CONFLUENTES
MATHEMATICI

Barbara SCHAPIRA

A short proof of unique ergodicity of horospherical foliations on infinite volume hyperbolic manifolds

Tome 8, n^{o} 1 (2016), p. 165-174.

<http://cml.cedram.org/item?id=CML_2016__8_1_165_0>

Confluentes Math.
8, 1 (2016) 165-174

A SHORT PROOF OF UNIQUE ERGODICITY OF HOROSPHERICAL FOLIATIONS ON INFINITE VOLUME HYPERBOLIC MANIFOLDS

BARBARA SCHAPIRA

Abstract. We provide a new proof of the fact that the horospherical group $N < G = SO_o(d,1)$ acting on the frame bundle $\Gamma\backslash G$ of a hyperbolic manifold admits a unique invariant ergodic measure (up to multiplicative constants) supported on the set of frames whose orbit under the geodesic flow comes back infinitely often in a compact set. This result is known, but our proof is more direct and much shorter.

1. Introduction

The (unstable) horocycle flow on the unit tangent bundle of compact hyperbolic surfaces is uniquely ergodic. Furstenberg [6] proved that the Liouville measure is the unique invariant measure under this flow. This result has been extended to many noncompact situations. On finite volume hyperbolic surfaces, Dani [4] proved that the Liouville measure is the unique finite invariant ergodic measure, except for the measures supported on periodic orbits. On convex-cocompact hyperbolic surfaces, Burger [2] proved that there is a unique locally finite ergodic invariant measure. It also follows from a result of Bowen-Marcus establishing the unique ergodicity of strong (un)stable foliations. Babillot-Ledrappier [1] and Sarig [10] described completely the set of invariant ergodic measures of the horocyclic flow on abelian covers of compact hyperbolic surfaces.

Roblin [9] proved in a much more general context that the unstable horocyclic flow admits a unique invariant ergodic Radon measure whose support is the set of vectors whose negative geodesic orbit comes back infinitely often to a compact set, as soon as the geodesic flow admits a finite measure of maximal entropy.

The goal of this note is to provide a new simpler proof of his result, inspired by the arguments of [3] in the case of surfaces of finite volume, with additional ingredients to deal with the fact that when the manifold has infinite volume, there is in general no ergodic invariant measure which is invariant and ergodic under both the geodesic flow and its strong (un)stable foliation.

Let us state the result more precisely.

Let $M = \Gamma\backslash\mathbb{H}^d$ be a hyperbolic manifold of dimension n with infinite volume. Let $G = SO^o(d,1)$ the group of isometries preserving orientation of $\mathbb{H}^d$, and $G = NAK$ its Iwasawa decomposition. The homogeneous space $\Gamma\backslash G$ is the frame bundle over $M = \Gamma\backslash G/K$. The action of $A = \{a_t, t \in \mathbb{R}\}$ by right multiplication on $\Gamma\backslash G$ is the natural lift of the action of the geodesic flow on the unit tangent bundle T^1M: a_t moves the first vector v_1 of a frame $\mathbf{x} = (v_1, \dots, v_d)$ as the geodesic flow g^t at time t on T^1M does, and the other vectors of the frame follow by parallel transport along

Math. classification: 22E40, 22D40, 28D15, 37A17, 37A25.
Keywords: unique ergodicity, horospherical group, frame bundle, in nite volume hyperbolic manifolds.

the geodesic of v_1. The N-orbits on $\Gamma\backslash G$ are the strong unstable manifolds of this A-action, and project to T^1M onto the strong unstable leaves of the geodesic flow.

Our result is the following.

THEOREM 1.1. — *Let $M = \Gamma\backslash\mathbb{H}^d$ be a hyperbolic manifold. Assume that Γ is Zariski dense and that the geodesic flow on T^1M admits a probability measure maximizing entropy. Then there is a unique (up to multiplicative constants) N-invariant conservative measure giving full measure to the set*

$$\mathcal{E}^{\mathcal{F}}_{rad} = \{\mathbf{x} \in \Gamma\backslash G : \quad \mathbf{x}a_{-t} \text{ returns infinitely often to a compact set}\}.$$

This result has been proved by Winter [12] on the frame bundle $\Gamma\backslash G$ as a (non-trivial) corollary of Roblin's work on $T^1M = \Gamma\backslash G/L$. Here, we provide a direct proof on $\Gamma\backslash G$, much simpler than Roblin's proof.

Let us emphasize the differences between our result and Roblin's work. Roblin [9] deals with the geodesic flow on the unit tangent bundle of quotients of $CAT(-1)$-spaces, that is metric spaces whose geometry is similar to the geometry of a negatively curved manifold. This setting is much more general than the geodesic flow on the unit tangent bundle of a hyperbolic manifold.

His result as our result is true under the assumption that the geodesic flow admits a probability measure maximizing entropy. He proves that the strong unstable foliation of the geodesic flow admits a unique (up to multiplicative constants) transverse invariant measure which gives full measure to the set $\mathcal{E}_{rad}$ of vectors v whose geodesic orbit $(g^{-t}v)_{t\geqslant 0}$ returns infinitely often in a compact set.

