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ERGODIC DILATION OF A QUANTUM DYNAMICAL SYSTEM

CARLO PANDISCIA

Abstract. Using the Nagy dilation of linear contractions on Hilbert space and the Stinespring's theorem for completely positive maps, we prove that any quantum dynamical system admits a dilation in the sense of Muhly and Solel which satisfies the same ergodic properties of the original quantum dynamical system.

1. Introduction

A quantum dynamical system is a pair $(\mathfrak{M}, \Phi)$ consisting of a von Neumann algebra $\mathfrak{M}$ and a normal, i.e. σ-weakly continuous, unital completely positive map $\Phi : \mathfrak{M} \to \mathfrak{M}$.

In this work we will prove that is possible to dilate any quantum dynamical system to a quantum dynamical system where the dynamics Φ is a *-homomorphism of a larger von Neumann algebra.

The existence of a dilation for a quantum dynamical system has been proven by Muhly and Solel [8, Prop. 2.24] using the minimal isometric dilation of completely contractive covariant representations of particular W*-correspondences over von Neumann algebras. In contrast, we prove the existence of a dilation for a quantum dynamical system using the Nagy dilations for linear contractions on Hilbert spaces (see [9]) and a particular representation obtained by the Stinespring theorem for completely positive maps (see [13]).

Throughout this paper we will use the abbreviation ucp-map for unital completely positive maps, and we denote by $\mathfrak{B}(\mathcal{H})$ the C*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$.

In the present paper by a *dilation* of a quantum dynamical system $(\mathfrak{M}, \Phi)$, with $\mathfrak{M}$ defined on a Hilbert space $\mathcal{H}$ we mean a quadruple $(\mathfrak{R}, \Theta, \mathcal{K}, Z)$ where $(\mathfrak{R}, \Theta)$ is a quantum dynamical system with $\mathfrak{R}$ defined on Hilbert space $\mathcal{K}$ and Θ is a **-homomorphism* of $\mathfrak{R}$; and $Z : \mathcal{H} \to \mathcal{K}$ is an isometry satisfying the following properties (see [8]):

- $Z\mathfrak{M}Z^* \subset \mathfrak{R}$;
- $Z^*\mathfrak{R}Z \subset \mathfrak{M}$;
- $\Phi^n(A) = Z^*\Theta^n(ZAZ^*)Z$ for $A \in \mathfrak{M}$ and $n \in \mathbb{N}$;
- $Z^*\Theta^n(X)Z = \Phi^n(Z^*XZ)$ for $X \in \mathfrak{R}$ and $n \in \mathbb{N}$.

Hence, we have the following commutative diagram:

$$\begin{array}{ccccc} & \mathfrak{R} & \xrightarrow{\Theta^n} & \mathfrak{R} & \\ Z \cdot Z^* & \uparrow & & \downarrow & Z^* \cdot Z \\ & \mathfrak{M} & \xrightarrow{\Phi^n} & \mathfrak{M} & \end{array}$$

Notice that in the literature of dynamical systems the dilation problem has taken meanings different from that used here, see e.g. [2, 3, 4, 12].

By a *representation* of a quantum dynamical system $(\mathfrak{M}, \Phi)$ we mean a triple $(\pi, \mathcal{H}, V)$, where $\pi : \mathfrak{M} \to \mathfrak{B}(\mathcal{H})$ is a normal faithful representation on the Hilbert space $\mathcal{H}$ and V is an isometry on $\mathcal{H}$ such that

$$\pi(\Phi(A)) = V^*\pi(A)V \quad \text{for} \quad A \in \mathfrak{M}.$$

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Since π is faithful and normal, we identify the quantum dynamical system $(\mathfrak{M}, \Phi)$ with $(\pi(\mathfrak{M}), \Phi_\bullet)$ where $\Phi_\bullet$ is the ucp-map $\Phi_\bullet(\pi(A)) = V^*\pi(A)V$, for any $A \in \mathfrak{M}$. This this leads us to the study of invariant algebras under the action of isometries.

In fact, in Section 3, we consider a concrete C*-algebra $\mathfrak{A}$ with unit of $\mathfrak{B}(\mathcal{H})$ and an isometry V of $\mathcal{H}$ such that

$$V^*\mathfrak{A}V \subset \mathfrak{A}.$$

If $(\widehat{V}, \widehat{\mathcal{H}}, Z)$ is the minimal unitary dilation of the isometry V, we will prove that there is a C*-algebra $\widehat{\mathfrak{A}}$ of $\mathfrak{B}(\widehat{\mathcal{H}})$ with the following properties:

- $Z\mathfrak{A}Z^* \subset \widehat{\mathfrak{A}}$;
- $Z^*\widehat{\mathfrak{A}}Z \subset \mathfrak{A}$;
- $\widehat{V}^*\widehat{\mathfrak{A}}\widehat{V} \subset \widehat{\mathfrak{A}}$;
- $Z^*\widehat{V}^*X\widehat{V}Z = V^*Z^*XZV$ for $X \in \widehat{\mathfrak{A}}$;
- $Z^*\widehat{V}^*(ZAZ^*)\widehat{V}Z = V^*AV$ for $A \in \mathfrak{A}$.

A dilation of a quantum dynamical system $(\pi(\mathfrak{M}), \Phi_\bullet)$ is given by $(\widehat{\pi(\mathfrak{M})}, \Theta, \widehat{\mathcal{H}}, Z)$, where the *-homomorphism Θ is defined by

$$\Theta(X) := \widehat{V}^*X\widehat{V} \quad \text{for} \quad X \in \widehat{\pi(\mathfrak{M})}.$$

In Section 4 we prove a Stinespring-type theorem for ucp-maps between C*-algebras with unit, fundamental for the proof of the main result of this paper.

In Section 5 we discuss the ergodic properties of the dilation of a quantum dynamical system. To this end it is worth recalling the notion of φ-adjointness. Let $(\mathfrak{M}, \Phi)$ be a quantum dynamical system and let φ be a faithful normal state on $\mathfrak{M}$ with $\varphi \circ \Phi = \varphi$. The dynamics Φ admits a φ-adjoint (see [6]) if there is a normal ucp-map $\Phi_\natural : \mathfrak{M} \to \mathfrak{M}$ such that for each $A, B \in \mathfrak{M}$

$$\varphi(\Phi(A)B) = \varphi(A\Phi_\natural(B)),$$

(see [1, 5, 7, 10] for the relation between reversible processes, modular operators and φ-adjointness). If $(\mathfrak{R}, \Theta)$ is our dilation of the quantum dynamical system $(\mathfrak{M}, \Phi)$, we shall prove that if the dynamics Φ admits a φ-adjoint and

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)| = 0 \quad \text{for} \quad A, B \in \mathfrak{M},$$

then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(Z^*X\Theta^k(Y)Z) - \varphi(Z^*XZ)\varphi(Z^*YZ)| = 0 \quad \text{for} \quad X, Y \in \mathfrak{R}.$$

Before proving the existence of a dilation of a quantum dynamical system, it is necessary to recall the fundamental Nagy dilation theorem. This is the subject of the next section.

2. Nagy dilation theorem

If V is an isometry on a Hilbert space $\mathcal{H}$, there is a triple $(\widehat{V}, \widehat{\mathcal{H}}, Z)$ where $\widehat{\mathcal{H}}$ is a Hilbert space, $Z : \mathcal{H} \to \widehat{\mathcal{H}}$ is an isometry and $\widehat{V}$ is a unitary operator on $\widehat{\mathcal{H}}$ with

$$\widehat{V}Z = ZV \tag{2.1}$$

satisfying the following minimal property:

$$\widehat{\mathcal{H}} = \bigvee_{k\in Z} \widehat{V}^k Z\mathcal{H}, \tag{2.2}$$

see [9]. However, for our purposes it is still useful to recall here the structure of the unitary minimal dilation of an isometry.

