $\mathscr{S}$ is clearly the minimal formal model. In [3], the authors generalize this result to the case of the plane curve singularity A_n for every even integer $n \geqslant 1$.

6.2. **Example.** We describe a family of examples generalizing slightly explicit examples presented in [17, 11]. Let $n, d \geqslant 1$ be integers, and F be a polynomial with n indeterminates. We consider the hypersurface $V \ : \ Y\, X_{n+1} + F(X_1, \dots, X_n) = 0$ and the arc $\gamma(T) = (0, \dots, 0, T^d, 0)$.

Let A be a test-ring and

$$((x_{i,A}(T))_{1 \leqslant i \leqslant n+1}, y_A(T)) \in \mathscr{L}_\infty(V)_\gamma(A).$$

Let $x_{n+1,A}(T) = q_A(T) u_A(T)$, with $q_A(T) \in \mathscr{W}(A, d)$ and $u_A(T) \in 1 + \mathfrak{M}_A[[T]]$, be the Weierstrass factorization of $x_{n+1,A}(T)$. For every integer $i \in \{1, \dots, n\}$, let $\tilde{x}_{i,A}(T)$ be the remainder of the euclidean division of $x_{i,A}(T)$ by $q_A(T)$.

Using the Taylor expansion and the uniqueness and functoriality of the Weiestrass factorization, one obtains that the equations of a pointed k-scheme (S, s) realizing isomorphism (1.1) for (V, γ) are determined by the relation

$$q_A(T) \text{ divides } F(\tilde{x}_{i,A}(T))_{1 \leqslant i \leqslant n}. \tag{6.1}$$

When $d = 1$, the elements $\tilde{x}_{i,A}(T) = \tilde{x}_{i,A}$ are constant polynomials and (6.1) is equivalent to $F(\tilde{x}_{i,A})_{1 \leqslant i \leqslant n} = 0$. This is the example given in [11], of which [17, Examples 1 and 2] are particular cases. See also [8].

When $d = 2$ and $\deg(F) = 2$, writing $\tilde{x}_{i,A} = \tilde{x}_{i,A}^{(0)} + \tilde{x}_{i,A}^{(1)} T$ and $q_A(T) = q_A^{(0)} + q_A^{(1)} T + T^2$ and considering the involved degrees, we see that (6.1) is equivalent to

$$q_A(T)\, F(\tilde{x}_{i,A}^{(1)})_{1 \leqslant i \leqslant n} = F(\tilde{x}_{i,A}^{(0)} + \tilde{x}_{i,A}^{(1)} T)_{1 \leqslant i \leqslant n}$$

Identifying coefficients, we find that the latter equation is equivalent to the two relations

$$\begin{cases} q_A^{(0)}\, F(\tilde{x}_{i,A}^{(1)})_{1 \leqslant i \leqslant n} & = & F(\tilde{x}_{i,A}^{(0)})_{1 \leqslant i \leqslant n} \\ q_A^{(1)}\, F(\tilde{x}_{i,A}^{(1)})_{1 \leqslant i \leqslant n} & = & \sum_{1 \leqslant j \leqslant n} \tilde{x}_{j,A}^{(1)} (\partial_{X_j} F)(\tilde{x}_{i,A}^{(0)})_{1 \leqslant i \leqslant n}. \end{cases}$$

When $F = \sum_{1 \leqslant i \leqslant n} X_i^2$, we recover [17, Example 3].

6.3. **Example.** Let k be a field whose characteristic is not 2 and V be the affine k-surface defined by the polynomial $Y\, Z + X^2 \in k[X, Y, Z]$ in $\mathbf{A}_k^3$.

Let $\gamma(T) = (0, 0, T)$. By example 6.2, one has an isomorphism $\mathscr{L}_\infty(V)_\gamma \cong \mathscr{S} \times_k \mathrm{Spf}(k[[(T_i)_{i \in \mathbf{N}}]])$ where $\mathscr{S} = \mathrm{Spf}(k[[X_0]]/\langle X_0^2 \rangle)$; hence, the nilpotency index $m_\gamma(V) = 2$ (see section 8.1 for a definition of this integer).

