CONFLUENTES
MATHEMATICI

Heinrich FREISTÜHLER and Matthias KOTSCHOTE

Models of Two-Phase Fluid Dynamics à la Allen-Cahn, Cahn-Hilliard, and ... Korteweg!

Tome 7, nº 2 (2015), p. 57-66.

<http://cml.cedram.org/item?id=CML_2015__7_2_57_0>

Confluentes Math.
7, 2 (2015) 57-66

MODELS OF TWO-PHASE FLUID DYNAMICS À LA ALLEN-CAHN, CAHN-HILLIARD, AND ... KORTEWEG!

HEINRICH FREISTÜHLER AND MATTHIAS KOTSCHOTE

Abstract. One purpose of this paper on the Navier-Stokes-Allen-Cahn (NSAC), the Navier-Stokes-Cahn-Hilliard (NSCH), and the Navier-Stokes-Korteweg (NSK) equations consists in surveying solution theories that one of the authors, M. K., has developed for these three evolutionary systems of partial differential equations. All three theories start from a Helmholtz free energy description of the compressible two-phase fluids whose dynamics they describe in various ways. While a diphasic fluid composed from two constituents of individually constant density is still compressible as long as these two densities are different from each other, the abovementioned solution theories for NSAC and NSCH do not apply in this "quasi-incompressible" case, as the Helmholtz-energy framework degenerates. The second purpose of the paper is to present an observation made by both authors together that shows how to fill these gaps. As 'by-products' one obtains (a) in the case that the phases can transform into each other, a justification of NSK, and (b) in the case that they cannot, a new Korteweg type system with non-local 'viscosity'.

Pensez à REDESSINER !

This paper is dedicated to DENIS SERRE on the occasion of his 60th birthday

1. Three models of two-phase fluid dynamics

We begin by recapitulating three systems of evolutionary partial differential equations that have been proposed for describing the dynamics of two-phase fluids.

1.1. The Navier-Stokes-Allen-Cahn system. The Navier-Stokes-Allen-Cahn system has the form

$$\begin{aligned}\partial_t\rho + \nabla\cdot(\rho u) &= 0,\\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u - \mathbf{T}) &= 0,\\ \partial_t\mathcal{E} + \nabla\cdot((\mathcal{E}\mathbf{I} - \mathbf{T})u) - \nabla\cdot(\beta\nabla\theta) &= 0,\\ \partial_t(\rho\chi) + \nabla\cdot(\rho\chi u) - J &= 0,\end{aligned} \tag{1.1}$$

where $\rho > 0$ and $u \in \mathbb{R}^3$ are the two-phase fluid's density and velocity, $\chi \in (0,1)$ denotes the concentration of one of the phases, and

$$\mathcal{E} \equiv \rho(E + \frac{1}{2}|u|^2) \quad \text{and} \quad \mathbf{T} \equiv -p\mathbf{I} + \mathbf{C} + \mathbf{S}$$

are the total energy and the total Cauchy stress. In this and the next subsection, we assume that the fluid posseses a Helmholtz energy

$$F(\tau,\theta,\chi,\nabla\chi) = \check{F}(\tau,\theta,\chi,|\nabla\chi|^2) \tag{1.2}$$

which satisfies

$$\partial_\tau^2 F > 0 \quad \text{and} \quad \partial_\theta^2 F \ > \ 0; \tag{1.3}$$

in particular, the internal energy E can be obtained from F through the Legendre transform

$$E(\tau,S,\chi,\nabla\chi) \equiv F(\tau,\theta,\chi,\nabla\chi) + \theta S$$

with temperature θ and specific entropy $S = -\partial_\theta F$ as dual variables. Viscous stress

$$\mathbf{S} = \eta(Du)^s + \zeta\nabla\cdot u\,\mathbf{I}, \quad (Du)^s \equiv \frac{1}{2}(Du + (Du)^\top)$$

Math. classification: 76N10, 35Q35, 82B24, 82B26, 35M10.

and heat flux, $-\beta\nabla\theta$, are quantified by means of the coefficients η, ζ, β of shear viscosity, bulk viscosity, and thermal conductivity, functions of ρ, θ, χ, and $|\nabla\chi|^2$, that satisfy

$$\eta, 2\eta + \zeta, \beta > 0 \quad \text{for all values of } (\rho, \theta, \chi, \nabla\chi) \in (0, \infty) \times (0, \infty) \times [0, 1] \times \mathbb{R}^n.$$

Obviously, the first three equations in (1.1) express the conservation of mass, momentum, and energy. The forth equation, in view of the first equivalent to its counterpart

$$\partial_t(\rho(1-\chi)) + \nabla\cdot(\rho(1-\chi)u) + J = 0$$

for the other phase, encodes the exchange between the phases.

The Navier-Stokes-Allen-Cahn equations[1] (NSAC) are now obtained by closing the system through specifying $\mathbf{C}$ and J in terms of F as

$$\mathbf{C} = \mathbf{C}_E, \qquad J = J_{AC} \tag{1.4}$$

with [2]

$$\mathbf{C}_E = -\nabla\chi \otimes \frac{\partial}{\partial\nabla\chi}(\rho F) \quad \text{and} \quad J_{AC} = \frac{\theta}{\epsilon}\bigg(-\frac{\partial}{\partial\chi}\Big(\frac{\rho}{\theta}F\Big) + \nabla\Big(\frac{\partial}{\partial\nabla\chi}\Big(\frac{\rho}{\theta}F\Big)\Big)\bigg). \tag{1.5}$$

Here

$$\epsilon = \epsilon(\rho, \theta, \chi, |\nabla\chi|^2) > 0, \quad (\rho, \theta, \chi, \nabla\chi) \in (0, \infty) \times (0, \infty) \times [0, 1] \times \mathbb{R}^n$$

is a relaxation time.