For a Hilbert space $\mathcal{K}$ recall that $l^2(\mathcal{K})$ denotes the Hilbert space $\{\xi : \mathbb{N} \to \mathcal{K} : \sum_{n\geqslant 0} |\xi(n)|^2 < \infty\}$. Consider the Hilbert space

$$\widehat{\mathcal{H}} = \mathcal{H} \oplus l^2(F\mathcal{H}) \tag{2.3}$$

and the unitary operator on $\widehat{\mathcal{H}}$ defined as

$$\widehat{V} = \begin{vmatrix} V & F\Pi_0 \\ 0 & W \end{vmatrix}, \tag{2.4}$$

where $F = I - VV^*$ and $\Pi_j : l^2(F\mathcal{H}) \to \mathcal{H}$ is the canonical projection

$$\Pi_j(\xi_0, \xi_1 ... \xi_n ...) = \xi_j \quad \text{for} \quad j \in \mathbb{N},$$

while $W : l^2(F\mathcal{H}) \to l^2(F\mathcal{H})$ is the operator

$$W(\xi_0, \xi_1 ... \xi_n ...) = (\xi_1, \xi_{,2} ...), \quad \text{for} \quad (\xi_0, \xi_1 ... \xi_n ...) \in l^2(F\mathcal{H}).$$

If $Z : \mathcal{H} \to \widehat{\mathcal{H}}$ is the isometry defined by $Zh = h \oplus 0$ for all $h \in \mathcal{H}$, it is simple to prove that the relations (2.1) and (2.2) are verified.

We observe that for each $n \in \mathbb{N}$ we have

$$\widehat{V}^n = \begin{vmatrix} V^n & C(n) \\ 0 & W^n \end{vmatrix}, \tag{2.5}$$

where $C(n) : l^2(F\mathcal{H}) \to \mathcal{H}$ are the following operators:

$$C(n) := \sum_{j=1}^{n} V^{n-j} F \Pi_{j-1} \quad \text{for } n \geqslant 1.$$

Furthermore, for each $n, m \in \mathbb{N}$ we obtain:

$$\Pi_n W^{m^*} = \Pi_{n+m} \quad \text{and} \quad \Pi_n W^{m^*} = \begin{cases} \Pi_{n-m} & \text{if } n \geqslant m \\ 0 & \text{if } n < m \end{cases}, \tag{2.6}$$

since

$$W^{m^*}(\xi_0, \xi_1 \ldots \xi_n \ldots) = (0, 0 \ldots 0, \overbrace{\xi_0}^{m+1}, \xi_1 \ldots),$$

while for each $k, p \in \mathbb{N}$ we obtain:

$$\Pi_p C(k)^* = \begin{cases} FV^{(k-p-1)^*} & \text{if } k > p \\ 0 & \text{elsewhere} \end{cases} \tag{2.7}$$

since for each $h \in \mathcal{H}$ we have:

$$C(k)^* h = (\overbrace{FV^{(k-1)^*} h \ldots FV^* h, Fh}^{k \text{ times}}, 0, 0 \ldots). \tag{2.8}$$

3. Isometric dilation and invariant algebras

In this section we consider a concrete unital C*-algebra $\mathfrak{A}$ of $\mathfrak{B}(\mathcal{H})$ and an isometry V on the Hilbert space $\mathcal{H}$ such that

$$V^* \mathfrak{A} V \subset \mathfrak{A}.$$

If $(\widehat{V}, \widehat{\mathcal{H}}, Z)$ denotes the minimal unitary dilation of the isometry V, we will prove the following proposition:

Proposition 3.1. — *There exists a unital C*-algebra $\widehat{\mathfrak{A}} \subseteq \mathfrak{B}(\widehat{\mathcal{H}})$ such that:*

(a) $Z \mathfrak{A} Z^* \subset \widehat{\mathfrak{A}}$;
(b) $Z^* \widehat{\mathfrak{A}} Z \subset \mathfrak{A}$;
(c) $\widehat{V}^* \widehat{\mathfrak{A}} \widehat{V} \subset \widehat{\mathfrak{A}}$;
(d) $Z^* \widehat{V}^* X \widehat{V} Z = V^* Z^* X Z V$ *for* $X \in \widehat{\mathfrak{A}}$;
(e) $Z^* \widehat{V}^* (Z A Z^*) \widehat{V} Z = V^* A V$ *for* $A \in \mathfrak{A}$.

The statements (d) and (e) are straightforward consequences of (a) and (b) and of the relationship $\widehat{V}Z = ZV$. In order to prove the other statements, we must study two classes of operators on the Hilbert space $\mathcal{H}$, associated to the pair $(\mathfrak{A}, V)$ defined above, which we shall call the gamma and the napla operators.

3.1. **Gamma operators.** We consider the sequences

$$\alpha := (n_1, n_2 \dots n_r, A_1, A_2 \dots A_r),$$

with $n_j \in \mathbb{N}$ and $A_j \in \mathfrak{A}$ for $j = 1, 2, \dots, r$. These elements α are called *strings* of $\mathfrak{A}$ of *length* $l(\alpha) := r$ and *weight* $\dot{\alpha} := \sum_{i=1}^{r} n_i$.

To any string α of $\mathfrak{A}$ correspond two operators of $\mathfrak{B}(\mathcal{H})$ defined by

$$|\alpha) := A_1 V^{n_1} A_2 V^{n_2} \cdots A_r V^{n_r} \quad \text{and} \quad (\alpha| := V^{n_r^*} A_r V^{n_{r-1}^*} A_{r-1} \cdots V^{n_1^*} A_1.$$

Furthermore for each natural number n we define the sets

$$|n) := \{|\alpha) \in \mathfrak{B}(\mathcal{H}) \ : \ \dot{\alpha} = n\},$$

and

$$|n)\mathfrak{A} = \{|\alpha)A \in \mathfrak{B}(\mathcal{H}) \ : \ A \in \mathfrak{A} \text{ and } \alpha\text{-string of } \mathfrak{A} \text{ with } \dot{\alpha} = n\}.$$

The symbols $(n|$ and $\mathfrak{A}(n|$ have analogous meanings.

Proposition 3.2. — *Let α and β be strings of $\mathfrak{A}$. For each $R \in \mathfrak{A}$ we have:*

$$(\alpha|R|\beta) \in \begin{cases} \mathfrak{A}(\dot{\alpha} - \dot{\beta}| & \text{if } \dot{\alpha} \geqslant \dot{\beta} \\ |\dot{\beta} - \dot{\alpha})\mathfrak{A} & \text{if } \dot{\alpha} < \dot{\beta} \end{cases}, \tag{3.1}$$

and

$$|\alpha)R|\beta) \in |\dot{\alpha} + \dot{\beta}). \tag{3.2}$$

Proof. — For each $m, n \in \mathbb{N}$ and $R \in \mathfrak{A}$ we have:

$$V^{m^*} R V^n \in \begin{cases} V^{(m-n)^*}\mathfrak{A} & \text{if } m \geqslant n \\ \mathfrak{A}V^{(n-m)} & \text{if } m < n \end{cases} \tag{3.3}$$

Given $\alpha = (m_1, m_2 \dots m_r, A_1, A_2 \dots A_r)$ and $\beta = (n_1, n_2 \dots n_s, B_1, B_2 \dots B_s)$ we have that

$$(\alpha|R|\beta) = V^{m_r^*} A_r \cdots V^{m_1^*} A_1 R B_1 V^{n_1} \cdots B_s V^{n_s} = (\widetilde{\alpha}|I|\widetilde{\beta}),$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are strings of $\mathfrak{A}$ with $l(\widetilde{\alpha}) + l(\widetilde{\beta}) = l(\alpha) + l(\beta) - 1$. Moreover if $\dot{\alpha} \geqslant \dot{\beta}$ then $\dot{\widetilde{\alpha}} \geqslant \dot{\widetilde{\beta}}$, while if $\dot{\alpha} < \dot{\beta}$ then $\dot{\widetilde{\alpha}} < \dot{\widetilde{\beta}}$. In fact if $m_1 \geqslant n_1$ we obtain:

$$(\alpha|R|\beta) = V^{m_r^*} A_r \cdots A_2 V^{(m_1 - n_1)^*} R_1 B_2 V^{n_2} \cdots B_s V^{n_s} = (\widetilde{\alpha}|I|\widetilde{\beta}),$$

where

$$\begin{aligned} R_1 &= V^{n_1^*} A_1 R B_1 V^{n_1}, \\ \widetilde{\alpha} &= (m_1 - n_1, m_2 \dots m_r, R_1, A_2 \dots A_r), \quad \text{and} \\ \widetilde{\beta} &= (n_2 \dots n_s, B_2 \dots B_S). \end{aligned}$$

If $m_1 < n_1$ then we can write:

$$(\alpha|R|\beta) = V^{m_r^*} A_r \cdots V^{m_2^*} A_2 R_1 V^{(n_1 - m_1)} B_2 \cdots B_s V^{n_s} = (\widetilde{\alpha}|I|\widetilde{\beta}),$$

where

$$\begin{aligned} R_1 &= V^{m_1^*} A_1 R B_1 V^{m_1}, \\ \widetilde{\alpha} &= (m_2 \dots m_r, A_2 \dots A_r) \quad \text{and} \\ \widetilde{\beta} &= (n_1 - m_1, n_2 \dots n_s, R_1, B_2 \dots B_S). \end{aligned}$$

The proof of (3.1) follows by induction on the number $\nu = l(\alpha) + l(\beta)$. The equation (3.2) follows by a direct calculation. □

Now, given the orthogonal projection $F = I - VV^*$ (see Section 2), for each string α of $\mathfrak{A}$ with $\dot\alpha \geqslant 1$ we define

$$\Gamma(\alpha) := (\alpha|F\Pi_{\dot\alpha-1},$$

which we call the *gamma operator* associated to $(\mathfrak{A}, V)$. The linear space generated by all gamma operators $\Gamma(\alpha)$ for $\dot\alpha \geqslant 1$ will be denoted by $\mathtt{G}(\mathfrak{A}, V)$.