Now let $\eta(T) = (0, 0, T^2)$. Again by example 6.2 one has an isomorphism $\mathscr{L}_\infty(V)_\gamma \cong \mathscr{S} \times_k \mathrm{Spf}(k[[(T_i)_{i \in \mathbf{N}}]])$ where

$$\mathscr{S} = \mathrm{Spf}(k[[X_0, X_1, Q_0, Q_1]]/\langle Q_0\, X_1^2 - X_0^2, Q_1 X_1^2 - 2\, X_1\, X_0 \rangle).$$

We will show that $m_\eta(V) \geqslant 3$; thus the finite dimensional models associated with γ and η are different.

One checks easily (with SAGE or by direct computation) that $(X_1\, Q_1 - 2\, X_0)^3$ lies in the ideal of $k[X_0, X_1, Q_0, Q_1]$ generated by $Q_0\, X_1^2 - X_0^2$ and $Q_1 X_1^2 - 2\, X_1\, X_0$. Thus to prove $m_\eta(V) \geqslant 3$ it suffices to show that $(X_1\, Q_1 - 2\, X_0)^2$ is not in the ideal I of $k[[X_0, X_1, Q_0, Q_1]]$ generated by $Q_0\, X_1^2 - X_0^2$ and $Q_1 X_1^2 - 2\, X_1\, X_0$. One has

$$(X_1\, Q_1 - 2\, X_0)^2 = X_1^2\, Q_1^2 + 4\, X_0^2 - 4\, X_0\, X_1 Q_1 = X_1^2\, Q_1^2 + 4\, Q_0 X_1^2 - 2\, Q_1^2 X_1^2 \mod I;$$

hence $(X_1\,Q_1 - 2\,X_0)^2 = (4\,Q_0 - Q_1)\,X_1^2 \mod I$. Now assume that there exist $A, B \in k[[X_0, X_1, Q_0, Q_1]]$ such that

$$(4\,Q_0 - Q_1)\,X_1^2 = (Q_0\,X_1^2 - X_0^2)\,A + (Q_1 X_1^2 - 2\,X_1\,X_0)\,B.$$

Then X_1 must divide A and, replacing A by $X_1\,A$, we are led to the relation

$$(4\,Q_0 - Q_1)\,X_1 = (Q_0\,X_1^2 - X_0^2)\,A + (Q_1 X_1 - 2\,X_0)\,B.$$

Identifying homogeneous components, we see that one must have $B(0) = 0$ and

$$(4\,Q_0 - Q_1)\,X_1 = -X_0^2\,A(0) - 2\,X_0\,B_1.$$

where B_1 is the linear component of B. Thus we have a contradiction.

7. Gabber's Cancellation Theorem and minimal formal models

The aim of this appendix is to present Theorem 7.1, which is a cancellation theorem in the context of formal geometry (in arbitrary dimension). As indicated in introduction, it is, in particular, an important piece in the interpretation of Theorem 4.1 with respect to singularity theory. This result is, to the best of our knowledge, new, and its statement, as well as its proof, were kindly communicated to us by O. Gabber. We would like to thank him for allowing us to reproduce them in the present article.

Let us fix some notation used in this section. Let us consider the category $\mathfrak{Lncp}$, formed by the *complete local noetherian k-algebras with residue field k-isomorphic to k* and continuous (local) morphisms. One says that a k-algebra $A \in \mathfrak{Lncp}$ is *cancellable* (in $\mathfrak{Lncp}$) if there exists a k-algebra $B \in \mathfrak{Lncp}$ such that A is isomorphic to $B[[T]]$. Let us note that, obviously, for every k-algebra $A \in \mathfrak{Lncp}$, there exists $N \in \mathbf{N}$ and a non-cancellable k-algebra $B \in \mathfrak{Lncp}$ such that A is isomorphic to $B[[T_1, \dots, T_N]]$.