The Navier-Stokes-Allen-Cahn equations seem to have first been formulated by Blesgen [6].

1.2. The Navier-Stokes-Cahn-Hilliard system. Mathematically, the Navier-Stokes-Cahn-Hilliard equations[3] (NSCH) are just another closure of system (1.1). They are obtained by letting

$$\mathbf{C} = \mathbf{C}_E, \qquad J = J_{CH} \equiv \nabla\cdot\mathcal{J} \tag{1.6}$$

with $\mathbf{C}_E$ as above and

$$\mathcal{J} = \gamma\nabla\big(\frac{1}{\theta}\mu\big) \tag{1.7}$$

with mobility

$$\gamma = \gamma(\rho, \theta, \chi, |\nabla\chi|^2) > 0, \quad (\rho, \theta, \chi, \nabla\chi) \in (0, \infty) \times (0, \infty) \times [0, 1] \times \mathbb{R}^n$$

and

$$\frac{\rho}{\theta}\mu = \partial_\chi\left(\frac{\rho}{\theta}F\right) - \nabla\cdot\left(\partial_{\nabla\chi}\left(\frac{\rho}{\theta}F\right)\right). \tag{1.8}$$

Physically speaking, the principal difference between NSAC and NSCH is — in direct analogy to the difference between the Allen-Cahn and the Cahn-Hilliard models of phase dynamics without convection [2, 7]— that NSAC permits transformations between the phases while the divergence form of J in NSCH represents spatial redistribution without transformation.

The Navier-Stokes-Cahn-Hilliard equations, though in a version from which ours differs[4], have been formulated by Lowengrub and Truskinovsky [22].

---

[1] In the other case $\eta = \zeta = \beta = 0$, system (1.1) is referred to as the Euler-Allen-Cahn equations.
[2] $\mathbf{C}_E$ is the Ericksen tensor [10].
[3] or, in the other case $\eta = \zeta = \beta = 0$, Euler-Cahn-Hilliard equations
[4] For a discussion of this difference, see [13].

1.3. The Navier-Stokes-Korteweg system. Differently from both NSAC and NSCH, the Navier-Stokes-Korteweg equations (NSK) do not have a concentration variable, but only the density ρ as an 'order parameter'. They read

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho u) = 0,\\ \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u - \bar{\mathbf{T}}) = 0,\\ \partial_t\bar{\mathcal{E}} + \nabla\cdot((\bar{\mathcal{E}}\mathbf{I} - \bar{\mathbf{T}})u) - \nabla\cdot(\beta\nabla\theta) = 0, \end{aligned} \tag{1.9}$$

with

$$\bar{\mathcal{E}} \equiv \rho(\bar{E} + \frac{1}{2}|u|^2) \quad \text{and} \quad \bar{\mathbf{T}} \equiv -p\mathbf{I} + \mathbf{K} + \mathbf{S}.$$

Based on Korteweg's classical idea, capillarity is now reflected in the fluid's Helmholtz energy

$$\bar{F}(\rho, \theta, \nabla\rho) = \check{\bar{F}}(\rho, \theta, |\nabla\rho|^2), \tag{1.10}$$

and its internal energy $\bar{E}$,

$$\bar{E}(\rho, S, \nabla\rho) \equiv \bar{F}(\rho, \theta, \nabla\rho) + \theta S, \quad S \equiv -\partial_\theta \bar{F},$$

by their dependence on $\nabla\rho$. While $\mathbf{S}$ is the same viscous stress as above and $\beta > 0$ again the thermal conductivity[5], the 'Korteweg tensor' should be chosen as

$$\mathbf{K} = \theta\left[\rho\nabla\cdot\left(\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right)\mathbf{I} - \nabla\rho \otimes \partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right]. \tag{1.11}$$

Going back to Korteweg [15], the Navier-Stokes-Korteweg equations have been intensely studied by Dunn and Serrin [9], though with an interesting difference[6] regarding $\mathbf{K}$.

2. Solution theories

2.1. Solution theory for NSAC. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with compact boundary $\Gamma := \partial\Omega$ of class C^2 decomposing disjointly as $\Gamma = \Gamma_d \,\dot\cup\, \Gamma_s$, where each set may be empty. The outer unit normal of Γ at position x is denoted by $\nu(x)$. Further, let $J = [0, T], T \in (0, \infty]$, be a time interval. The partial differential equations (1.1) have to be complemented by initial conditions

$$\rho(0, x) = \rho_0(x), \quad u(0, x) = u_0(x), \quad \theta(0, x) = \theta_0(x), \quad \chi(0, x) = \chi_0(x), \quad x \in \Omega, \tag{2.1}$$

and boundary conditions. Two natural boundary conditions are of interest for u, namely the non-slip condition

$$u(t, x) = 0, \quad (t, x) \in J \times \Gamma_d \tag{2.2}$$

and the pure slip condition

$$(u(t, x)\,|\,\nu(x)) = 0, \quad \mathcal{Q}(\nu(x))\mathbf{S}(t, x) \cdot \nu(x) = 0, \quad (t, x) \in J \times \Gamma_s \tag{2.3}$$

with $(\cdot\,|\,\cdot)$ denoting the inner product of $\mathbb{R}^n$. The matrix $\mathcal{Q}(\nu(x)) := \mathcal{I} - \nu(x) \otimes \nu(x)$ projects a vector field on the boundary Γ to its tangential part. As for boundary conditions for θ and χ, we prescribe

$$\theta(t, x) = g_d(t, x), \quad \chi(t, x) = l_d(t, x), \quad (t, x) \in J \times \Gamma_d, \tag{2.4}$$

and

$$(\nabla\theta(t, x)\,|\,\nu(x)) = 0, \quad (\nabla\chi(t, x)\,|\,\nu(x)) = 0, \quad (t, x) \in J \times \Gamma_s. \tag{2.5}$$

[5]In the other case $\eta = \zeta = \beta = 0$, one calls (1.9) the Euler-Korteweg eqautions.