Proposition 3.3. — *For any strings α and β of $\mathfrak{A}$ with $\dot\alpha, \dot\beta \geqslant 1$, we have*

$$\Gamma(\alpha)\Gamma(\beta)^* \in \mathfrak{A}.$$

Proof. — Note that

$$\Gamma(\alpha)\Gamma(\beta)^* = (\alpha|F\Pi_{\dot\alpha-1}\Pi^*_{\dot\beta-1}F|\beta) = \begin{cases} (\alpha|F|\beta) & \text{if } \dot\alpha = \dot\beta \\ 0 & \text{if } \dot\alpha \neq \dot\beta \end{cases}.$$

In fact if $\dot\alpha = \dot\beta$ we have that

$$(\alpha|F|\beta) = (\alpha|(I - VV^*)|\alpha) = (\alpha|I|\alpha) - (\alpha|VV^*|\alpha) \in \mathfrak{A},$$

since $(\alpha|V \in (\dot\alpha - 1|$ and $V^*|\alpha) \in |\dot\alpha - 1)$, and $(\dot\alpha - 1|I|\dot\alpha - 1) \subset \mathfrak{A}$ by relationship (3.1). □

The gamma operators associated to $(\mathfrak{A}, V)$ define an operator system Σ of $\mathfrak{B}(l^2(F\mathcal{H}))$ by

$$\Sigma := \{T \in \mathfrak{B}(l^2(F\mathcal{H})) : \Gamma_1 T \Gamma_2^* \in \mathfrak{A} \quad \text{for all } \Gamma_1, \Gamma_2 \in \mathtt{G}(\mathfrak{A}, V)\}. \tag{3.4}$$

We observe that the unit I belongs to Σ and that

$$\Gamma_1^* A \Gamma_2 \in \Sigma \quad \text{for} \quad A \in \mathfrak{A},$$

for any pair of gamma operators Γ_1, Γ_2. Furthermore, it is easy to prove that Σ is norm closed, and it is weakly closed if $\mathfrak{A}$ is a W*-algebra.

3.2. **Napla operators.** For strings α and β of $\mathfrak{A}$, any $A \in \mathfrak{A}$ and $k \in \mathbb{N}$ we define

$$\Delta_k(A, \alpha, \beta) := \Pi^*_{\dot\alpha+k}F|\alpha)A(\beta|F\Pi_{\dot\beta+k}.$$

We call these operators of $\mathfrak{B}(l^2(F\mathcal{H}))$ the *napla* operators associated to the pair $(\mathfrak{A}, V)$.

In the next lines we show that the linear space generated by the napla operators form a $*$-algebra. To this end, it is easily seen that $\Delta_k(A, \alpha, \beta)^* = \Delta_k(A^*, \beta, \alpha)$ for any $h, k \geqslant 0$. Moreover we have the following two relationships: if $k + \dot\beta \neq h + \dot\gamma$, then

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = 0, \tag{3.5}$$

while if $k + \dot\beta = h + \dot\gamma$, then there is ϑ and $R \in \mathfrak{A}$ with

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \begin{cases} \Delta_k(R, \alpha, \vartheta) & \text{if } h - k \geqslant 0, \text{ where } \dot\vartheta = \dot\delta + h - k \\ \Delta_h(R, \vartheta, \delta) & \text{if } h - k < 0, \text{ where } \dot\vartheta = \dot\delta + k - h. \end{cases} \tag{3.6}$$

In fact, notice that

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \Pi^*_{\dot\alpha+k}F|\alpha)A(\beta|F\Pi_{\dot\beta+k}\Pi^*_{\dot\gamma+h}F|\gamma)B(\delta|F\Pi_{\dot\delta+h}.$$

If $k+\dot\beta \neq h+\dot\gamma$ it follows that $\Pi_{\dot\beta+k}\Pi^*_{\dot\gamma+h} = 0$, and this shows (3.5). If $k+\dot\beta = h+\dot\gamma$, without lost of generality we can assume that $h \geqslant k$. So $\dot\beta = \dot\gamma + h - k \geqslant \dot\gamma$ and, by relationship (3.1), we have that $(\beta|F|\gamma) \in \mathfrak{A}(\dot\beta - \dot\gamma|$. Consequently, $A(\beta|F|\gamma)B(\delta| \in \mathfrak{A}(\dot\delta + \dot\beta - \dot\gamma|$, and there exists a ϑ string of $\mathfrak{A}$ and an element $R \in \mathfrak{A}$ such that $\dot\vartheta = \dot\delta + \dot\beta - \dot\gamma$ and $A(\beta|F|\gamma)B(\delta| = R(\vartheta|$. Now, since $\dot\vartheta = \dot\delta + h - k$ we have:

$$\begin{aligned}\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) &= \Pi^*_{\dot\alpha+k}F|\alpha)R(\vartheta|F\Pi_{\dot\delta+h} \\ &= \Pi^*_{\dot\alpha+k}F|\alpha)R(\vartheta|F\Pi_{\dot\vartheta+k} = \Delta_k(R, \alpha, \vartheta),\end{aligned}$$

showing relationship (3.6).

Proposition 3.4. — *The linear space $\mathfrak{X}_o$ generated by the napla operators is a *-subalgebra of $\mathfrak{B}(l^2(F\mathcal{H}))$ included in the operator systems Σ defined in (3.4).*

Proof. — From relationships (3.5),(3.6) the linear space $\mathfrak{X}_o$ is a *-algebra. Furthermore for each pair $\Gamma(\alpha)$, $\Gamma(\beta)$ of gamma operators we obtain:

$$\Gamma(\alpha)\Delta_k(A,\gamma,\delta)\Gamma(\beta)^* = (\alpha|F\Pi_{\dot\alpha-1}\Pi^*_{\dot\gamma+k}F|\gamma)A(\delta|F\Pi_{\dot\delta+k}\Pi_{\dot\beta-1}F|\beta) \in \mathfrak{A},$$

since by the relationships (3.1) and (3.2) we have

$$(\alpha|F\Pi_{\dot\alpha-1}\Pi^*_{\dot\gamma+k}F|\gamma)A(\delta|F\Pi_{\dot\delta+k}\Pi_{\dot\beta-1}F|\beta) \in \begin{cases} (k+1|\mathfrak{A}|k+1) & \text{if } \begin{cases}\dot\alpha-1=\dot\gamma+k,\\ \dot\beta-1=\dot\delta+k\end{cases} \\ 0 & \text{elsewhere}\end{cases}$$

In fact, if $\dot\alpha=\dot\gamma+k+1$ we can write

$$(\alpha|F\Pi_{\dot\alpha-1}\Pi^*_{\dot\gamma+k}F|\gamma) = (\alpha|F|\gamma) = (\alpha|I|\gamma) - (\alpha|VV^*|\gamma) \in \mathfrak{A}(k+1|,$$

since $(\alpha|I|\gamma) \in \mathfrak{A}(k+1|$ and $(\alpha|VV^*|\gamma) \in \mathfrak{A}(k+1|$. If $\dot\beta=\dot\delta+k+1$ we have $(\delta|F\Pi_{\dot\delta+k}\Pi_{\dot\beta-1}F|\beta) \in (k+1|\mathfrak{A}$, completing the proof. □

The next result is concerned with W-invariance.