7.1. Gabber's Theorem.

We state this important result.

Theorem 7.1 (O. Gabber [15]). — *Let k be a field. Let $A, B \in \mathfrak{Lncp}$, and let I, J be sets (possibly infinite). Assume that the admissible k-algebras $A[[(T_i)_{i\in I}]]$ and $B[[(U_j)_{j\in J}]]$ are isomorphic. Then, up to exchanging A and B, there exists a finite subset $I' \subset I$ such that $A[[(T_i)_{i\in I'}]]$ and B are isomorphic (in $\mathfrak{Lncp}$). In particular, if both A and B are non-cancellable, then they are isomorphic.*

A slighty weaker statement of this theorem appears in [7]. We would like to thank A. Bouthier for interesting discussions on this topic. Theorem 7.1 also generalizes an older version valid in case the sets I, J are finite, see [21, Theorem 4].

Remark 7.2. — Let $A \in \mathfrak{Lncp}$ and I be an infinite set. Before engaging into the proof, a word of explanation is in order regarding the definition of the admissible k-algebra $A' = A[[(T_i)_{i\in I}]]$. We endow the k-algebra $k[[(T_i)_{i\in I}]]$ defined to be $\varprojlim_N k[(T_i)_{i\in I}]/\langle (T_i)_{i\in I}\rangle^N$, with the inverse limit topology, for which it is complete. Recall that this topology on the local k-algebra $k[[(T_i)_{i\in I}]]$ does not coincide with its adic topology (see remark 2.3). Then A' may be defined as the completed tensor product $A\hat{\otimes}_k k[[(T_i)_{i\in I}]]$. In particular, if $\mathfrak{M}_{A'}$ is the maximal ideal of A', $\mathfrak{M}_{A'}^2$ is not closed for this topology and its closure $\overline{\mathfrak{M}_{A'}^2}$ coincides with the kernel of the projection $A[[(T_i)_{i\in I}]] \to A[(T_i)_{i\in I}]/\langle \mathfrak{M}_A, (T_i)_{i\in I}\rangle^2$.

7.2. **Proof of Gabber's Theorem.** Let $A, B \in \mathfrak{Lcnp}$. Let $\mathfrak{M}_A$ (resp. $\mathfrak{M}_B$) be the maximal ideal of A (resp. B). Clearly, we may assume that A, B are non-cancellable. We set $A' := A[[(T_i)_{i\in I}]]$, $B' := B[[(U_j)_{i\in J}]]$, and let $\varphi : A' \to B'$ be an isomorphism.

We have a natural injective morphism $\iota_A : A \to A'$ admitting a retraction ρ_A given by $T_i \mapsto 0$. We define analogously ι_B and ρ_B.

Let $\mathfrak{M}_{A'}$ (resp. $\mathfrak{M}_{B'}$) be the maximal ideal of A' (resp. B') and $\overline{\mathfrak{M}^2_{A'}}$ (resp. $\overline{\mathfrak{M}^2_{B'}}$) be the closure of $\mathfrak{M}^2_{A'}$ (resp. $\mathfrak{M}^2_{B'}$).

Identifying, via ι_A, $\mathfrak{M}_A/\mathfrak{M}_A^2$ with a subvector space of $\mathfrak{M}_{A'}/\overline{\mathfrak{M}^2_{A'}}$, we have a decomposition

$$\mathfrak{M}_{A'}/\overline{\mathfrak{M}^2_{A'}} \cong \mathfrak{M}_A/\mathfrak{M}_A^2 \bigoplus \underset{i\in I}{\oplus} k\, t_i \tag{7.1}$$

(where we denote by t_i the class of T_i modulo $\overline{\mathfrak{M}^2_{A'}}$) and a similar decomposition for B'.