Our result is valid under a much more restrictive geometric assumption (hyperbolic manifolds). However, first, it is valid for the strong unstable foliation (parametrized by the orbits of the group N) of the geodesic flow (parametrized by the diagonal group A) on the frame bundle, and not only on the unit tangent bundle. As the frame bundle $\Gamma\backslash G$ is a fiber bundle over the unit tangent bundle $T^1(\Gamma\backslash\mathbb{H}^d)$, with fiber isomorphic to $L = SO(d-1)$, any transverse invariant measure to the strong unstable foliation of the geodesic flow on the unit tangent bundle can be lifted (by taking the product locally with the Haar measure of $SO(d-1)$) to a transverse invariant measure to the strong unstable foliation of the A-orbits on $\Gamma\backslash G$. Taking the product locally with the Haar measure of N gives a N-invariant measure. Therefore, additional work (done by Winter in [12]) is needed to get a unique ergodicity result on the frame bundle $\Gamma\backslash G$. Thus, in the setting of hyperbolic manifolds, our result (the same as Winter's result) is stronger than Roblin's theorem.

But the main interest of our result is the proof, which is short, shorter than Roblin's proof and gives directly Winter's result. The main ingredient is the same in both cases, the mixing property of the measure of maximal entropy of the geodesic flow. However, we follow an idea of Coudène [3], which allows to get a much simpler argument, and could maybe allow in the future to remove the assumption of existence of a finite measure of maximal entropy. The key additional ingredients are a trick already used in preceding works of the author, allowing to compare the measure of maximal entropy of the geodesic flow with a natural N-invariant measure supported in $\mathcal{E}^{\mathcal{F}}_{rad}$, with an ergodic tool provided in [7]. These arguments allow to generalize Coudene's argument to manifolds of infinite volume.

Note also that Hochman's result can be used only in constant negative curvature, when the horospherical group N is isomorphic to $\mathbb{R}^{n-1}$. In variable negative curvature, this proof could probably work in dimension $n = 2$, when N is parametrized by a horocyclic flow, so that classical Hopf's ergodic theorem can be used instead of Hochman's result. In this case, the frame bundle coincides with the unit tangent bundle, and we would recover exactly Roblin's result.

We will comment further the difference between the two proofs at the end of the paper.

2. Infinite volume manifolds, actions of A and N

Let $G = SO^o(d, 1)$ be the group of direct isometries of the hyperbolic n-space $\mathbb{H}^d$. Let $M = \Gamma\backslash\mathbb{H}^d$ be a hyperbolic manifold, where Γ is a discrete group without torsion.

The *limit set* $\Lambda_\Gamma := \overline{\Gamma.x} \setminus \Gamma.x$ is the set of accumulation points of any orbit of Γ in the boundary $\partial\mathbb{H}^d$. We assume Γ to be non elementary, i.e. it is not virtually abelian, or equivalently, the set Λ_Γ of its limit points in the boundary is infinite.

Let $K = SO(n)$ be the stabilizer in G of the point $o = (0, \dots, 0, 1) \in \mathbb{H}^d$, and $L = SO(d-1)$ the stabilizer of the unit vector $(0, \dots, 0, 1)$ based at o. Then the unit tangent bundle T^1M is identified with $\Gamma\backslash G/L$ whereas the homogeneous space $\Gamma\backslash G$ is identified with the frame bundle over T^1M, whose fiber at any point is isomorphic to L. Denote by $\pi : T^1M \to M$ the natural projection.

The Busemann function is defined on $\partial\mathbb{H}^d \times \mathbb{H}^d \times \mathbb{H}^d$ by

$$\beta_\xi(x, y) = \lim_{z\to\xi} d(x, z) - d(y, z).$$

The following map is a homeomorphism from $T^1\mathbb{H}^d$ to $\partial\mathbb{H}^d \times \partial\mathbb{H}^d \setminus \{diagonal\} \times \mathbb{R}$:

$$v \mapsto (v^-, v^+, \beta_{v^-}(\pi(v), o)).$$

In these coordinates, the geodesic flow acts by translation on the $\mathbb{R}$-coordinate, and an isometry γ of $\mathbb{H}^d$ acts as follows: $\gamma.(v^-, v^+, s) = (\gamma.v^-, \gamma.v^+, s + \beta_{v^-}(o, \gamma^{-1}o))$. This homeomorphism induces on the quotient a homeomorphism from T^1M to $\Gamma\backslash \left(\partial\mathbb{H}^d \times \partial\mathbb{H}^d \setminus \{diagonal\} \times \mathbb{R}\right)$.

The strong unstable manifold

$$W^{su}(v) = \{w \in T^1M, d(g^{-t}v, g^{-t}w) \to 0 \text{ when } t \to +\infty\}$$

of a vector $v = (v^-, v^+, s) \in T^1\mathbb{H}^d$ under the geodesic flow is exactly the set of vectors $w = (v^-, w^+, s)$, for $w^+ \in \partial\mathbb{H}^d$.

As said in the introduction, the action of the geodesic flow on T^1M lifts into the A-action by right multiplication on $\Gamma\backslash G$. Denote by $a_t \in A$, for $t \in \mathbb{R}$, the element whose action moves the first vector of a frame as the geodesic flow g^t does on T^1M. The strong unstable manifold of a frame $\mathbf{x} \in G$ under the action of A is exactly its N-orbit $\mathbf{x}N$.

The nonwandering set of the geodesic flow $\Omega \in T^1M$ is exactly the set of vectors $v \in T^1M$ such that $v^\pm \in \Lambda_\Gamma$. The nonwandering set $\Omega^{\mathcal{F}}$ of the action of A in $\Gamma\backslash G$ is simply the set of frames whose first vector is in Ω. The point is that $\Omega^{\mathcal{F}}$ is *not* N-invariant. Let $\mathcal{E}^{\mathcal{F}} = \Omega^{\mathcal{F}}.N$ and $\mathcal{E}$ its projection on T^1M, that is the set of vectors $v \in T^1M$ such that $v^- \in \Lambda_\Gamma$.