Proposition 3.5. — *The *-algebra $\mathfrak{X}_o$ and the operator system Σ are W-invariants:*

$$W^*\mathfrak{X}_oW \subset \mathfrak{X}_o \quad \textit{and} \quad W^*\Sigma W \subset \Sigma.$$

Proof. — The first inclusion follows by (2.6). Concerning the second one, let $T\in\Sigma$. For each pair $\Gamma(\alpha)$, $\Gamma(\beta)$ of gamma operators

$$\begin{aligned}\Gamma(\alpha)(W^*TW)\Gamma(\beta)^* &= (\alpha|F\Pi_{\dot\alpha-1}W^*TW\Pi_{\dot\beta-1}F|\beta)\\ &= (\alpha|F\Pi_{\dot\alpha-2}T\Pi_{\dot\beta-2}F|\beta) \in \mathfrak{A}V^*\Gamma_1(\alpha_o)T\Gamma_2(\beta_o)V\mathfrak{A},\end{aligned}$$

where α_o and β_o are strings of $\mathfrak{A}$ with $\dot\alpha_o=\dot\alpha-1$ and $\dot\beta_o=\dot\beta-1$. In fact if $\alpha=(m_1,m_2\ldots m_r,A_1,A_2\ldots A_r)$, then, by definition of the gamma operator, there is $i\leqslant r$ with $m_i\geqslant 1$ such that

$$(\alpha|F\Pi_{\dot\alpha-2} = A_1\cdots A_iV^*(\alpha_o|F\Pi_{\dot\alpha-2} = A_1\cdots A_iV^*\Gamma(\alpha_o),$$

where

$$\alpha_o = (0,\ldots 0, m_i-1, m_{i+1}\ldots m_r, A_1,A_2\ldots A_r)$$

with $\dot\alpha_o=\dot\alpha-1$. Consequently

$$\Gamma(\alpha)(W^*TW)\Gamma(\beta)^* \subset V^*\mathfrak{A}V \subset \mathfrak{A},$$

completing the proof. □

3.3. **The algebra generated by the napla and gamma operators.** Let $\mathfrak{X}$ be the closure in norm of the *-algebra $\mathfrak{X}_o$ of the apla operators previously defined. Since the operator system Σ defined in (3.4) is a norm closed set, we have $\mathfrak{X}\subset\Sigma$. Notice that in case $\mathfrak{A}$ is a von Neumann algebra of $\mathfrak{B}(\mathcal{H})$, the operator system Σ is weakly closed and $\mathfrak{X}''_o\subset\Sigma$.

Proposition 3.6. — *The set*

$$\mathcal{S} = \left\{ \begin{vmatrix} A & \Gamma_1 \\ \Gamma_2^* & T \end{vmatrix} : A\in\mathfrak{A}, T\in\mathfrak{X} \textit{ and } \Gamma_1,\Gamma_2 \in \mathcal{G}(\mathfrak{A},V) \right\} \tag{3.7}$$

is an operator system of $\mathfrak{B}(\widehat{\mathcal{H}})$ such that:

$$\widehat{V}^*\mathcal{S}\widehat{V} \subset \mathcal{S}.$$

Furthermore

$$\widehat{V}^*\mathcal{A}^*(\mathcal{S})\widehat{V} \subset \mathcal{A}^*(\mathcal{S}),$$

*where $\mathcal{A}^*(\mathcal{S})$ is the *-algebra generated by the set $\mathcal{S}$.*

Proof. — From relationship (2.4) we obtain:

$$\widehat{V}^*\mathcal{S}\widehat{V}=\begin{vmatrix} V^*AV & V^*AC(1)+V^*\Gamma_1 W \\ C(1)^*AV+W^*\Gamma_2^*V & C(1)^*AC(1)+W^*\Gamma_2^*C(1)+C(1)^*\Gamma_1 W+W^*TW \end{vmatrix}$$

We observe that $V^*\Gamma(\alpha)W$ and $V^*AC(1)$ are gamma operators associated to the pair $(\mathfrak{A}, V)$, while $C(1)^*AC(1)$, $C(1)^*\Gamma(\alpha)W$ and W^*TW are operators belonging to $\mathfrak{X}$. In fact we have $V^*AC(1) = V^*AF\Pi_0 = \Gamma(\vartheta)$ with $\vartheta = (1, A)$; while if

$$\alpha = (m_1, m_2 \dots m_r, A_1, A_2 \dots A_r),$$

then $V^*\Gamma(\alpha)W = V^*(\alpha|F\Pi_{\dot{\alpha}-1}W = \Gamma(\vartheta)$, with

$$\vartheta = (m_1+1, m_2 \dots m_r, A_1, A_2 \dots A_r)$$

since $\Pi_{\dot{\alpha}-1}W = \Pi_{\dot{\alpha}}$. Furthermore

$$C(1)^*AC(1) = \Pi_0^*FAF\Pi_0 = \Delta_0(A, \alpha, \beta),$$

with $\alpha = \beta = (0, I)$; while

$$C(1)^*\Gamma(\alpha)W = \Pi_0^*F(\alpha|F\Pi_{\dot{\alpha}-1}W = \Pi_0^*F|\gamma)(\alpha|F\Pi_{\dot{\alpha}+0} = \Delta_0(I, \gamma, \alpha)$$

with $\gamma = (0, I)$, where the last statement follows from the fact that $\widehat{V}$ is unitary. □

We observe that $\mathcal{A}^*(\mathcal{S})$, the *-algebra generated by the operator system $\mathcal{S}$ defined in (3.7), is the linear space generated by the following elements of $\mathfrak{B}(\widehat{\mathcal{H}})$:

$$\begin{vmatrix} A_1 & A_2\Gamma_1 T_1 \\ T_2\Gamma_2^*A_3 & T_3 \end{vmatrix}$$

with $A_i \in \mathfrak{A}$, $\Gamma_j \in \mathtt{G}(\mathfrak{A}, V)$ and $T_k \in \mathfrak{X}$ for all $i, k = 1, 2, 3$ and $j = 1, 2$. We list here some easy properties of the *-algebra $\mathcal{A}^*(\mathcal{S})$:

(a) $Z\mathfrak{A}Z^* \subset \mathcal{A}^*(\mathcal{S})$;
(b) $Z^*\mathcal{A}^*(\mathcal{S})Z \subset \mathfrak{A}$;
(c) $\widehat{V}^*\mathcal{A}^*(\mathcal{S})\widehat{V} \subset \mathcal{A}^*(\mathcal{S})$.

Furthermore, since $\widehat{V}Z = ZV$ we have:

(d) $Z^*\widehat{V}^*X\widehat{V}Z = V^*Z^*XZV$;
(e) $Z^*\widehat{V}^*(ZAZ^*)\widehat{V}Z = V^*AV$.

Using these results we prove the Proposition 3.1.

Proof of Proposition 3.1. — Let $\widehat{\mathfrak{A}}$ be the C*-subalgebra of $\mathcal{B}(\widehat{\mathcal{H}})$ generated by

$$\bigcup_{k=0}^{\infty} \widehat{V}^{k^*}ZAZ^*\widehat{V}^k \quad \text{for} \quad A \in \mathfrak{A}. \tag{3.8}$$

For each natural number k we have that $\widehat{V}^{k^*}Z\mathfrak{A}Z^*\widehat{V}^k \subset \widehat{V}^{k^*}\mathcal{S}\widehat{V}^k \subset \mathcal{S}$, since $Z\mathfrak{A}Z^* \subset \mathcal{S}$; so $\widehat{\mathfrak{A}} \subset C^*(\mathcal{S})$, the norm closure of the *-algebra $\mathcal{A}^*(\mathcal{S})$. It is easily seen that $\widehat{\mathfrak{A}}$ satisfies the conditions of Proposition 3.1, completing the proof. □

Remark 3.7. — It is straightforward to show that if $\mathfrak{A}$ is a von Neumann algebra of $\mathcal{B}(\mathcal{H})$, then the Proposition 3.1 still holds true, with $\widehat{\mathfrak{A}}$ the von Neumann algebra of $\mathcal{B}(\widehat{\mathcal{H}})$ generated by the elements (3.8).