[6]For a discussion of this difference, see [13].

We are looking for solutions (ρ, u, θ, χ) of problem (1.1), (1.4), (2.1)-(2.5) in the regularity class $\mathbb{Z}_1(J) \times \mathbb{Z}_2(J) \times \mathbb{Z}_3(J) \times \mathbb{Z}_4(J)$, where these spaces are defined by

$$\begin{aligned}
\mathbb{Z}_1(J) &:= \mathrm{H}^2_p(J; \mathrm{H}^{-1}_p(\Omega)) \cap \mathrm{C}^1(J; \mathrm{L}_p(\Omega)) \cap \mathrm{C}(J; \mathrm{H}^1_p(\Omega)), \quad \mathrm{H}^{-1}_p(\Omega) := (\mathring{\mathrm{H}}^1_p(\Omega))',\\
\mathbb{Z}_2(J) &:= Z(J; \mathbb{R}^n), \quad \mathbb{Z}_3(J) := Z(J; \mathbb{R}), \quad \mathbb{Z}_4(J) := Z(J; \mathbb{R}),\\
Z(J; E) &:= \mathrm{H}^1_p(J; \mathrm{L}_p(\Omega; E)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\Omega; E)), \quad E \in \{\mathbb{R}^n, \mathbb{R}\}, \quad p \in (1, \infty).
\end{aligned}$$

Here and in the sequel, H^s_p denote the Bessel potential spaces and W^s_p the Slobodeckij spaces ($W^s_p \equiv B^s_{pp}$ Besov spaces), see [23], [24]. We shall also need the function spaces

$$Y_{j,k}(J; E) := \mathrm{W}^{1-j/2-1/2p}_p(J; \mathrm{L}_p(\Gamma_k; E)) \cap \mathrm{L}_p(J; \mathrm{W}^{2-j-1/p}_p(\Gamma_k; E)), \; j = 0, 1, k = d, s. \tag{2.6}$$

The Helmholtz energy F is assumed to be of class C^3, the coefficients $\eta, \zeta, \beta, \epsilon$ of class C^2.

Theorem 2.1 ([18]). — *Assume the situation described in Subsection 1.1 and the hypotheses above. Then, for each initial data $(\rho_0, u_0, \theta_0, \chi_0)$ in*

$$\begin{aligned}
\mathcal{V} := \{(\varrho, \upsilon, \vartheta, c) \in\ & \mathrm{H}^1_p(\Omega) \times \mathrm{W}^{2-2/p}_p(\Omega; \mathbb{R}^n) \times \mathrm{W}^{2-2/p}_p(\Omega) \times \mathrm{W}^{2-2/p}_p(\Omega) :\\
& (\upsilon(y) \,|\, \nu(y)) \geqslant 0 \quad \forall y \in \Gamma, \quad \varrho(x) > 0, \quad \vartheta(x) > 0, \quad c(x) \in [0, 1] \quad \forall x \in \overline{\Omega}\}
\end{aligned}$$

and boundary data

$$g_d, l_d \in Y_{0,d}(J; \mathbb{R}), \quad g_d(t, x) > 0, \quad l_d(t, x) \in (0, 1), \quad \forall (t, x) \in J \times \Gamma_d,$$

satisfying the compatibility conditions

$$\begin{aligned}
& u_{0|\Gamma_d} = 0, \quad (u_0 \,|\, \nu)_{|\Gamma_s} = 0, \quad \mathcal{Q}\mathbf{S}_{|t=0} \cdot \nu_{|\Gamma_s} = 0,\\
& \theta_{0|\Gamma_d} = g_{d|t=0}, \quad \chi_{0|\Gamma_d} = l_{d|t=0}, \quad (\nabla\theta_0 \,|\, \nu)_{|\Gamma_s} = 0, \quad (\nabla\chi_0 \,|\, \nu)_{|\Gamma_s} = 0,
\end{aligned} \tag{2.7}$$

there is a unique solution (ρ, u, θ, χ) of (1.1),(1.4),(2.1)-(2.5) on a maximal time interval, which is $J^* = [0, T^*)$, $T^* := T^*(\rho_0, u_0, \theta_0, \chi_0) \in (0, T]$. The solution (ρ, u, θ, χ) belongs to the class $\mathbb{Z}_1(J_*) \times \mathbb{Z}_2(J_*) \times \mathbb{Z}_3(J_*) \times \mathbb{Z}_4(J_*)$ for each interval $J_* = [0, T_*]$, $0 < T_* < T^*$. If finite, the maximal time T^* is characterised by the property:

$$\lim_{t \to T^*} (\rho, u, \theta, \chi)(t) \quad \text{does not exist in } \mathcal{V}.$$

In the autonomous case, the solution map $(\rho_0, u_0, \theta_0, \chi_0) \mapsto (\rho, u, \theta, \chi)(t)$ generates a local semiflow on the phase space

$$\mathcal{V}_p := \{\varphi := (\rho_0, u_0, \theta_0, \chi_0) \in \mathcal{V} : \ \varphi \text{ satisfies } (2.7)\}.$$