The difficulty in general is to deal with $\Omega^{\mathcal{F}}$ and $\mathcal{E}^{\mathcal{F}}$, to get informations on the dynamics of N on $\mathcal{E}^{\mathcal{F}}$ thanks to the knowledge of the A-action on $\Omega^{\mathcal{F}}$ and vice versa.

3. Ergodic theory

3.1. **Measures on the boundary.** The Patterson-Sullivan δ_Γ-conformal density is a family $(\nu_x)_{x\in\mathbb{H}^d}$ of equivalent measures on the boundary giving full measure to Λ_Γ, and satisfying the two crucial properties, for all $\gamma\in\Gamma$ and ν_x-almost all $\xi\in\Lambda_\Gamma$:

$$\gamma_*\nu_x=\nu_{\gamma x}\quad\text{and}\quad \frac{d\nu_y}{d\nu_x}(\xi)=\exp(\delta_\Gamma\beta_\xi(x,y)).$$

Given any $x\in\mathbb{H}^d$, there is a canonical homeomorphism between the unit sphere $T^1_x\mathbb{H}^d$ and the boundary at infinity, associating to a vector v the endpoint of the geodesic that it defines. The family (λ_x) of Lebesgue measures on the unit spheres $T^1_x\mathbb{H}^d$, seen as measures on the boundary, satisfy a similar property as the Patterson-Sullivan family of measures: for all $g\in G$ and λ_x-almost all $\xi\in\Lambda_\Gamma$:

$$g_*\lambda_x=\lambda_{gx}\quad\text{and}\quad \frac{d\lambda_y}{d\lambda_x}(\xi)=\exp((d-1)\beta_\xi(x,y)).$$

3.2. **Invariant measures under the geodesic flow on T^1M and the A-action on $\Gamma\backslash G$.** We define a Γ-invariant measure $\tilde m_{BM}$ on $T^1\mathbb{H}^d=G/L$, in terms of the Hopf coordinates, by

$$d\tilde m_{BM}(v)=e^{\left(\delta_\Gamma\beta_{v^+}(o,\pi(v))+\delta_\Gamma\beta_{v^-}(o,\pi(v))\right)}d\nu_o(v^-)\,d\nu_o(v^+)dt.$$

This measure is also invariant under the geodesic flow. We denote by m_{BM} the induced measure on T^1M, known as the *Bowen-Margulis measure*.

This measure lifts in a natural way to the frame bundle $\Gamma\backslash G$. Indeed, as said in the introduction, the frame bundle is a fiber bundle over $T^1(\Gamma\backslash G)$, with fiber isomorphic to $L=SO(d-1)$. Thus, we take locally the product of the Bowen-Margulis measure with the Haar measure on L. We denote the resulting measure, still called the *Bowen-Margulis measure*, by $m^{\mathcal{F}}_{BM}$.

It is well known that when this measure $m^{\mathcal{F}}_{BM}$ is finite (or equivalently m_{BM} on T^1M is finite), it gives full measure to the set $\Omega^{\mathcal{F}}_{rad}$ ([1]) of frames whose A-orbit return infinitely often in the future and in the past to a compact set.

3.3. **Invariant measures under the action of N on $\Gamma\backslash G$.** The measure $m^{\mathcal{F}}_{BM}$ on $\Gamma\backslash G$ is not invariant under the N-action. However, its product structure allows to build such a measure. The *Burger-Roblin measure* is defined on $T^1\mathbb{H}^d$ by

$$d\tilde m_{BR}(v)=e^{\left((d-1)\beta_{v^+}(o,\pi(v))+\delta_\Gamma\beta_{v^-}(o,\pi(v))\right)}d\nu_o(v^-)\,d\lambda_o(v^+)dt.$$

It is Γ-invariant, quasi-invariant under the geodesic flow, and we denote by m_{BR} the induced measure on the quotient, called Burger-Roblin measure.

Its lift $m^{\mathcal{F}}_{BR}$ to the frame bundle (by doing the local product with the Haar measure of L, as above) is N-invariant. Indeed, it can be written locally as follows.

[1] The notation $\Omega^{\mathcal{F}}_{rad}$ comes from the fact that the frames of $\Omega^{\mathcal{F}}_{rad}$ are exactly the frames whose positive and negative endpoints in the boundary belong to the so-called *radial limit set*.

Let $v \in T^1(\Gamma\backslash G)$ be a vector. Its fiber in $\Gamma\backslash G$ is isomorphic to $L = SO(d-1)$. The lift of the measure m_{BR} to $\Gamma\backslash G$ can be written locally as

$$dm^{\mathcal{F}}_{BR} = e^{\delta_\Gamma \beta_{v^-}(o,\pi(v))}\, d\nu_o(v^-)\, e^{(d-1)\beta_{v^+}(o,\pi(v))}\, d\lambda_o(v^+)\, dt\, dl$$

and an elementary computation shows that the measure $e^{(d-1)\beta_{v^+}(o,\pi(v))}\, d\lambda_o(v^+)$ on the strong unstable manifold of a frame coincides exactly with the Haar measure dn of N.