4. Stinespring representation and quantum dynamical systems

We consider a concrete C*-algebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ with unit and a ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$.
On the algebraic tensor product $\mathfrak{A} \otimes \mathcal{H}$ we can define a semi-inner product by

$$\langle A_1 \otimes h_1, A_2 \otimes h_2\rangle_\Phi := \langle h_1, \Phi(A_1^*A_2)h_2\rangle_{\mathcal{H}},$$

for all $A_1, A_2 \in \mathfrak{A}$ and $h_1, h_2 \in \mathcal{H}$. We denote by $\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H}$ the Hilbert space completion of the quotient space of $\mathfrak{A}\otimes\mathcal{H}$ by the linear subspace $\{\mathtt{T} \in \mathfrak{A} \otimes \mathcal{H} : \langle \mathtt{T}, \mathtt{T}\rangle_\Phi = 0\}$, with inner product induced by $\langle\cdot,\cdot\rangle_\Phi$. Furthermore, we denote the image of $A\otimes h\in\mathfrak{A}\otimes\mathcal{H}$ in $\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H}$ by $A\overline{\otimes}_\Phi h$; so

$$\langle A_1\overline{\otimes}_\Phi h_1, A_2\overline{\otimes}_\Phi h_2\rangle_{\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H}} = \langle h_1, \Phi(A_1^*A_2)h_2\rangle_\mathcal{H}$$

for all $A_1, A_2\in\mathfrak{A}$ and $h_1, h_2\in\mathcal{H}$.

Moreover, we define a representation $\sigma_\Phi:\mathfrak{A}\to\mathcal{B}(\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H})$ by

$$\sigma_\Phi(A)(X\overline{\otimes}_\Phi h) := AX\otimes_\Phi h \quad\text{for}\quad A\in\mathfrak{A} \text{ and } X\overline{\otimes}_\Phi h\in\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H},$$

and a linear isometry $V_\Phi:\mathcal{H}\to\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H}$ by

$$V_\Phi h := 1\overline{\otimes}_\Phi h \quad\text{for}\quad h\in\mathcal{H},$$

satisfying the equation

$$\Phi(A) = V_\Phi^*\sigma_\Phi(A)V_\Phi \quad\text{for}\quad A\in\mathfrak{A}. \tag{4.1}$$

The triple $(V_\Phi,\sigma_\Phi,\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H})$ is the Stinespring representation of the ucp-map Φ (see [13]).

Our aim is to analyze the behaviour of the isometry V_Φ and of its adjoint V_Φ^* on the multiplicative domain of the ucp-map Φ. To this end note that the adjoint V_Φ^* verifies $V_\Phi^*A\overline{\otimes}_\Phi h = \Phi(A)h$ for any $A\in\mathfrak{A}$ and $h\in\mathcal{H}$. Furthermore, recall that the multiplicative domain of the ucp-map $\Phi:\mathfrak{A}\to\mathfrak{A}$ is the C*-subalgebra with unit of $\mathfrak{A}$ defined as

$$\mathcal{D}_\Phi = \{A\in\mathfrak{A} : \Phi(A^*)\Phi(A) = \Phi(A^*A) \text{ and } \Phi(A)\Phi(A^*) = \Phi(AA^*)\},$$

see [11]. The multiplicative domain is characterized by the following relationship

$$A\in\mathcal{D}_\Phi \iff \sigma_\Phi(A)V_\Phi V_\Phi^* = V_\Phi V_\Phi^*\sigma_\Phi(A). \tag{4.2}$$

In fact, we first note that

$$A\overline{\otimes}_\Phi h = 1\overline{\otimes}_\Phi\Phi(A)h \quad\text{for all } h\in\mathcal{H} \iff \Phi(A^*A) = \Phi(A^*)\Phi(A),$$

since

$$|A\overline{\otimes}_\Phi h - 1\overline{\otimes}_\Phi\Phi(A)h|^2 = \langle h,\Phi(A^*A)h\rangle - \langle h,\Phi(A^*)\Phi(A)h\rangle.$$

Consequently, for any $A\in\mathcal{D}_\Phi$ and $B\overline{\otimes}_\Phi h\in\mathfrak{A}\overline{\otimes}_\Phi\mathcal{H}$ we have

$$\begin{aligned}\sigma_\Phi(A)V_\Phi V_\Phi^*B\overline{\otimes}_\Phi h &= A\overline{\otimes}_\Phi\Phi(B)h = 1\overline{\otimes}_\Phi\Phi(A)\Phi(B)h\\ &= 1\overline{\otimes}_\Phi\Phi(AB)h = V_\Phi V_\Phi^*\sigma_\Phi(A)B\overline{\otimes}_\Phi h,\end{aligned}$$

where we have used the property of the multiplicative domain $\Phi(A)\Phi(B) = \Phi(AB)$ (see [13]). Conversely, if $\sigma_\Phi(A)V_\Phi V_\Phi^* = V_\Phi V_\Phi^*\sigma_\Phi(A)$ then

$$\begin{aligned}\Phi(A^*A) &= V_\Phi^*\sigma_\Phi(A^*A)V_\Phi = V_\Phi^*\sigma_\Phi(A^*)\sigma_\Phi(A)V_\Phi V_\Phi^*V_\Phi\\ &= V_\Phi^*\sigma_\Phi(A^*)V_\Phi V_\Phi^*\sigma_\Phi(A)V_\Phi = \Phi(A^*)\Phi(A),\end{aligned}$$

and this completes the proof of (4.2).

It is easily seen from (4.2) that Φ is a *homomorphism if, and only if, V_Φ is a unitary operator.

The next steps provides some simple applications of the Stinespring representation of ucp-maps.

Let $\mathfrak{A}$ be a concrete C*-subalgebra with unit of $\mathcal{B}(\mathcal{H})$ and $\Phi:\mathfrak{A}\to\mathfrak{A}$ a ucp-map. By the Stinespring's theorem we obtain a triple $(V_0,\sigma_1,\mathcal{H}_1)$, with $\mathcal{H}_1 = \mathfrak{A}\overline{\otimes}_\Phi\mathcal{H}$ such that $\Phi(A) = V_0^*\sigma_1(A)V_0$ for all $A\in\mathfrak{A}$. Moreover the application $\Phi_1:\mathfrak{A}\to\mathcal{B}(\mathcal{H}_1)$ defined by $\Phi_1(A) := \sigma_1(\Phi(A))$, for $A\in\mathfrak{A}$, is a ucp-map because it is a composition of ucp-maps. By applying the Stinespring's theorem to Φ_1, we have a new triple $(V_1,\sigma_2,\mathcal{H}_2)$, with $\mathcal{H}_2 = \mathfrak{A}\overline{\otimes}_{\Phi_1}\mathcal{H}_1$ such that $\Phi_1(A) = V_1^*\sigma_2(A)V_1$ for all $A\in\mathfrak{A}$. So, iterating this procedure we obtain, for each natural number $n\geqslant 1$, a ucp-map $\Phi_n:\mathfrak{A}\to\mathfrak{B}(\mathcal{H}_n)$ such that

$$\Phi_n(A) = \sigma_n(\Phi(A)) \quad\text{for}\quad A\in\mathfrak{A}, \tag{4.3}$$

and a new triple $(V_n, \sigma_{n+1}, \mathcal{H}_{n+1})$, where $\mathcal{H}_{n+1} = \mathfrak{A}\overline{\otimes}_{\Phi_n}\mathcal{H}_n$, and an isometry $V_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ such that $\Phi_n(A) = V_n^*\sigma_{n+1}(A)V_n$ for all $A \in \mathfrak{A}$.

Now we prove the following Stinespring-type theorem (see [14]):

Proposition 4.1. — *Let $\mathfrak{A}$ be a concrete C*-algebra with unit of $\mathcal{B}(\mathcal{H})$ and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a ucp-map. There exists an injective representation $(\pi_\infty, \mathcal{H}_\infty)$ of $\mathfrak{A}$ and a linear isometry V_∞ on the Hilbert Space $\mathcal{H}_\infty$ such that*

$$\pi_\infty(\Phi(A)) = V_\infty^*\pi_\infty(A)V_\infty \quad \textit{for} \quad A \in \mathfrak{A}.$$

Furthermore, $A \in \mathcal{D}_\Phi$ if, and only if, $V_\infty V_\infty^*\pi_\infty(A) = \pi_\infty(A)V_\infty V_\infty^*$.