Now the crucial point of the proof is the following fact:

LEMMA 7.3. — *Identifying, via φ, $\mathfrak{M}_{A'}/\overline{\mathfrak{M}^2_{A'}}$ and $\mathfrak{M}_{B'}/\overline{\mathfrak{M}^2_{B'}}$, the images of $\mathfrak{M}_A/\mathfrak{M}_A^2$ by ι_A and $\mathfrak{M}_B/\mathfrak{M}_B^2$ by ι_B coincide.*

The proof of Lemma 7.3 is given below. Using the lemma we obtain that the composition

$$h : A \overset{\iota_A}{\to} A' \overset{\varphi}{\cong} B' \overset{\rho_B}{\to} B$$

induces an isomorphism $\mathfrak{M}_A/\mathfrak{M}_A^2 \cong \mathfrak{M}_B/\mathfrak{M}_B^2$. By this and a straightforward induction, we infer that, for every integer $n \in \mathbf{N}$, the morphism h induces a surjection of finite dimensional k-vector spaces :

$$h_n : \mathfrak{M}_A^n/\mathfrak{M}_A^{n+1} \to \mathfrak{M}_B^n/\mathfrak{M}_B^{n+1}.$$

Thus, one has $\dim(\mathfrak{M}_B^n/\mathfrak{M}_B^{n+1}) \leqslant \dim(\mathfrak{M}_A^n/\mathfrak{M}_A^{n+1})$. Exchanging the roles of A in B, we get the opposite inequalities, hence the equality of the dimensions for all n. Thus, the morphism h_n is an isomorphism for every integer $n \in \mathbf{N}$; hence, the morphism h is an isomorphism.

Proof of Lemma 7.3. — Assume that there exists an element $f \in \mathfrak{M}_A$ such that the class of the element $\varphi(f)$ in $\mathfrak{M}_{B'}/\overline{\mathfrak{M}^2_{B'}}$ does not belong to $\mathfrak{M}_B/\mathfrak{M}_B^2$. Thus, there exists $j_0 \in J$ such that $\partial_{U_{j_0}}\varphi(f)$ is invertible, thanks to decomposition (7.1).

Now, let us define the morphism $\psi : B' \to B'[[T]]$ by

$$\begin{cases} \psi_{|B} & = & \mathrm{Id}_B \\ \psi(U_j) & = & U_j \text{ for every element } j \neq j_0 \\ \psi(U_{j_0}) & = & U_{j_0} + T. \end{cases}$$

In particular, we observe that the formula $\partial_T\psi = \psi\partial_{U_{j_0}}$ holds true.

Let us denote by $\mathrm{ev}_A : A[[T]]{\to}A$ the evaluation morphism given by $T \mapsto 0$. Composing with $\rho_A \circ \varphi^{-1} : B' \to A$, we get the following morphism

$$\mu : A \overset{\iota_A}{\to} A' \overset{\varphi}{\cong} B' \overset{\psi}{\to} B'[[T]] \to A[[T]]$$

which has the following properties:

$$\begin{cases} \mathrm{ev}_A \circ \mu & = & \mathrm{Id}_A \\ \partial_T(\mu(f)) & \in & (A[[T]])^\times. \end{cases}$$

Let us denote by $\mathrm{ev}_{A/fA} : (A/fA)[[T]] \to A/fA$ the evaluation morphism given by $T \mapsto 0$. Then, composing the morphism μ with $A[[T]] \to (A/fA)[[T]]$, one gets a new morphism:

$$\mu' \,:\, A \to (A/fA)[[T]]$$

such that the morphism $\mathrm{ev}_{A/fA} \circ \mu'$ coincides with the quotient morphism $A \to A/fA$, and $\mu'(f) = T\,u(T)$ with $u(T) \in (A/fA)[[T]]$ satisfying $u(0) \in (A/fA)^{\times}$.