It is well known that, when the Bowen-Margulis measure is finite, the Burger-Roblin measure gives full measure to the set $\mathcal{E}^{\mathcal{F}}_{rad}$ of frames whose negative A-orbit returns infinitely often in the past to a compact set.

3.4. **Comparison of the Bowen-Margulis and the Burger-Roblin measure.** In any local chart $\mathcal{B}$ of the strong unstable foliation of A, whose leaves are here the N-orbits, these measures have a very similar form, the product of the same transverse measure ν_T, by a measure on the leaves. The transverse measure (ν_T) is a collection of measures on all transversals to the foliation by N-orbits, which is invariant by holonomy.

A natural family of transversals to the strong unstable foliation of the geodesic flow on $T^1(\Gamma\backslash\mathbb{H}^d)$ is the family of weak stable manifolds of the geodesic flow. On the frame bundle $\Gamma\backslash G$, the weak stable manifolds of the A-action are the preimage (by the natural projection $\Gamma\backslash G \to T^1(\Gamma\backslash\mathbb{H}^d)$) of the precedings.

As said above, the Burger-Roblin and the Bowen-Margulis measures can both be written locally as the (noncommutative) "product" of a family of measures on the N-orbits by a transverse measure ν_T, defined on any weak stable manifold of a frame $\mathbf{x}$ whose first vector is $v = (v^-, v^+, t)$ by $d\nu_T(\mathbf{x}) = e^{\delta_\Gamma \beta_{v^-}(o,\pi(v))}\, d\nu_o(v^-)\, dt\, dl$.

In the chart $\mathcal{B}$ of the foliation, denote by T a transversal, and for $\mathbf{z} \in T$, let $N_{\mathcal{B}}(\mathbf{z})$ be the local leaf of the foliation intersected with $\mathcal{B}$. For all continuous functions $\varphi : \Gamma\backslash G \to \mathbb{R}$ supported in $\mathcal{B}$, we have

$$\int \varphi\, dm^{\mathcal{F}}_{BR} = \int_T \int_{N_{\mathcal{B}}(\mathbf{z})} \varphi(\mathbf{x}) d\lambda_{\mathbf{z}N}(\mathbf{x}) d\nu_T(\mathbf{z}),$$

whereas

$$\int \varphi\, dm^{\mathcal{F}}_{BM} = \int_T \int_{N_{\mathcal{B}}(\mathbf{z})} \varphi(\mathbf{x}) d\mu^{BM}_{\mathbf{z}N}(\mathbf{x}) d\nu_T(\mathbf{z}),$$

where $d\lambda_{\mathbf{z}N} = dn$ and $d\mu^{BM}_{\mathbf{z}N}$ are respectively the conditional measures of $m^{\mathcal{F}}_{BR}$ and $m^{\mathcal{F}}_{BM}$ on the N-orbits.

3.5. **Mixing of the geodesic flow.** When it is finite, the measure m_{BM} on T^1M is mixing. Filling a gap in [5] Flaminio-Spatzier, Winter [12] proved that its lift $m^{\mathcal{F}}_{BM}$ to $\Gamma\backslash G$ is also mixing, as soon as the group Γ is Zariski dense.

As a corollary, he gets the following equidistribution result of averages pushed by the flow. Let $\mathbf{x} \in \Omega^{\mathcal{F}}$ be a frame, and $\varphi : \Gamma\backslash G \to \mathbb{R}$ be a continuous map with compact support. Define the following averages

$$M_1^t(\varphi)(\mathbf{x}) = \frac{1}{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_1)} \int_{\mathbf{x}N_1} \varphi(\mathbf{y}.a_t)\, d\mu^{BM}_{\mathbf{x}N}(\mathbf{y}),$$

where $N_1 = \{n \in N, |n| \leqslant 1\}$.

Theorem 3.1 (Winter [12]). — *Let $\Gamma < SO^o(d,1)$ be a Zariski dense discrete subgroup, such that the Bowen-Margulis measure $m^{\mathcal{F}}_{BM}$ is finite. Then it is mixing. As a consequence, for all $\mathbf{x} \in \Omega^{\mathcal{F}}$ and φ continuous with compact support, the averages $M^t_1(\varphi)(\mathbf{x})$ converge towards $\int_{\Gamma\backslash G} \varphi \, dm^{\mathcal{F}}_{BM}$ when $t \to +\infty$.*

3.6. **Higher dimensional version of Hopf's ratio ergodic theorem.** As the group N is isomorphic to $\mathbb{R}^{d-1}$, with $d \geqslant 2$ arbitrarily large, we will need a higher dimensional version of the ergodic theorem, for the N-action on $\Gamma\backslash G$. As the natural N-invariant measure $m^{\mathcal{F}}_{BR}$ is infinite, we need a version of Hopf ratio ergodic theorem for actions of $\mathbb{R}^d$. The desired result is the following.

Theorem 3.2 (Hochman [7]). — *Consider a free, ergodic, measure-preserving action $(\phi_t)_{t\in\mathbb{R}^d}$ of $\mathbb{R}^d$ on a standard σ-finite measure space $(X, \mathcal{B}, \nu)$. Let $\|.\|$ be any norm on $\mathbb{R}^d$ and $B_r = \{t \in \mathbb{R}^d, \|t\| \leqslant r\}$. Then for every $f, g \in L^1(X, \nu)$, with $\int g \, d\nu \neq 0$ we have*

$$\frac{\int_{B_r} f(\phi^t x)dt}{\int_{B_r} g(\phi^t x)dt} \to \frac{\int_X f d\nu}{\int_X g d\nu}$$

almost surely.