Proof. — We consider for each natural number n the ucp-map $\Phi_n : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_n)$ defined in (4.3) and its Stinespring representation $(V_n, \sigma_{n+1}, \mathcal{H}_{n+1})$ with $\mathcal{H}_0 = \mathcal{H}$ and $\sigma_0 = id$. Then, we obtain a faithful representation $\pi_\infty : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_\infty)$ on the Hilbert space $\mathcal{H}_\infty = \bigoplus_{n\geqslant 0} \mathcal{H}_n$ by defining

$$\pi_\infty(A) := \bigoplus_{n\geqslant 0} \sigma_n(A) \quad \text{for} \quad A \in \mathfrak{A}.$$

Now, let $V_\infty : \mathcal{H}_\infty \to \mathcal{H}_\infty$ be the isometry defined by

$$V_\infty(h_0, h_1 \ldots h_n \ldots) := (0, V_0h_0, V_1h_1 \ldots V_nh_n \ldots), \tag{4.4}$$

for all $h_n \in \mathcal{H}_n$ and $n \in \mathbb{N}$. Note that the adjoint of V_∞ is

$$V_\infty^*(h_0, h_1, \ldots h_n \ldots) = (V_0^*h_1, V_1^*h_2 \ldots V_{n-1}^*h_n \ldots) \tag{4.5}$$

for all $h_n \in \mathcal{H}_n$ and $n \in \mathbb{N}$. Hence, for any n and $h_n \in \mathcal{H}_n$ we have

$$V_\infty^*\pi_\infty(A)V_\infty \bigoplus_{n\geqslant 0} h_n = \bigoplus_{n\geqslant 0} \Phi_n(A)h_n = \bigoplus_{n\geqslant 0} \sigma_n(\Phi(A))h_n = \pi_\infty(\Phi(A)) \bigoplus_{n\geqslant 0} h_n.$$

Finally, the last statement easily follows by 4.2.
In fact if $A \in \mathcal{D}_\Phi$ then $A \in \mathcal{D}_{\Phi_n}$ for all natural number n, where $\mathcal{D}_{\Phi_n}$ is the multiplicative domain of the ucp-map (4.3), then

$$V_\infty V_\infty^* \in \pi_\infty(\bigcap_{n\geqslant 0} \mathcal{D}_{\Phi_n})' \subset \pi_\infty(\mathcal{D}_\Phi)'.$$

□

We have the following remark on the existence of a representation of a quantum dynamical system:

Remark 4.2. — Let $(\mathfrak{M}, \Phi)$ be a quantum dynamical system. The injective representation $\pi_\infty(A) : \mathfrak{M} \to \mathfrak{B}(\mathcal{H}_\infty)$ defined in proposition 4.1 is normal, since the Stinespring representation $\sigma_\Phi : \mathfrak{A} \to \mathcal{B}(\mathcal{L}_\Phi)$ is a normal map. Then $(\pi_\infty, \mathcal{H}_\infty, V_\infty)$ is a representation of the quantum dynamical system $(\mathfrak{M}, \Phi)$.

4.1. **Dilation of a quantum dynamical system.** We use the results of the previous section to analyze the problem of dilation of quantum dynamical systems.

Consider a ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$ with $\mathfrak{A}$ a concrete C*-algebra with unit of $\mathfrak{B}(\mathcal{H})$. If $(\mathcal{H}_\infty, \pi_\infty, V_\infty)$ is the Stinespring representation of Proposition 4.1, then

$$V_\infty^*\pi_\infty(\mathfrak{A})V_\infty \subset \pi_\infty(\Phi(\mathfrak{A}) \subset \pi_\infty(\mathfrak{A}).$$

Hence, we can define a normal ucp-map $\Phi_\infty : \pi_\infty(\mathfrak{A})'' \to \pi_\infty(\mathfrak{A})''$ as

$$\Phi_\infty(B) := V_\infty^*BV_\infty \quad \text{for} \quad B \in \pi_\infty(\mathfrak{A})''.$$

Clearly we have that $\Phi_\infty(\pi_\infty(A)) = \pi_\infty(\Phi(A))$ for all $A \in \mathfrak{A}$.

Now, if $(\widehat{V}, \widehat{\mathcal{H}}, Z)$ is minimal unitary dilation of the isometry $V_\infty : \mathcal{H}_\infty \to \mathcal{H}_\infty$, then by Proposition 3.1 there is a C*-algebra with unit $\widehat{\mathfrak{A}}$ of $\mathcal{B}(\widehat{\mathcal{H}})$ such that:

(a) $Z\pi_\infty(\mathfrak{A})Z^* \subset \widehat{\mathfrak{A}}$,
(b) $Z^*\widehat{\mathfrak{A}}Z = \pi_\infty(\mathfrak{A})$,
(c) $\widehat{V}^*\widehat{\mathfrak{A}}\widehat{V} \subset \widehat{\mathfrak{A}}$.

Furthermore, we have a *-homomorphism $\widehat{\Phi}:\widehat{\mathfrak{A}}\to\widehat{\mathfrak{A}}$ defined by

$$\widehat{\Phi}(X)=\widehat{V}^{*}X\widehat{V}\quad\text{for}\quad X\in\widehat{\mathfrak{A}},\tag{4.6}$$

such that for any $A\in\mathfrak{A}$, $X\in\widehat{\mathfrak{A}}$ and any natural number n we have:

$$\pi_{\infty}(\Phi^{n}(A))=Z^{*}\widehat{\Phi}^{n}(ZAZ^{*})Z,$$

and

$$\pi_{\infty}(\Phi^{n}(Z^{*}XZ))=Z^{*}\widehat{\Phi}^{n}(X)Z.$$

In conclusion, it is straightforward to prove that $(\widehat{\mathfrak{A}}'',\Theta,\widehat{\mathcal{H}},Z)$, with $\Theta:\widehat{\mathfrak{A}}''\to\widehat{\mathfrak{A}}''$ the normal *-homomorphism

$$\Theta(X):=\widehat{V}^{*}X\widehat{V}\quad\text{for}\quad X\in\widehat{\mathfrak{A}}'',$$

is a dilation of the quantum dynamical system $(\pi_{\infty}(\mathfrak{A})'',\Phi_{\infty})$ above defined.

Summarizing, the quantum dynamical system $(\mathfrak{M},\Phi)$ can be identified with its associated quantum dynamical system $(\pi_{\infty}(\mathfrak{M}),\Phi_{\infty})$ which admits the dilation $(\widehat{\pi_{\infty}(\mathfrak{M})},\Theta,\widehat{\mathcal{H}},Z)$.

4.2. **The deterministic part of a quantum dynamical system and its dilations.** In this section we study which relationships there are between the dilations and the deterministic part of a quantum dynamical system.

Let $\Phi:\mathfrak{A}\to\mathfrak{A}$ be a ucp-map as described in previous section and $C^{*}(\mathcal{S})$ the C*-algebra generated by the operator systems $\mathcal{S}$ defined in (3.7).

We recall that $\mathcal{S}\subset\mathcal{A}^{*}(\mathcal{S})\subset C^{*}(\mathcal{S})\subset\mathfrak{B}(\widehat{\mathcal{H}})$ where $\widehat{\mathcal{H}}=\mathcal{H}_{\infty}\oplus l^{2}(F\mathcal{H}_{\infty}))$ with $F=I-V_{\infty}V_{\infty}^{*}$. By relationships (a), (b) and (c) of Section 3.3, we can define a *-homomorphism $\Lambda:C^{*}(\mathcal{S})\to C^{*}(\mathcal{S})$ as follows:

$$\Lambda(X)=\widehat{V}^{*}X\widehat{V}\quad\text{for}\quad X\in C^{*}(\mathcal{S}).\tag{4.7}$$

Furthermore, we have a ucp-map $\mathcal{E}:C^{*}(\mathcal{S})\to\mathfrak{A}$ such that

$$\pi_{\infty}(\mathcal{E}(X))=Z^{*}XZ\quad\text{for}\quad X\in C^{*}(\mathcal{S})$$

and for any natural number $n\in\mathbb{N}$

$$\mathcal{E}\circ\Lambda^{n}=\Phi^{n}\circ\mathcal{E}.$$

Hence, we have the following diagram:

$$\begin{array}{ccc} C^{*}(\mathcal{S}) & \xrightarrow{\Lambda^{n}} & C^{*}(\mathcal{S})\\ \mathcal{E}\downarrow & & \downarrow\mathcal{E}\\ \mathfrak{A} & \xrightarrow{\Phi^{n}} & \mathfrak{A}\end{array}$$

where $\mathcal{E}(ZAZ^{*})=A$ for all $A\in\mathfrak{A}$.