2.2. Solution theory for NSCH. Let $J = [0, T]$ as before and $\Omega \subset \mathbb{R}^n$ be a bounded domain with compact boundary $\Gamma := \partial\Omega$ of class C^4 decomposing disjointly as $\Gamma = \Gamma_0 \,\dot\cup\, \Gamma_s$, where each set may be empty. For the velocity field u, natural boundary conditions are the non-slip and pure slip condition

$$\begin{aligned}
u = 0, & \text{ on } J \times \Gamma_0,\\
(u \,|\, \nu) = 0, \quad \mathcal{Q}\mathbf{S} \cdot \nu = 0, & \text{ on } J \times \Gamma_s,
\end{aligned} \tag{2.8}$$

where by $(\cdot \,|\, \cdot)$, ν, and $\mathcal{Q}$ mean the same as above. Note that in view of the boundary conditions the total mass and the total phase are conserved,

$$\begin{aligned}
\int_\Omega \rho(t, x)\, dx &= \int_\Omega \rho(0, x)\, dx, \quad \forall t \in J,\\
\int_\Omega (\rho\chi)(t, x)\, dx &= \int_\Omega \rho(0, x)\chi(0, x)\, dx, \quad \forall t \in J.
\end{aligned} \tag{2.9}$$

As for boundary conditions for θ, one can prescribe both Dirichlet and Neumann boundary conditions. We therefore assume that Γ also splits as

$$\Gamma = \Gamma_d \,\dot\cup\, \Gamma_n \tag{2.10}$$

and require

$$\theta = g_d,\ (t,x) \in J \times \Gamma_d, \qquad (\nabla\theta \,|\, \nu) = g_n,\ (t,x) \in J \times \Gamma_n. \tag{2.11}$$

Finally, we consider the following boundary conditions for χ

$$\Big(\nabla\big(\frac{1}{\theta}\mu\big) \,|\, \nu\Big) = 0, \quad (\nabla\chi \,|\, \nu) = 0, \quad (t,x) \in J \times \Gamma, \tag{2.12}$$

meaning that no diffusion through the boundary occurs and a possibly present diffuse interface is orthogonal to the boundary of the domain. We also note that in case of $\Gamma_d = \varnothing$ and $g_n \equiv 0$, also the total energy is conserved.

We are interested in strong solutions, now to (1.1) with (1.6), in the L_p-setting. More precisely, we are looking for solutions $(\rho,\theta,\chi,u) \in \mathbb{E}_1 \times \mathbb{E}_2 \times \mathbb{E}_3 \times \mathbb{E}_4$ with

$$\begin{aligned}
\mathbb{E}_1 &:= \mathrm{H}_p^{2+1/4}(J;\mathrm{L}_p(\Omega)) \cap \mathrm{C}^1(J;\mathrm{H}_p^2(\Omega)) \cap \mathrm{C}(J;\mathrm{H}_p^3(\Omega)),\\
\mathbb{E}_2 &:= \mathrm{H}_p^1(J;\mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J;\mathrm{H}_p^3(\Omega)),\\
\mathbb{E}_3 &:= \mathrm{H}_p^1(J;\mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J;\mathrm{H}_p^4(\Omega)),\\
\mathbb{E}_4 &:= \mathrm{H}_p^{3/2}(J;\mathrm{L}_p(\Omega;\mathbb{R}^n)) \cap \mathrm{H}_p^1(J;\mathrm{H}_p^2(\Omega;\mathbb{R}^n)) \cap \mathrm{L}_p(J;\mathrm{H}_p^4(\Omega;\mathbb{R}^n)),
\end{aligned} \tag{2.13}$$

and

$$p \in (\hat{p},\infty), \quad \hat{p} := \max\{4,n\}. \tag{2.14}$$

We also write $\mathbb{E}_i(J)$ to indicate the time interval.

The Helmholtz energy F is now supposed to be of class C^5 and the coefficients η,ζ,β,γ of class C^4.

Theorem 2.2 ([20]). — *Considering the situation described in Subsection 1.2, assume the above and that*

(i) the initial data $(\rho_0,\theta_0,\chi_0,u_0)$ lie in

$$\begin{aligned}
\mathcal{V} := \{(\varrho,\vartheta,c,v) \in \mathrm{H}_p^3(\Omega) \times \mathrm{W}_p^{3-\frac{2}{p}}(\Omega) \times \mathrm{W}_p^{4-\frac{4}{p}}(\Omega) \times \mathrm{W}_p^{4-\frac{2}{p}}(\Omega;\mathbb{R}^n) :&\\
\varrho(x) > 0,\ \vartheta(x) > 0,\ \forall x \in \overline{\Omega}\},&
\end{aligned}$$

(ii) the subsequent compatibility conditions hold:

$$\begin{aligned}
u_{0|\Gamma_0} = 0, \quad (u_0 \,|\, \nu)_{|\Gamma_s} = 0, \quad \mathcal{Q}\mathbf{S}_{|t=0}\cdot\nu_{|\Gamma_s} = 0, \quad \partial_\nu\chi_{0|\Gamma} = 0, \quad \partial_\nu\big(\tfrac{1}{\theta_0}\mu_0\big)_\Gamma = 0,&\\
-\nabla\cdot\mathbf{S}_{|t=0,\Gamma_0} = (\nabla\cdot\mathbf{C})_{|t=0,\Gamma_0} \in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_0;\mathbb{R}^n),&\\
-\big(\nabla\cdot\mathbf{S}_{|t=0} \,|\, \nu\big)_{|\Gamma_s} = (\nabla\cdot\mathbf{C} - \rho\nabla u\cdot u \,|\, \nu)_{|t=0,\Gamma_s} \in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_s),&\\
-\mathcal{Q}\mathbf{S}(\nabla\cdot\mathbf{S})_{|t=0}\cdot\nu_{|\Gamma_s} = \mathcal{Q}\mathbf{S}(\nabla\cdot\mathbf{C} - \rho\nabla u\cdot u)_{|t=0,\Gamma_s}\cdot\nu_{|\Gamma_s} \in \mathrm{W}_p^{1-\frac{3}{p}}(\Gamma_s;\mathbb{R}^n),&\\
\theta_{0|\Gamma_d} = g_{d|t=0} \in \mathrm{W}_p^{3-\frac{3}{p}}(\Gamma_d), \quad \partial_\nu\theta_{0|\Gamma_n} = g_{n|t=0} \in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_n),&
\end{aligned}$$