4. Proof of the unique ergodicity

Assume that ν is a N-invariant ergodic and conservative measure on $\mathcal{E}^{\mathcal{F}}_{rad} = \{\mathbf{x} \in \mathcal{E}^{\mathcal{F}} \subset \Gamma\backslash G : \mathbf{x}a_{-t}$ comes back i.o. in a compact set$\}$.

We will show that the Burger-Roblin measure $m^{\mathcal{F}}_{BR}$ is absolutely continuous w.r.t ν, and conservative for the N-action. It will show that m_{BR} is ergodic (consider for ν the restriction of m_{BR} to any invariant set of positive measure), and imply the theorem.

More precisely, we will show that for any relatively compact chart $\mathcal{B}$ of the foliation in N-orbits of $\mathcal{E}^{\mathcal{F}}_{rad}$, there exists a constant $C_{\mathcal{B}} > 0$ such that for any continuous map φ with support in $\mathcal{B}$, $\int \varphi \, dm^{\mathcal{F}}_{BR} \leqslant C_{\mathcal{B}} \int \varphi \, d\nu$.

Choose a generic frame $\mathbf{x} \in \mathcal{E}^{\mathcal{F}}_{rad}$ w.r.t ν, i.e. a frame whose N-orbit becomes equidistributed towards ν. Without loss of generality, translating $\mathbf{x}$ along its N-orbit, we can assume that

$$\mathbf{x} \in \Omega \cap \mathcal{E}^{\mathcal{F}}_{rad} = \{\mathbf{x} \in \Omega^{\mathcal{F}} \subset \Gamma\backslash G, \mathbf{x}a_{-t} \text{ comes back i.o. in a compact set}\}.$$

Therefore, we know that there exists a sequence $t_k \to +\infty$, such that $\mathbf{x}a_{-t_k}$ converges to some frame $\mathbf{x}_\infty \in \Omega^{\mathcal{F}}$.

Without loss of generality, we can assume that the boundaries of the following unit balls on N-orbits have measure zero:

$$\mu_{\mathbf{x}_\infty N}(\partial(\mathbf{x}_\infty N_1)) = 0\,, \quad \text{and for all } k \in \mathbb{N} \quad \mu_{\mathbf{x}a_{-t_k}N}(\partial(\mathbf{x}a_{-t_k}N_1)) = 0.$$

Indeed, if it were not the case, as $\mu_{\mathbf{x}_\infty N}$ and $\mu_{\mathbf{x}a_{-t_k}N}$ for all k are Radon measure, there are at most countably many radii r s.t. $\mu_{\mathbf{x}_\infty N}(\partial(\mathbf{x}_\infty N_r)) > 0$ or $\mu_{\mathbf{x}a_{-t_k}N}(\partial(\mathbf{x}a_{-t_k}N_r)) > 0$. Choose ρ close to 1 such that all these measures of boundaries of balls of radius ρ are zero, and change t_k into $t_k + \log\rho$, $\mathbf{x}_\infty$ into $\mathbf{x}_\infty a_{-\log\rho}$. As the measures $\mu_{\mathbf{x}_\infty . a_{-\log\rho}N}$ and $(a_{-\log\rho})_*\mu_{\mathbf{x}_\infty N}$ are proportional, we get $\mu_{\mathbf{x}_\infty a_{-\log\rho}N}(\partial(\mathbf{x}_\infty a_{-\log\rho}N_1)) = 0$, and $\mu_{\mathbf{x}a_{-\log\rho}a_{-t_k}N}(\partial(\mathbf{x}a_{-\log\rho}a_{-t_k}N_1)) = 0$.

Choose any nonnegative continuous map φ with compact support in $\mathcal{B}$ s.t. $\int \varphi d\nu > 0$. We will prove that for some positive finite constant $C_{\mathcal{B}}$, the following inequality holds.

$$\frac{\int \varphi \, dm^{\mathcal{F}}_{BR}}{m^{\mathcal{F}}_{BR}(\mathcal{B})} \leqslant C_{\mathcal{B}} \frac{\int \varphi \, d\nu}{\int \mathbf{1}_{\mathcal{B}} \, d\nu}.$$

This will imply the theorem.

The first important ingredient is an equicontinuity argument.

Let $\mathbf{x}N_1$ be the N-ball around $\mathbf{x}$ inside $\mathbf{x}N$. For any continuous map φ, $\mathbf{x} \in \Gamma\backslash G$ and $t \geqslant 0$, define

$$M_1^t(\varphi)(\mathbf{x}) = \frac{1}{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_1)} \int_{\mathbf{x}N_1} \varphi(\mathbf{y}a_t) d\mu^{BM}_{\mathbf{x}N}(\mathbf{y}).$$

Proposition 4.1 (Equicontinuity). — *Let φ be any uniformly continuous function. For all $\mathbf{x} \in \Gamma\backslash G$ such that $\mu_{\mathbf{x}N}(\partial(\mathbf{x}N_1)) = 0$, the maps $\mathbf{x} \mapsto M_1^t(\varphi)(\mathbf{x})$ are equicontinuous in $t \geqslant 0$.*

Proof. — The result is relatively classical for surfaces at least. This is written in details on T^1M in any dimension here [11]. The assumptions in this reference are slightly stronger, but the compactness assumption of [11] was used only to ensure uniform continuity of φ.