We consider now the C*-algebra $\mathcal{D}:=\bigcap_{n\geqslant 0}\mathcal{D}_{\Phi^{n}}$ where the set $\mathcal{D}_{\Phi^{n}}$ is the multiplicative domain of the ucp-map $\Phi^{n}:\mathfrak{A}\to\mathfrak{A}$ for all natural numbers n. The restriction of Φ to $\mathcal{D}$ is a *-homomorphism $\Phi_{\circ}:\mathcal{D}\to\mathcal{D}$ of C*-algebras. It is said to be the *deterministic part* of the ucp-map $\Phi:\mathfrak{A}\to\mathfrak{A}$.

The *-homomorphism Λ defined above is related to the deterministic part of Φ in the following way:

Proposition 4.3. — *There is an injective *-homomorphism $i:\mathcal{D}\to C^{*}(\mathcal{S})$ such that for each natural number n and $D\in\mathcal{D}$ we have:*

$$\mathcal{E}(\Lambda^{n}(i(D)))=\Phi^{n}(D)$$

and

$$\Lambda^{n}(i(D))=i(\Phi^{n}(D)).$$

Proof. — Since $F \in \pi_\infty(\mathcal{D}_\Phi)' \subset \pi_\infty(\mathcal{D})'$ by Proposition 4.1, the map $\Xi : \mathcal{D} \to \mathfrak{B}(l^2(F\mathcal{H}_\infty))$ defined by

$$\Xi(D) = \sum_{k\geqslant 0} \Pi_k^* F\pi_\infty(\Phi_r^{-(k+1)}(D)F\Pi_k \qquad D \in \mathcal{D}$$

is a representation. Furthermore for any $D \in \mathcal{D}$ we have that $\Xi(D)$ belongs to $\mathfrak{X}_0$, the linear space generated by the napla operators defined in Proposition 3.4, since $\Pi_k^* F\pi_\infty(\Phi_r^{-(k+1)}(D)F\Pi_k$ is the napla operator $\Delta_k(\pi_\infty(\Phi_r^{-(k+1)}(D)), \alpha, \beta)$ with the strings $\alpha = \beta = (0, I)$.

We define a *-homomorphism $i : \mathcal{D} \to C^*(\mathcal{S})$ as follows

$$i(D) = \pi_\infty(D) \oplus \Xi(D) \quad \text{for} \quad D \in \mathcal{D},$$

and by relationship (2.5) we obtain that

$$\Lambda^n(i(D)) = \begin{vmatrix} V^{n^*}\pi_\infty(D)V^n, & V^{n^*}\pi_\infty(D)C_n \\ C_n^*\pi_\infty(D)V^n, & C_n^*\pi_\infty(D)C_n + W^{n^*}\Xi(D)W^n \end{vmatrix}.$$

It is straightforward to prove that

$$C_n^*\pi_\infty(D)C_n + W^{n^*}\Xi(D)W^n = \Xi(\Phi^n(D))$$

and $C_n^*\pi_\infty(D)V^n = 0$, since by relationship (2.8) we have

$$FV^{(n-k)^*}\pi_\infty(D)V^n = \pi_\infty(\Phi^{(n-k)}(D))FV^k = 0$$

for all $1 \leqslant k \leqslant n$, completing the proof. □

Finally, we observe that there is the following relationship between dilations and the deterministic part of a quantum dynamical system:

If $(\mathfrak{R}, \Theta, \mathcal{K}, Z)$ is any dilation of quantum dynamical system $(\mathfrak{M}, \Phi)$, then for any natural number n and $D \in \mathcal{D}$ we have :

$$\Theta^n(ZDZ^*)Z = Z\Phi_\circ^n(D),$$

since if $Y = \Theta^n(ZDZ^*)Z - Z\Phi^n(D)$, then $Y^*Y = 0$.

5. Ergodic properties

Let $\mathfrak{A}$ be a concrete C*-algebra of $\mathcal{B}(\mathcal{H})$ with unit, $\Phi : \mathfrak{A} \to \mathfrak{A}$ a ucp-map and φ a state on $\mathfrak{A}$ such that $\varphi \circ \Phi = \varphi$. We recall that φ is an ergodic state, relative to the ucp-map Φ (see [10]), if for each $A, B \in \mathfrak{A}$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} (\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)) = 0,$$

and that φ is weakly mixing if for each $A, B \in \mathfrak{A}$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)| = 0.$$

By Proposition 4.1 we can assume that $\mathfrak{A}$ is a concrete C*-algebra of $\mathfrak{B}(\mathcal{H})$, and that there is an isometry V on $\mathcal{H}$ such that:

$$\Phi(A) = V^*AV \quad \text{for} \quad A \in \mathfrak{A}.$$

Let $(\widehat{V}, \widehat{\mathcal{H}}, Z)$ be the minimal unitary dilation of $(V, \mathcal{H})$ defined in (2.4), let $\widehat{\mathfrak{A}}$ be the C*-algebra included in $\mathfrak{B}(\widehat{\mathcal{H}})$ defined in Proposition 3.1, and let $\widehat{\Phi} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ be the ucp-map defined in (4.6).

Proposition 5.1. — *If the ucp-map Φ admits a φ-adjoint and φ is an ergodic state, then:*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} [\varphi(Z^*X\widehat{\Phi}^k(Y)Z) - \varphi(Z^*XZ)\varphi(Z^*YZ))] = 0$$

for all $X, Y \in \widehat{\mathfrak{A}}$*, while if* φ *is weakly mixing, then:*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(Z^* X \widehat{\Phi}^k(Y) Z) - \varphi(Z^* X Z)\varphi(Z^* Y Z)| = 0$$

for all $X, Y \in \widehat{\mathfrak{A}}$*.*

The proof of this proposition is a straightforward consequence of the next lemma.

To this purpose, we make a preliminary observation. Recall that $\widehat{\mathcal{H}} = \mathcal{H} \oplus l^2(F\mathcal{H})$ and that, writing an element X of $\mathcal{B}(\widehat{\mathcal{H}})$ in matrix representation

$$X = \begin{vmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{vmatrix},$$

the following relationship holds:

$$\varphi(Z^* X \widehat{\Phi}^k(Y) Z) = \varphi(X_{1,1}\Phi^k(Y_{1,1})) + \varphi(X_{1,2}C(k)^* Y_{1,1} V^k) + \varphi(X_{1,2} W^{k^*} Y_{2,1} V^k).$$

Lemma 5.2. — *Let* $X \in \mathcal{A}^*(\mathcal{S})$*, the *-algebra generated by the operator system* $\mathcal{S}$ *defined in (3.7) and* $Y \in \widehat{\mathfrak{A}}$*. The following relations hold:*

(a) *If* φ *is an ergodic state then we have:*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^* Y_{1,1} V^k + X_{1,2} W^{k^*} Y_{2,1} V^k) = 0, \tag{5.1}$$

(b) *If* φ *is weakly mixing then we have:*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(X_{1,2}C(k)^* Y_{1,1} V^k + X_{1,2} W^{k^*} Y_{2,1} V^k)| = 0. \tag{5.2}$$

Proof. — Since $X \in \mathcal{A}^*(\mathcal{S})$, we can assume without loss of generality that $X_{1,2} = A\Gamma(\gamma)\Delta_m(B, \alpha, \beta)$ with $A, B \in \mathfrak{A}$ and α, β, γ strings of $\mathfrak{A}$. Then we can write

$$X_{1,2} = \begin{cases} A(\gamma|F|\alpha)B(\beta|F\Pi_{\dot\beta+m} & \text{if } \dot\gamma - 1 = \dot\alpha + m \\ 0 & \text{elsewhere} \end{cases} \tag{5.3}$$

since

$$X_{1,2} = A(\gamma|F\Pi_{\dot\gamma-1}\Pi^*_{\dot\alpha+m}F|\alpha)B(\beta|F\Pi_{\dot\beta+m}.$$

Observe that we can find a natural number k_o such that the relation

$$X_{1,2} W^{k^*} Y_{2,1} V^k = 0 \tag{5.4}$$

holds for each $k > k_o$. In fact

$$W^{k^*}(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (\overbrace{0 \ldots, 0}^{k-time}, \xi_0, \xi_1, \ldots),$$

for all vectors $(\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in l^2(F\mathcal{H})$; so $\Pi_{\beta+m} W^{k^*} = 0$ for all $k > \dot\beta + m$. Then by equation (5.4) it follows that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^* Y_{1,1} V^k + X_{1,2} W^{k^*} Y_{2,1} V^k)$$
$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^* Y_{1,1} V^k).$$