where μ_0 is defined as $\mu_{|t=0} = \mu[\rho_0,\theta_0,\chi_0]$. Then the problem (1.1), (1.6), (2.1), (2.8), (2.11), (2.12) *possesses a unique strong solution (ρ,θ,χ,u) on a maximal time interval $J^* := [0,T^*), T^* \in (0,T]$. This solution belongs to the class $\mathbb{E}_1(J_*) \times \mathbb{E}_2(J_*) \times \mathbb{E}_3(J_*) \times \mathbb{E}_4(J_*)$ for each interval $J_* = [0,T_*]$ with $0 < T_* < T^*$. If finite, the maximal time T^* is characterized by the property:*

$$\lim_{t\to T^*}(\rho,\theta,\chi,u)(t) \quad \text{does not exist in } \mathcal{V}_p,$$

where $\mathcal{V}_p := \{\omega \in \mathcal{V} :\ \omega$ fulfils the compatibility conditions in (ii)$\}$. Moreover, the solution map $(\rho_0,\theta_0,\chi_0,u_0) \mapsto (\rho,\theta,\chi,u)(\cdot)$ generates a local semiflow on the phase space $\mathcal{V}_p$.

2.3. Solution theory for NSK. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with C^3- boundary, $\Gamma := \partial\Omega$, and $J \equiv [0,T], T \in (0,\infty]$, a time interval.

The equations (1.9) have to be supplemented with initial and boundary conditions

$$\begin{aligned} u &= g_D(t,x), && (t,x) \in J \times \partial\Omega, \\ \partial_\nu \rho &= g_N(t,x), && (t,x) \in J \times \partial\Omega, \\ \theta &= g_d(t,x), && (t,x) \in J \times \partial\Omega, \\ u &= u_0(x), && (t,x) \in \{0\} \times \Omega, \\ \rho &= \rho_0(x), && (t,x) \in \{0\} \times \Omega, \\ \theta &= \theta_0(x), && (t,x) \in \{0\} \times \Omega. \end{aligned} \tag{2.15}$$

Assume F, η, ζ, β are of class C^2.

Theorem 2.3 ([19]). — *Let $n+2 < p < \infty$ and suppose that*

(i) the initial data (u_0, ρ_0, θ_0) belong to

$$\mathcal{V} := \{(\upsilon, \varrho, \vartheta) \in \mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n) \times \mathrm{B}^{3-2/p}_{pp}(\Omega;\mathbb{R}_+) \times \mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}_+) : \\ \varrho_0(x) > 0, \vartheta_0(x) > 0 \forall x \in \overline{\Omega}\};$$

(ii) compatibility conditions hold: $u_{0|\Gamma} = g_{D|t=0}$ in $\mathrm{B}^{2-3/p}_{pp}(\Gamma;\mathbb{R}^n)$, $\partial_\nu \rho_0 = g_{N|t=0}$ in $\mathrm{B}^{2-3/p}_{pp}(\Gamma)$, $\theta_{0|\Gamma} = g_{d|t=0}$ in $\mathrm{B}^{2-3/p}_{pp}(\Gamma)$.

Then there exists $T^* \in (0,T]$ such that for any $T_* \in (0,T^*)$, the nonlinear problem (1.9), (2.15) admits a unique solution (u,ρ,θ) on $J_* = [0,T_*]$ in the maximal regularity class $\mathbb{F}(J_*) := \mathbb{F}_1(J_*) \times \mathbb{F}_2(J_*) \times \mathbb{F}_3(J_*)$ with

$$\begin{aligned} \mathbb{F}_1(J) &:= \mathrm{H}^1_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n)) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega;\mathbb{R}^n)), \\ \mathbb{F}_2(J) &:= \mathrm{H}^{3/2}_p(J;\mathrm{L}_p(\Omega;\mathbb{R}_+)) \cap \mathrm{L}_p(J;\mathrm{H}^3_p(\Omega;\mathbb{R}_+)), \\ \mathbb{F}_3(J) &:= \mathrm{H}^1_p(J;\mathrm{L}_p(\Omega;\mathbb{R}_+)) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega;\mathbb{R}_+)). \end{aligned}$$

If finite, the maximal time T^* is characterized by the property:

$$\lim_{t \to T^*} (u, \rho\theta)(t) \quad \text{does not exist in } \mathcal{V}_p,$$

where $\mathcal{V}_p := \{\omega \in \mathcal{V} : \omega$ fulfils the compatibility conditions in (ii)$\}$. Moreover, the solution map $(u_0,\rho_0,\theta_0) \mapsto (u,\rho,\theta)(\cdot)$ generates a local semiflow on the phase space $\mathcal{V}_p$.

Remark 2.4. — Solution theories for NSK have earlier been given by Hattori and Li [14] and Danchin and Desjardins [8]. A solution theory for the Euler-Korteweg equations is due to Benzoni-Gavage and collaborators [4, 5].

Remark 2.5. — Certain heteroclinic traveling wave solutions of NSK, NSAC, NSCH model diffuse interphase interfaces. Such solutions and their stability have been studied in [22, 3, 5, 11, 12, 16].