The extension to $\Gamma\backslash G$ does not change anything. Indeed, the fibers of the fiber bundle $\Gamma\backslash G \to T^1M$ are included in the (weak) stable leaves of the A action. Therefore, the argument still applies. We refer to [11], but the idea is as follows. If $\mathbf{x}$ and $\mathbf{x}'$ are very close along a weak stable leaf, the sets $\mathbf{x}N_1a_t$ and $\mathbf{x}'N_1a_t$ remain at distance roughly $d(\mathbf{x}, \mathbf{x}')$ when $t \geqslant 0$.

If they are close and belong to the same N-orbit, then the assumption on the boundary of the balls allows to ensure that their averages stay close for all $t \geqslant 0$.

If $\mathbf{x}$ and $\mathbf{x}'$ are close, in general, there exists a frame $\mathbf{x}'' \in \mathbf{x}N$ on the weak stable leaf of $\mathbf{x}'$, so that combining both arguments above allows to conclude equicontinuity. □

The next ingredient is mixing. As said in section 3 above, when $t \to +\infty$, for all $\mathbf{y} \in \Omega^{\mathcal{F}}_{rad}$, we have $M_1^t(\varphi)(\mathbf{y}) \to \int_{\Gamma\backslash G} \varphi \, dm^{\mathcal{F}}_{BM}$ and $M_1^t(\psi)(\mathbf{y}) \to \int_{\Gamma\backslash G} \psi \, dm^{\mathcal{F}}_{BM}$, uniformly on compact sets.

From the above equicontinuity argument, we deduce relative compactness. In particular, consider the compact set $K = \{\mathbf{x}_\infty\} \cup \{\mathbf{x}a_{-t_k}, k \in \mathbb{N}\}$. Each frame of K satisfies the assumption on the measure of the boundary of the N-ball of radius 1. Therefore, there exists a subsequence of t_k, still denoted by t_k, such that $M_1^{t_k}(\varphi)(\mathbf{x}')$ converges to $\int \varphi dm_{BM}$ uniformly in **x'** $\in K$.

Let us now observe that

$$M^0_{e^t}(\phi)(\mathbf{x}) = M_1^t(\phi)(\mathbf{x}a^{-t}) \quad \text{and} \quad M^0_{e^t}(\psi)(\mathbf{x}) = M_1^t(\phi)(\mathbf{x}a^{-t}).$$

Therefore, as $\mathbf{x}a^{-t_k}$ converges to $\mathbf{x}_\infty$, and by the above uniform convergence on the compact $K = \{\mathbf{x}_\infty\} \cup \{\mathbf{x}a_{-t_k}, k \in \mathbb{N}\}$, we have for all $\varphi \in C_c(\Gamma\backslash G)$

$$\frac{1}{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_{e^{t_k}})} \int_{\mathbf{x}N_{e^{t_k}}} \varphi d\mu^{BM}_{\mathbf{x}N} \to \int \varphi \, dm^{\mathcal{F}}_{BM} \tag{4.1}$$

Of course, we deduce that for all φ, ψ continuous with compact support, and $\int \psi \, dm^{\mathcal{F}}_{BM} > 0$, the following convergence holds.

$$\frac{\int_{\mathbf{x}N_{e^{t_k}}} \varphi d\mu^{BM}_{\mathbf{x}N}}{\int_{\mathbf{x}N_{e^{t_k}}} \psi \, d\mu^{BM}_{\mathbf{x}N}} \to \frac{\int \varphi \, dm^{\mathcal{F}}_{BM}}{\int \psi \, dm^{\mathcal{F}}_{BM}} . \tag{4.2}$$

By a standard approximation argument, the above convergence also holds for $\psi = \mathbf{1}_{\mathcal{B}}$, with $\mathcal{B}$ any compact set satisfying $m^{\mathcal{F}}_{BM}(\mathcal{B}) > 0$ and $m^{\mathcal{F}}_{BM}(\partial\mathcal{B}) = 0$.

Now, consider a small chart of the foliation $\mathcal{B}$, with boundary of measure zero for both measures $m^{\mathcal{F}}_{BM}$ and $m^{\mathcal{F}}_{BR}$. It is possible as they are both Radon measures, therefore finite on compact sets. Let φ and ψ be two continuous maps supported in $\mathcal{B}$. Then, the integral $M^{t_k}_1(\varphi)(\mathbf{x}a_{-t_k}) = M^0_{e^{t_k}}(\varphi)(\mathbf{x})$ can be rewritten as

$$\int_T \int_{N_{\mathcal{B}}(\mathbf{z})} \varphi \, d\mu^{BM}_{\mathbf{z}N} \, d\nu_{T,t_k} + R(t_k, \varphi),$$

where

$$\nu_{T,t_k} = \frac{1}{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_{e^{t_k}})} \sum_{\mathbf{z} \in T \cap \mathbf{x}N_{e^{t_k}}} \delta_{\mathbf{z}}.$$

As we chose $\mathbf{x}_\infty$ such that $\mu^{BM}_{\mathbf{x}_\infty N}(\partial(\mathbf{x}_\infty N_1)) = 0$, we deduce, as in [8], that the error term $R(t_k, \varphi)$, which is bounded from above by $\|\varphi\|_\infty . \frac{\mu^{BM}_{\mathbf{x}N}((\mathbf{x}N_{e^{t_k+r_0}}) \setminus (\mathbf{x}N_{e^{t_k-r_0}})}{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_{e^{t_k}})}$, goes to 0 when $t_k \to +\infty$.