Let us show that μ' is an isomorphism, which will contradict the fact that A is non-cancellable; hence, it will prove the lemma. In order to do so, we consider the f-adic filtration on A and the T-adic filtration on $(A/fA)[[T]]$. By [18, 0/§7, Proposition 7.2.4], A and $(A/fA)[[T]]$ are separated and complete for the topologies induced by these filtrations. Hence, since μ' respects these filtrations, we are reduced to show that μ' induces an isomorphism on the level of the homogeneous parts of the associated graded rings. But, for every integer $n \in \mathbf{N}$, we may identify canonically $T^n(A/fA)[[T]]/T^{n+1}(A/fA)[[T]]$ with A/fA. Then, via this identification, the morphism induced by μ' on the homogeneous parts of degree n reads as:

$$\begin{array}{rcl} f^n\,A/f^{n+1}\,A & \longrightarrow & A/fA \\ [f^n\,a] & \longmapsto & u(0)^n\,[a] \end{array}$$

whose inverse is defined by $[b] \mapsto [f^n u(0)^{-n} b]$, since $u(0)$ is invertible. □

7.3. **A remark.** Keep the notation of the previous section. If the k-algebra A happens to be the completion of a local k-algebra $\tilde{A}$, the element $f \in \mathfrak{M}_A$ in the proof of Lemma 7.3 may be assumed to belong to $\mathfrak{M}_{\tilde{A}}$, since every element of $f + \mathfrak{M}_A^2$ will have the required properties. Since one has $\widehat{\tilde{A}/f\tilde{A}} \cong A/fA$, it shows that if A and B are assumed to be completions of local k-algebras essentially of finite type, the last assertion in the statement of Theorem 7.1 still holds true under the weaker hypothesis that A and B are non-cancellable in the full subcategory of $\mathfrak{Lncp}$ consisting of elements which are completion of local k-algebras essentially of finite type.

7.4. **Stable invariance of finite dimensional formal models.** With every local noetherian complete k-algebra A, whose residue field is k-isomorphic to k, one can associate such a k-algebra $A_{\min}$, which is *non-cancellable*, unique (up to isomorphism) by Theorem 7.1, and satisfies $A \cong A_{\min}[[T_1, \ldots, T_n]]$ for some integer $n \in \mathbf{N}$. This remark in particular shows that there exist a way to define finite dimensional formal models canonically. Precisely, a direct application of Theorem 7.1 provides the following important corollary:

Corollary 7.4. — *Let V be a R-space. If $\mathscr{S}, \mathscr{S}'$ are two formal affine noetherian adic k-scheme which realize isomorphism* (4.1)*. Then, the complete local noetherian k-algebras $\mathcal{O}(\mathscr{S})_{\min}, \mathcal{O}(\mathscr{S}')_{\min}$ are isomorphic (as admissible local k-algebras). In particular, there exists two integers $m, m' \in \mathbf{N}$ and an isomorphism of formal k-schemes $\mathscr{S} \times_k \mathrm{Spf}(k[[T_1, \ldots, T_m]]) \cong \mathscr{S}' \times_k \mathrm{Spf}(k[[T_1, \ldots, T_{m'}]])$.*

8. A numerical invariant in singularity theory

In this section, we study consequences of Theorem 4.1 in the direction of singularity theory. This construction can be seen as the starting point of an exegesis of the Drinfeld-Grinberg-Kazhdan Theorem with respect to singularity theory.

8.1. **Nilpotence index.** If A is a ring whose nilradical $\mathscr{N}(A)$ is nilpotent, then the *nilpotency index* of A is the smallest positive integer m such that $\mathscr{N}(A)^m = \{0\}$.

Proposition 8.1. — *Let V be a R-space, and let $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$. Then the nilradical of the ring $\mathscr{L}_\infty(V)_\gamma$ is finitely generated, and in particular nilpotent. Moreover, the nilpotency index of the ring $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ equals the nilpotency index of $\mathcal{O}(\mathscr{S})$ (or $(\mathcal{O}(\mathscr{S}))_{\min}$), for every formal k-scheme $\mathscr{S}$ which realizes isomorphism (4.1).*