3. Reduction of phase-field models to Korteweg-type models.

In this section we consider fluids that consist of two incompressible phases of different temperature-independent[7] specific volumes. Such a fluid is properly described by a Gibbs energy of the form

$$G(p,\theta,\chi,\nabla\chi) = T(\chi)p + W(\theta,\chi,|\nabla\chi|^2), \tag{3.1}$$

where

$$T(\chi) = \chi\tau_1 + (1-\chi)\tau_2 \tag{3.2}$$

[7]The case of two different *temperature-dependent* specific volumes can also be treated, but we refrain from doing this *here*.

with constants $\tau_1, \tau_2 > 0$ satisfying

$$\tau_* \equiv \tau_1 - \tau_2 \neq 0. \tag{3.3}$$

If G satisfied $-\partial_p^2 G < 0$, this description would be related to a Helmholtz energy F through a Legendre transform,

$$F(\tau, \theta, \chi, \nabla\chi) = G(p, \theta, \chi, \nabla\chi) - p\tau.$$

However, as G is affine, (the Legendre transform degenerates and) a description based on a Helmholtz energy is not available. While the capillarity tensor and the exchange rates can be defined refering to G instead of F, namely using

$$\mathbf{C}_E = -\nabla\chi \otimes \rho \frac{\partial G}{\partial \nabla\chi}, \tag{3.4}$$

$$J_{AC} = \frac{\theta}{\epsilon}\left(-\rho \frac{\partial}{\partial\chi}\left(\frac{1}{\theta}G\right) + \nabla\cdot\left(\rho\frac{\partial}{\partial\nabla\chi}\left(\frac{1}{\theta}G\right)\right)\right), \tag{3.5}$$

and, for J_{CH},

$$\frac{\rho}{\theta}\mu = \frac{\rho}{\theta}\partial_\chi G - \nabla\cdot\left(\frac{\rho}{\theta}\partial_{\nabla\chi} G\right), \tag{3.6}$$

the argumentations of Subsections 2.1, 2.2, and thus Theorems 2.1 and 2.2, need F for many more purposes, and therefore do not apply. Instead we find the following.

3.1. Reduction of the Navier-Stokes-Allen-Cahn system.

Theorem 3.1. — *In the case of two molecularly immiscible incompressible phases of different, temperature-independent specific volumes,* (3.1), (3.2), (3.3), *the Navier-Stokes-Allen-Cahn equations* (1.1) *with* $\mathbf{C} = \mathbf{C}_E, J = J_{AC}$ *from* (3.4), (3.5) *can be written as the Navier-Stokes-Korteweg system*

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho u) &= 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u) - \nabla\cdot(-\bar{p}\mathbf{I} + \mathbf{K} + \mathbf{S}_*) &= 0, \\ \partial_t\bar{\mathcal{E}} + \nabla\cdot(\bar{\mathcal{E}}u - (-\bar{p}\mathbf{I} + \mathbf{K} + \mathbf{S}_*)u) - \nabla\cdot(\beta\nabla\theta) &= 0, \end{aligned} \tag{3.7}$$

with $\bar{\mathcal{E}}, \bar{p}, \mathbf{K}$ *derived as in 2.3 from the Helmholtz energy*

$$\bar{F}(\theta, \rho, \nabla\rho) = W(\theta, \bar{\chi}(\rho), [\bar{\chi}'(\rho)]^2|\nabla\rho|^2) \quad \textit{with} \quad \bar{\chi}(\rho) := \frac{1/\rho - \tau_2}{\tau_*} \tag{3.8}$$

and with viscous stress

$$\mathbf{S}_* = \eta(Du)^s + (\zeta + \zeta_*)\nabla\cdot u\,\mathbf{I} \quad \textit{where} \quad \zeta_* \equiv \frac{\epsilon}{\rho\tau_*^2}. \tag{3.9}$$

Proof. — First, note that by relations (3.2), (3.3), the unknown χ can be regarded as a function of ρ. We therefore set

$$\chi = \bar{\chi}(\rho) = \frac{1/\rho - \tau_2}{\tau_1 - \tau_2} = \frac{1/\rho - \tau_2}{\tau_*} \tag{3.10}$$

with derivative

$$\bar{\chi}'(\rho) = -\frac{1}{\tau_*\rho^2}.$$

Moreover, we shall need the relations

$$\begin{gathered} \partial_\chi G = \tau_* p + \partial_\chi W, \\ \partial_{\nabla\chi}\left(\frac{\rho}{\theta}G\right) = \partial_{\nabla\chi}\left(\frac{\rho}{\theta}W\right) = \frac{1}{\bar{\chi}'(\rho)}\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right), \quad \nabla\chi = \bar{\chi}'(\rho)\nabla\rho, \\ \partial_\rho\bar{F} = \bar{\chi}'(\rho)\partial_\chi W + \frac{\bar{\chi}''(\rho)}{\bar{\chi}'(\rho)}\nabla\rho\cdot\partial_{\nabla\rho}\bar{F}, \end{gathered} \tag{3.11}$$

the last of which implies

$$\frac{1}{\tau_*}\partial_\chi W = -\rho^2\partial_\rho\bar{F} + \rho^2\frac{\bar{\chi}''(\rho)}{\bar{\chi}'(\rho)}\nabla\rho\cdot\partial_{\nabla\rho}\bar{F}. \tag{3.12}$$