As $M^0_{e^{t_k}}$ converges to $m^{\mathcal{F}}_{BM}$ it implies that for all transversals T to the foliation of $\Gamma\backslash G$ in N orbits, ν_{T,t_k} converges weakly to ν_T.

Coming back to our assumptions, thanks to theorem 3.2 we know that for all continuous maps $\varphi, \psi \in C_c(\Gamma\backslash G)$ with $\int \psi \, d\nu > 0$, we have

$$\frac{\int_{\mathbf{x}N_r} \varphi(\mathbf{x}n) dn}{\int_{\mathbf{x}N_r} \psi(\mathbf{x}n) dn} \to \frac{\int_{\Gamma\backslash G} \varphi d\nu}{\int_{\Gamma\backslash G} \psi \, d\nu}.$$

By a standard approximation argument, this convergence also holds for $\psi = \mathbf{1}_{\mathcal{B}}$ where $\mathcal{B}$ is a relatively compact chart of the foliation with $\nu(\partial\mathcal{B}) = 0$.

Consider now such a box $\mathcal{B}$, with $\nu(\partial\mathcal{B}) = m^{\mathcal{F}}_{BR}(\partial\mathcal{B}) = 0$ and a transversal T to the foliation into N-orbits inside $\mathcal{B}$, and a map φ with support in $\mathcal{B}$. Let $r_0 = r_0(\mathcal{B}) > 0$ be the maximal diameter of the connected components of N-orbits inside $\mathcal{B}$. By compactness, r_0 is finite.

Observe that the following inequality holds for all nonnegative continuous maps $\varphi \in C_c(\mathcal{B})$.

$$\frac{\sum_{\mathbf{z} \in T \cap \mathbf{x}N_{e^{t_k}}} \int_{N_{\mathcal{B}}(\mathbf{z})} \varphi \, dn}{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_{e^{t_k}})} \times \frac{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_{r_k})}{\int_{\mathbf{x}N_{r_k+r_0}} \mathbf{1}_{\mathcal{B}} \, dn} \leqslant \frac{\int_{\mathbf{x}N_{r_k+r_0}} \varphi \, dn}{\int_{\mathbf{x}N_{r_k+r_0}} \mathbf{1}_{\mathcal{B}} \, dn} \tag{4.3}$$

This inequality comes from the fact that for $\mathbf{z} \in T \cap \mathbf{x}N_{r_k}$, the connected component of $\mathbf{x}N \cap \mathcal{B}$ containing $\mathbf{z}$ is certainly included in a ball (in $\mathbf{z}N$) of radius r_0 around $\mathbf{z}$.

Now, by the above work and the convergence of the transverse measure ν_{T,t_k} towards ν_T, the first ratio on the left converges to $\frac{1}{m^{\mathcal{F}}_{BR}(\mathcal{B})} \int \varphi dm^{\mathcal{F}}_{BR}$. By assumption on ν, the ratio on the right converges to $\frac{1}{\nu(\mathcal{B})}\int \varphi\, d\nu$. Therefore, necessarily, the ratio

$$\frac{\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_{r_k})}{\int_{\mathbf{x}N_{r_k+r_0}} \mathbf{1}_{\mathcal{B}}\, dn}$$

converges to some positive finite limit, depending on $\mathcal{B}$.

Let us now conclude the proof. We just proved that for any relatively compact chart of the foliation satisfying $\nu(\mathcal{B}) > 0$, $m^{\mathcal{F}}_{BR}(\mathcal{B}) > 0$, $\nu(\partial\mathcal{B}) = m^{\mathcal{F}}_{BR}(\partial\mathcal{B}) = m^{\mathcal{F}}_{BM}(\partial B) = 0$, there exists $0 < C_{\mathcal{B}} < \infty$ such that for all continuous nonnegative maps with support in $\mathcal{B}$, $\int \varphi\, dm^{\mathcal{F}}_{BR} \leqslant C_B \int \varphi\, d\nu$.

It implies that $m^{\mathcal{F}}_{BR}$ is absolutely continuous w.r.t ν which is ergodic. If we knew that $m^{\mathcal{F}}_{BR}$ is conservative, for any N-invariant set Y of positive $m^{\mathcal{F}}_{BR}$, we could apply the above to $\nu = (m^{\mathcal{F}}_{BR})_{|Y}$, and get that $m^{\mathcal{F}}_{BR}$ is absolutely continuous w.r.t $(m^{\mathcal{F}}_{BR})_{|Y}$, and therefore equal. Therefore, if $m^{\mathcal{F}}_{BR}$ is conservative, it is ergodic, and has full support in $\mathcal{E}_{rad}$, and therefore is the unique N-invariant ergodic and conservative measure supported on $\mathcal{E}_{rad}$.

The conclusion of the proof follows from the following lemma.

Lemma 4.2. — *Under the assumptions of the theorem, the measure $m^{\mathcal{F}}_{BR}$ is conservative.*

Proof. — Indeed, choose a compact chart $\mathcal{B}$ of the foliation in N-orbits such that $m^{\mathcal{F}}_{BM}(\mathcal{B}) > 0$ and $m^{\mathcal{F}}_{BM}(\partial\mathcal{B}) = 0$. The convergence in (4.1) together with the fact that $\mu^{BM}_{\mathbf{x}N}(\mathbf{x}N_r) \to +\infty$ when $r \to +\infty$ implies that $\int_{\mathbf{x}N} \mathbf{1}_{\mathcal{B}}(\mathbf{x}n) d\mu^{BM}_{\mathbf{x}N}(\mathbf{x}n) \to +\infty$.