Proof. — Let $\mathscr{S}$ be a formal k-scheme which realizes isomorphism (4.1). One has an isomorphism of k-formal schemes $\mathscr{S} \cong \mathrm{Spf}(A/I)$ where A is a power series ring in a finite number of variables over k and I is a proper ideal of A, which is, by noetherianity, finitely generated. Then isomorphism (4.1) induces an isomorphism of formal k-schemes

$$\mathscr{L}_\infty(V)_\gamma \xrightarrow{\sim} \mathrm{Spf}(A[[(T_i)_{i\in\mathbf{N}}]]/I[[(T_i)_{i\in\mathbf{N}}]]).$$

Indeed, since the ideal I is finitely generated, we observe that $I \cdot A[[(T_i)_{i\in\mathbf{N}}]] = I[[(T_i)_{i\in\mathbf{N}}]]$. Let us set $B = A[[(T_i)_{i\in\mathbf{N}}]]$. By Lemma 8.2, one has $\sqrt{I\cdot B} = (\sqrt{I})\cdot B$, which shows that $\sqrt{I\cdot B}$ is finitely generated. Let m be the nilpotency index of A/I. Then one has $(\sqrt{I\cdot B})^m = (\sqrt{I})^m \cdot B \subset I \cdot B$. On the other hand, if n is a positive integer satisfying $(\sqrt{I\cdot B})^n \subset I\cdot B$, one has $(\sqrt{I})^n \cdot B \cap A \subset I \cdot B \cap A$; hence, $(\sqrt{I})^n \subset I$. That concludes the proof. □

Lemma 8.2. — *Let A be ring and I be a finitely generated ideal of A such that $\sqrt{I}$ is finitely generated.*

1. *Let $n \geqslant 1$ and $B = A[T_1, \dots, T_n]$. Then $\sqrt{I\cdot B} = (\sqrt{I})\cdot B$.*
2. *Let $B = A[[(T_i)_{i\in\mathbf{N}}]]$. Then $\sqrt{I\cdot B} = (\sqrt{I})\cdot B$.*

Proof. — For both assertions, let us note that we obviously have $\sqrt{I\cdot B} \supset (\sqrt{I})\cdot B$. Let us prove the first assertion. By induction on the integer n, we reduce immediately to the case $n = 1$. Let $F = \sum f_d T_1^d \in \sqrt{I\cdot B}$. We have to show that all the coefficients of F lie in $\sqrt{I}$. Assume that this is not the case and let D be the smallest degree for which $f_D \notin \sqrt{I}$; then, for an integer N big enough, one has $(F - \sum_{d\leqslant D-1} f_d T_1^d)^N \in I\cdot B = I[T_1]$, but this implies (identifying the coefficients) that $f_D^N \in I$.

Let us prove the second assertion. Recall that, since the ideals I and $\sqrt{I}$ are finitely generated, one has

$$I\cdot B = I[[(T_i)_{i\in\mathbf{N}}]] \quad \text{and} \quad \sqrt{I}\cdot B = (\sqrt{I})[[(T_i)_{i\in\mathbf{N}}]].$$

Let $F \in \sqrt{I\cdot B}$. Again, we have to show that all the coefficients of F lie in $\sqrt{I}$. For $d \geqslant 0$, let F_d be the d-homogeneous component of F. First we show that all the F_d's lie in $\sqrt{I\cdot B}$. Assume that this is not the case and let D be the smallest degree such that $F_D \notin \sqrt{I\cdot B}$; then, for an integer N big enough, one has $(F - \sum_{d\leqslant D-1} F_d)^N \in I\cdot B$, which implies (identifying the homogeneous components) that $F_D^N \in I\cdot B$. Now, let us note that, thanks to the very definition of the ring B, for every integer $d \geqslant 0$, there exists an integer $n(d)$ such that $F_d \in A[T_1, \dots, T_{n(d)}]$. It follows that $F_d \in \sqrt{I\cdot A[T_1, \dots, T_{n(d)}]}$. Using the first assertion, we see that $F_d \in (\sqrt{I})\cdot A[T_1, \dots, T_{n(d)}]$. That concludes the proof. □

Let V be a k-variety, and let $\gamma \in \mathscr{L}_\infty^\circ(V)(k)$. We define $m_\gamma(V)$ to be the nilpotency index of $\mathscr{L}_\infty(V)_\gamma$.