Looking at the forth equation of (1.1), we evaluate

$$\partial_t(\rho\chi) + \nabla\cdot(\rho\chi u) = \rho\dot{\chi} = \rho\bar{\chi}'(\rho)\dot{\rho} = -\rho^2\bar{\chi}'(\rho)\nabla\cdot u = \frac{1}{\tau_*}\nabla\cdot u$$

and J as

$$\frac{\theta}{\epsilon}\left(\nabla\cdot\left(\frac{\rho}{\theta}\partial_{\nabla\chi}G\right) - \frac{\rho}{\theta}\partial_\chi G\right) = \frac{\theta}{\epsilon}\left(\nabla\cdot\left([\bar{\chi}'(\rho)]^{-1}\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right) - \frac{\rho}{\theta}\tau^* p - \frac{\rho}{\theta}\partial_\chi W\right)$$

and find an explicit representation of the pressure,

$$p = -\frac{\epsilon}{\tau_*^2\rho}\nabla\cdot u - \theta\rho\bar{\chi}'(\rho)\nabla\cdot\left([\bar{\chi}'(\rho)]^{-1}\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right) - \frac{1}{\tau_*}\partial_\chi W. \tag{3.13}$$

Using this, we get

$$\begin{aligned}
-p\,\mathbf{I} + \mathbf{C} + \mathbf{S} &= -p\,\mathbf{I} - \nabla\chi\otimes\partial_{\nabla\chi}(\rho W) + \mathbf{S}\\
&= \frac{1}{\tau_*}\partial_\chi W\,\mathbf{I} + \theta\rho\bar{\chi}'(\rho)\nabla\cdot\left([\bar{\chi}'(\rho)]^{-1}\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right)\mathbf{I} - \nabla\rho\otimes\partial_{\nabla\rho}(\rho\bar{F}) + \mathbf{S}_*\\
&= \frac{1}{\tau_*}\partial_\chi W\,\mathbf{I} + \theta\rho\nabla\cdot\left(\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right)\mathbf{I} - \rho^2\frac{\bar{\chi}''(\rho)}{\bar{\chi}'(\rho)}\nabla\rho\cdot\partial_{\nabla\rho}\bar{F} - \nabla\rho\otimes\partial_{\nabla\rho}(\rho\bar{F}) + \mathbf{S}_*,
\end{aligned}$$

where we have used (3.10) and (3.9). Replacing now the term $\partial_\chi W$ according to the identity (3.12), we indeed find

$$\begin{aligned}
-p\mathbf{I} + \mathbf{C} + \mathbf{S} &= -\bar{p}\mathbf{I} + \theta\left[\rho\nabla\cdot\left(\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right)\mathbf{I} - \nabla\rho\otimes\partial_{\nabla\rho}\left(\frac{\rho}{\theta}\bar{F}\right)\right] + \mathbf{S}_*\\
&= -\bar{p}\mathbf{I} + \mathbf{K} + \mathbf{S}_*
\end{aligned}$$

with the Korteweg tensor $\mathbf{K}$ as defined in (1.11). □

Remark 3.2. — Theorem 3.1 shows that for fluids consisting of two immiscible incompressible phases of different specific volumes, the PDE theory of the Navier-Stokes-Korteweg system is an alternative to the "rather incompressible" description proposed in [22] and pursued (later) in [1].

3.2. Reduction of the Navier-Stokes-Cahn-Hilliard system

Theorem 3.3. — *In the case of two molecularly immiscible incompressible phases of different, temperature-independent specific volumes,* (3.1), (3.2), (3.3), *the Navier-Stokes-Cahn-Hilliard equations* (1.1) *with* $\mathbf{C} = \mathbf{C}_E$ *and* $J = J_{CH}$ *from* (3.4), (1.7), *and* (3.6) *can be written as the non-local Navier-Stokes-Korteweg system*

$$\begin{aligned}
\partial_t\rho + \nabla\cdot(\rho u) &= 0,\\
\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) - \nabla\cdot(-\bar{p}\mathbf{I} + \mathbf{K} + \mathbf{S}_\gamma) &= 0,\\
\partial_t\bar{\mathcal{E}} + \nabla\cdot(\bar{\mathcal{E}}u - (-\bar{p}\mathbf{I} + \mathbf{K} + \mathbf{S}_\gamma)u) - \nabla\cdot(\beta\nabla\theta) &= 0,
\end{aligned} \tag{3.14}$$

with $\bar{\mathcal{E}}, \bar{p}, \mathbf{K}$ *derived as in 2.3 from the Helmholtz energy* (3.8) *and the non-local viscous stress*

$$\mathbf{S}_\gamma = \eta(Du)^s + \zeta\nabla\cdot u\,\mathbf{I} + \frac{\theta}{\tau_*^2}\Lambda_\gamma(\nabla\cdot u)\mathbf{I}, \tag{3.15}$$

where Λ_γ *denotes the solution operator of the elliptic problem*

$$-\nabla\cdot(\gamma\nabla\phi) = \nabla\cdot u \quad \text{on } \Omega.$$

Proof. — The only difference from the proof of Theorem 3.1 consists in the form of the representation for the pressure p. Using (3.10) in the forth equation of (1.1) we obtain

$$\frac{1}{\tau_*}\nabla\cdot u = \nabla\cdot\left(\gamma\nabla\left(\frac{\mu}{\theta}\right)\right) = -\nabla\cdot\left(\gamma\nabla\left(-\frac{\mu}{\theta}\right)\right) \quad\Leftrightarrow\quad \frac{1}{\tau_*^2}\Lambda_\gamma\nabla\cdot u = -\frac{\mu}{\tau_*\theta}$$