Recall that $N_{\mathcal{B}}(\mathbf{z})$ is the connected component of $\mathbf{z}N \cap \mathcal{B}$ containing $\mathbf{z}$. As $\mathcal{B}$ is compact, we have

$$0 < \inf_{\mathbf{z}\in\mathcal{B}} \frac{\mu^{BM}_{\mathbf{z}N}(N_{\mathcal{B}}(\mathbf{z}))}{\int_{N_{\mathcal{B}}(\mathbf{z})} 1 dn} \leqslant \sup_{\mathbf{z}\in\mathcal{B}} \frac{\mu^{BM}_{\mathbf{z}N}(N_{\mathcal{B}}(\mathbf{z}))}{\int_{N_{\mathcal{B}}(\mathbf{z})} 1 dn} < +\infty.$$

This easily implies that $\int_{\mathbf{x}N} \mathbf{1}_{\mathcal{B}}(\mathbf{x}n) dn \to +\infty$ for all $\mathbf{x} \in \mathcal{E}^{\mathcal{F}}_{rad}$, so that any invariant measure supported on $\mathbf{x} \in \mathcal{E}^{\mathcal{F}}_{rad}$ is conservative. □

Let us now make a brief comparison with Roblin's argument. Once again, he works on the unit tangent bundle, on more general $CAT(-1)$-spaces. He also uses the convergence of subsequences $M_{e^{t_k}}(\varphi)(\mathbf{x})$ to $m^{\mathcal{F}}_{BM}$. And he proves that $m^{\mathcal{F}}_{BR}$ is absolutely continuous w.r.t ν. But to prove that, he first integrates $M_{e^{t_k}}(\varphi)(\mathbf{x})$ w.r.t ν. As the good subsequences $r_k = e^{t_k}$ *depend* strongly on the point $\mathbf{x}$, he needs to work very hard to find some r which is good enough for sufficiently many $\mathbf{x}$. This part of the proof is long and technical, and the approach of Yves Coudène, that we extend here in the infinite volume and higher dimension case, is much simpler.

Acknowledgments

I thank Hee Oh for the reference [7], which allows a proof valid in any dimension, and the referee for his/her comments, which allowed me to improve the end of the argument.

References

[1] Martine Babillot and François Ledrappier. Geodesic paths and horocycle flow on abelian covers. In *Lie groups and ergodic theory (Mumbai, 1996)*, volume 14 of *Tata Inst. Fund. Res. Stud. Math.*, pages 1–32. Tata Inst. Fund. Res., Bombay, 1998.

[2] Marc Burger. Horocycle flow on geometrically finite surfaces. *Duke Math. J.*, 61(3):779–803, 1990. URL: `http://dx.doi.org/10.1215/S0012-7094-90-06129-0`.

[3] Yves Coudene. A short proof of the unique ergodicity of horocyclic flows. In *Ergodic theory*, volume 485 of *Contemp. Math.*, pages 85–89. Amer. Math. Soc., Providence, RI, 2009. URL: `http://dx.doi.org/10.1090/conm/485/09494`.

[4] S. G. Dani. Invariant measures of horospherical flows on noncompact homogeneous spaces. *Invent. Math.*, 47(2):101–138, 1978.

[5] L. Flaminio and R. J. Spatzier. Geometrically finite groups, Patterson-Sullivan measures and Ratner's rigidity theorem. *Invent. Math.*, 99(3):601–626, 1990. URL: `http://dx.doi.org/10.1007/BF01234433`.

[6] Harry Furstenberg. The unique ergodicity of the horocycle flow. In *Recent advances in topological dynamics (Proc. Conf., Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund)*, pages 95–115. Lecture Notes in Math., Vol. 318. Springer, Berlin, 1973.

[7] Michael Hochman. A ratio ergodic theorem for multiparameter non-singular actions. *J. Eur. Math. Soc. (JEMS)*, 12(2):365–383, 2010. URL: `http://dx.doi.org/10.4171/JEMS/201`.

[8] François Maucourant and Barbara Schapira. Distribution of orbits in $\mathbb{R}^2$ of a finitely generated group of $\mathrm{SL}(2,\mathbb{R})$. *Amer. J. Math.*, 136(6):1497–1542, 2014.

[9] Thomas Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95):vi+96, 2003.

[10] Omri Sarig. Invariant Radon measures for horocycle flows on abelian covers. *Invent. Math.*, 157(3):519–551, 2004. URL: `http://dx.doi.org/10.1007/s00222-004-0357-4`.

[11] Barbara Schapira. On quasi-invariant transverse measures for the horospherical foliation of a negatively curved manifold. *Ergodic Theory Dynam. Systems*, 24(1):227–255, 2004. URL: `http://dx.doi.org/10.1017/S0143385703000166`.

[12] Dale Winter. Mixing of frame flow for rank one locally symmetric spaces and measure classification. *Isr. J. Math.*, 2015. URL: `http://dx.doi.org/10.1007/s11856-015-1258-5`.

Manuscript received April 30, 2015,
revised June 7, 2015,
accepted September 14, 2015.

Barbara SCHAPIRA
I.R.M.A.R. UMR CNRS 6625, UFR de mathématiques, Campus de Beaulieu, 263 avenue du Général Leclerc, CS 74205 35042 RENNES Cédex, France
barbara.schapira@univ-rennes1.fr