Hence we have to compute only $\varphi(X_{1,2}C(k)^* Y_{1,1} V^k)$. Notice that

$$X_{1,2}C(k)^* Y_{1,1} V^k = A(\gamma|F|\alpha)B(\beta|F\Pi_{\dot\beta+m}C(k)^* Y_{1,1} V^k$$

by relationship (5.3), and that

$$\Pi_{\dot\beta+m}C(k)^* = FV^{(k-\dot\beta-m-1)^*} \quad \text{for} \quad k > \dot\beta + m,$$

by relationship (2.7). It follows that

$$\begin{aligned} X_{1,2}C(k)^*Y_{1,1}V^k &= A(\gamma|F|\alpha)B(\beta|FV^{(k-\dot\beta-m-1)^*}Y_{1,1}V^k \\ &= A(\gamma|F|\alpha)B(\beta|F\Phi^{(k-\dot\beta-1)}(Y_{1,1})V^{\dot\beta+m+1}. \end{aligned}$$

Since $\dot\gamma = \dot\alpha + m + 1$, we have $A(\gamma|F|\alpha)B(\beta| \in \mathfrak{A}(\dot\beta + m + 1|$ by relationship (3.1). Hence there is a string ϑ of $\mathfrak{A}$ with $\dot\vartheta = \dot\beta + m + 1$ and an operator $R \in \mathfrak{A}$, such that $A(\gamma|F|\alpha)B(\beta| = R(\vartheta|$. So we can write

$$X_{1,2}C(k)^*Y_{1,1}V^k = R(\vartheta|F\Phi^{(k-\dot\beta-1)}(Y_{1,1})V^{\dot\beta+m+1}.$$

If we set $\vartheta = (n_1, n_2, \ldots n_r, A_1, A_2, \ldots A_r)$ then we have $n_1+n_2+\ldots+n_r = \dot\beta+m+1$ and

$$\begin{aligned} &R(\vartheta|F\Phi^{(k-\dot\beta-1)}(Y_{1,1})V^{\dot\beta+m+1} \\ &\qquad = RV^{n_r^*}A_rV^{n_{r-1}^*}A_{r-1}\cdots A_2V^{n_1^*}A_1F\Phi^{(k-\dot\beta-1)}(Y_{1,1})V^{\dot\beta+m+1} \\ &\qquad = R\Phi^{n_r}(A_r\Phi^{n_{r-1}}(A_{r-1}\cdots\Phi^{n_2}(A_2R_k))), \end{aligned}$$

where

$$R_k = \Phi^{n_1}(A_1\Phi^{(k-\dot\beta-1)}(Y_{1,1})) - \Phi^{n_1-1}(\Phi(A_1)\Phi^{(k-\dot\beta)}(Y_{1,1})) \in \mathfrak{A}.$$

Using the φ-adjont, we have

$$\varphi(X_{1,2}C(k)^*Y_{1,1}V^k) = \varphi(\Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2R_k). \tag{5.5}$$

In fact,

$$\begin{aligned} \varphi(X_{1,2}C(k)^*Y_{1,1}V^k) &= \varphi(R\Phi^{n_r}(A_r\Phi^{n_{r-1}}(A_{r-1}\cdots\Phi^{n_2}(A_2R_k)))) \\ &= \varphi(\Phi_\natural^{n_r}(R)A_r\Phi^{n_{r-1}}(A_{r-1}(\cdots\Phi^{n_2}(A_2R_k)))) \\ &= \varphi(\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)A_{r-1}(A_{r-2}\cdots A_3\Phi^{n_2}(A_2R_k)) \\ &= \varphi(\Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2R_k), \end{aligned}$$

and replacing R_k we obtain that

$$\begin{aligned} &\Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2R_k \\ &\quad = \Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2\Phi^{n_1}(A_1\Phi^{(k-\dot\beta-1)}(Y_{1,1}))- \\ &\quad - \Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2\Phi^{n_1-1}(\Phi\,(A_1)\,\Phi^{(k-\dot\beta)}(Y_{1,1})). \end{aligned}$$

Therefore

$$\begin{aligned} &\varphi(X_{1,2}C(k)^*Y_{1,1}V^k) \\ &\quad = \varphi(\Phi_\natural^{n_1}(\Phi_\natural^{n_2}(\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots)A_2)A_1\Phi^{(k-\dot\beta-1)}(Y_{1,1}))- \\ &\quad - \varphi(\Phi_\natural^{n_1-1}(\Phi_\natural^{n_2}(\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots)A_2)\Phi(A_1)\Phi^{(k-\dot\beta)}(Y_{1,1}))). \end{aligned}$$

Now, assume that φ is ergodic. Then we have that

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi(\Phi_\natural^{n_1}(\Phi_\natural^{n_2}(\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots)A_2)A_1\Phi^{(k-\dot\beta-1)}(Y_{1,1})) \\ &\quad = \varphi(\Phi_\natural^{n_1}(\Phi_\natural^{n_2}(\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots)A_2)A_1)\varphi(Y_{1,1}), \end{aligned}$$

and that

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi(\Phi_\natural^{n_1-1}(\Phi_\natural^{n_2}(\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots)A_2)\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})) \\ &\quad = \varphi(\Phi_\natural^{n_1-1}(\Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2)\Phi(A_1))\varphi(Y_{1,1}) \\ &\quad = \varphi(\Phi_\natural(\Phi_\natural^{n_1-1}(\Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2))A_1)\varphi(Y_{1,1}). \end{aligned}$$

Thus

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varphi(X_{1,2}C(k)^*Y_{1,1}V^k)=0,$$

completing the proof of item (a).

In the weakly mixing case, using relationship (5.5) we obtain:

$$\begin{aligned}&|\varphi(X_{1,2}C_k^*Y_{1,1}V^k)|\\ &\qquad=|\varphi\big(T\Phi^{n_1}(A_1)\Phi^{(k-\dot{\beta}-1)}(Y_{1,1})-\varphi(T\Phi^{n_1-1}(\Phi(A_1)\Phi^{k-\dot{\beta})}(Y_{1,1})))|,\end{aligned}$$

where $T=\Phi_\natural^{n_2}(\Phi_\natural^{n_3}\cdots\Phi_\natural^{n_{r-1}}(\Phi_\natural^{n_r}(R)A_r)\cdots A_3)A_2$.

Adding and subtracting the element $\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})$ we can write:

$$\begin{aligned}|\varphi(X_{1,2}C_k^*Y_{1,1}V^k)|\leqslant&\,|\varphi(T\Phi^{n_1}(A_1)\Phi^{(k-\dot{\beta}-1)}(Y_{1,1}))-\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|\\ &+|\varphi(T\Phi^{n_1-1}(\Phi(A_1)\Phi^{(k-\dot{\beta})}(Y_{1,1})))-\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|.\end{aligned}$$

Moreover

$$\begin{aligned}&|\varphi(T\Phi^{n_1-1}(\Phi(A_1)\Phi^{(k-\dot{\beta})}(Y_{1,1})))-\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|\\ &\quad=|\varphi(\Phi_\natural^{n_1-1}(T)\Phi(A_1)\Phi^{(k-\dot{\beta})}(Y_{1,1}))-\varphi(\Phi_\natural^{n_1-1}(T)\Phi(A_1)\varphi(Y_{1,1})|,\end{aligned}$$

and by the weakly mixing properties we obtain:

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}|\varphi(T\Phi^{n_1}(A_1)\Phi^{(k-\dot{\beta}-1)}(Y_{1,1}))-\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|=0,$$

and

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}|\varphi(\Phi_\natural^{n_1-1}(T)\Phi(A_1)\Phi^{(k-\dot{\beta})}(Y_{1,1}))-\varphi(\Phi_\natural^{n_1-1}(T)\Phi(A_1))\varphi(Y_{1,1})|=0$$

completing the proof of item (b). □

Finally, the proof of proposition Proposition 5.1 is a simple consequence of this lemma since the C*-algebra $\widehat{\mathfrak{A}}$ is included in $C^*(\mathcal{S})$, the norm closure of *-algebra $\mathcal{A}(\mathcal{S})$.

It is clear that Proposition 5.1 can be extended to a quantum dynamical system $(\mathfrak{M},\Phi)$ with φ a normal faithful state on $\mathfrak{M}$.

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