with

$$\frac{\mu}{\theta} = \Big(\partial_\chi\Big(\frac{1}{\theta}G\Big) - \frac{1}{\rho}\nabla\cdot\Big(\partial_{\nabla\chi}\Big(\frac{\rho}{\theta}G\Big)\Big) = \frac{\tau_*}{\theta}p + \frac{1}{\theta}\partial_\chi W - \frac{1}{\rho}\nabla\cdot\Big([\bar{\chi}'(\rho)]^{-1}\partial_{\nabla\rho}\Big(\frac{\rho}{\theta}\bar{F}\Big)\Big)$$

and thus

$$\frac{\theta}{\tau_*^2}\Lambda_\gamma\nabla\cdot u = -\frac{1}{\tau_*}\mu = -p - \theta\rho\bar{\chi}'(\rho)\nabla\cdot\Big([\bar{\chi}'(\rho)]^{-1}\partial_{\nabla\rho}\Big(\frac{\rho}{\theta}\bar{F}\Big)\Big) - \frac{1}{\tau_*}\partial_\chi W$$

instead of (3.13). □

Remark 3.4. — An existence theorem for the initial-boundary value problem of (3.14) with (3.15) has been given in [17].

Acknowledgments. H. F. is permanently grateful to Denis Serre for supporting him at early stages of his career. The research leading to this paper was supported by Priority Programme 1506 "Transport Processes at Fluidic Interfaces" of the German Science Foundation (DFG) through grants FR 822/8-1 and KO 4715/1-2.

References

[1] H. Abels, H. Garcke, G. Grün: Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities. *Math. Models Meth. Appl. Sci.* **22** (2012), 1150013.

[2] S. M. Allen, J. W. Cahn : A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metallurgica* **27** (1979), 1085–1095.

[3] S. Benzoni-Gavage: Stability of subsonic planar phase boundaries in a van der Waals fluid *Arch. Rational Mech. Anal.* **150** (1999) 23–55.

[4] S. Benzoni-Gavage, R. Danchin, S. Descombes: On the well-posedness for the Euler-Korteweg model in several space dimensions. *Indiana Univ. Math. J.* 56 (2007), 1499–1579.

[5] S. Benzoni-Gavage, R. Danchin, S. Descombes, D. Jamet: Structure of Korteweg models and stability of diffuse interfaces. *Interfaces Free Bound.* **7** (2005), 371–414.

[6] T. Blesgen: A generalisation of the Navier-Stokes equations to two-phase-flows. *J. Phys. D: Appl. Phys.* **32** (1999), 1119–1123.

[7] J. W. Cahn, J. E. Hilliard: Free energy of a non-uniform system I. Interfacial free energy. *J. Chem. Phys.* **28** (1958), 258–267.

[8] R. Danchin, B. Desjardins : Existence of solutions for compressible fluid models of Korteweg type. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 18 (2001), 97–133.

[9] J. E. Dunn, J. Serrin: On the thermomechanics of interstitial working. *Arch. Rational Mech. Anal.* **88** (1985), 95–133.

[10] J. L. Ericksen: Liquid crystals with variable degree of orientation *Arch. Rational Mech. Anal.* **113** (1990), 97–120.

[11] H. Freistühler: Phase transitions and traveling waves in compressible fluids. *Arch. Rational Mech. Anal.* **211** (2014), 189–204.

[12] H. Freistühler, M. Kotschote: Internal structure of dynamic phase-transition fronts in a fluid with two compressible or incompressible phases, *Bulletin Institute of Mathematics Academia Sinica* (2015).

[13] H. Freistühler, M. Kotschote: Phase-field and Korteweg-type models for the time-dependent flow of compressible two-phase fluids. Submitted.

[14] H. Hattori, D. Li: The existence of global solutions to a fluid dynamic model for materials of Korteweg type. *J. Partial Differential Equations* 9 (1996), 323–342.

[15] D.J. Korteweg: Sur la forme que prennent les équations des mouvements des fluides si l'on tient compte des forces capillaires par des variations de densité. *Arch. Néer. Sci. Exactes II* **6** (1901), 1–24.

[16] M. Kotschote: Spectral analysis for traveling waves in compressible two-phase fluids of Navier-Stokes-Allen-Cahn type. Submitted.

[17] M. Kotschote: Mixing rules and the Navier-Stokes-Cahn-Hilliard equations for compressible heat-conductive fluids. *Bulletin Brazilian Mathematical Society* (2015).

[18] M. Kotschote: Strong solutions of the Navier-Stokes equations for a compressible fluid of Allen-Cahn type. *Arch. Ration. Mech. Anal.* **206** (2012), 489–514.

[19] M. Kotschote: On compressible non-isothermal fluids of non-Newtonian Korteweg-type, *SIAM J. Math. Anal.* **44** (2012), 74–101.

[20] M. Kotschote: Strong solutions to the compressible Navier-Stokes-Cahn-Hilliard system for heat-conductive fluids, in preparation.

[21] M. Kotschote, R. Zacher: Strong solutions in the dynamical theory of compressible fluid mixtures, *Math. Models Methods Appl. Sci.* **25**, no. 7 (2015), 1217–1256.

[22] J. Lowengrub, L. Truskinovsky: Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *Proc. R. Soc. Lond. A* **454** (1998), 2617–2654.
[23] H. Triebel: Interpolation Theory, Function Spaces, Differential Operators. North-Holland, Amsterdam - New York - Oxford, 1978.
[24] H. Triebel: Theory of Function Spaces. Geest & Portig K.-G., Leipzig, 1983.

Manuscript received October 3, 2015,
accepted November 2, 2015.

Heinrich FREISTÜHLER & Matthias KOTSCHOTE
Department of Mathematics and Statistics, University of Konstanz, Germany.
heinrich.freistuehler@uni-konstanz.de (Corresponding author)
matthias.kotschote@uni-konstanz.de