8.2. **Absolute nilpotency index.** If $v \in V(k)$, we denote by $\mathscr{L}_\infty^\circ(V,v)$ the subset of $\mathscr{L}_\infty^\circ(V)(k)$ formed by the rational arcs γ on V with base-point $\gamma(0)=v$. This definition suggests to introduce the following invariant:

Definition 8.3. — Let k be a field. Let V be a k-variety, with $v \in V(k)$. The *absolute nilpotency index* of the pair (V,v) is the integer

$$m_{\mathrm{abs}}(V,v) := \inf_{\gamma \in \mathscr{L}_\infty^\circ(V,v)} (m_\gamma(V)).$$

The absolute nilpotency index of (V,v) only depends on the pointed k-variety (V,v) by construction. Besides, there exists an arc $\gamma_{\mathrm{abs}} \in \mathscr{L}_\infty^\circ(V)(k)$ such that $m_{\mathrm{abs}}(V,v) = m_{\gamma_{\mathrm{abs}}}(V)$.

Proposition 8.4. — *Let k be a field. Let V be a k-variety, with $v \in V(k)$. Then, the absolute nilpotency index $m_{\mathrm{abs}}(V,v)$ is a formal invariant of the singularity (V,v).*

Proof. — Let (V',v') be a pointed k-variety endowed with an isomorphism of complete local k-algebras: $f : \widehat{\mathcal{O}_{V',v'}} \xrightarrow{\sim} \widehat{\mathcal{O}_{V,v}}$. Let $\gamma_v \in \mathscr{L}_\infty(V)(k)$ be an arc such that $m_{\gamma_v}(V) = m_{\mathrm{abs}}(V,v)$ and $\gamma_{v'} \in \mathscr{L}_\infty(\mathscr{C}')(k)$ its image by f, i.e., $\gamma_{v'} = \gamma_v \circ f$. By corollary 2.9, the formal k-schemes $\mathscr{L}_\infty(\mathscr{C})_{\gamma_v}$ and $\mathscr{L}_\infty(\mathscr{C})_{\gamma_{v'}}$ are isomorphic. By this way, we deduce that $m_{\mathrm{abs}}(V,v) \geqslant m_{\mathrm{abs}}(V',v')$. We conclude, by symmetry, by applying the same arguments to f^{-1}. □

8.3. **Further comments.** Various questions arise from the previous remarks.

∘ We do not know, in general, whether such arcs $\gamma_{\mathrm{abs}} \in \mathscr{L}_\infty^\circ(V)(k)$ are, up to isomorphism, unique or not.

∘ The computation and interpretation of such nilpotency indexes seem to be non-trivial problems even in the case of plane curves. See [3] for some related results.

∘ The situation for singular (constant or non-constant) arcs seems to be very different from that of non-singular arcs. See [2, 31] for some related results.

∘ Let k be a field. Let V be a k-variety, with $v \in V(k)$. If v is smooth, then one easily observes that $m_{\mathrm{abs}}(V,v) = 1$. The example of the arc (T,T,T,T) on the variety $\mathrm{Spec}(k[X_0,X_1,X_2,X_3]/\langle X_0X_1 - X_2X_3\rangle)$, whose minimal formal model may be computed as $\mathrm{Spf}(k[[U,V]]/\langle UV\rangle)$, shows that the converse is false.

∘ If V is a k-curve and γ is a primitive parametrization at a unibranch singular point $v \in V(k)$, does the nilpotency index (or the absolute nilpotency index) associated with the ring $\widehat{\mathcal{O}_{\mathscr{L}_\infty(V),\gamma}}$ equal the singularity degree $\delta(V,v)$ plus one? (The first positive elements of answer in this direction can be found in [3].)

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