ERGODIC DILATION OF A QUANTUM DYNAMICAL SYSTEM

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Abstract. Using the Nagy dilation of linear contractions on Hilbert space and the Stinespring’s theorem for completely positive maps, we prove that any quantum dynamical system admits a dilation in the sense of Muhly and Solel which satisfies the same ergodic properties of the original quantum dynamical system.

1. Introduction

A quantum dynamical system is a pair \((\mathcal{M}, \Phi)\) consisting of a von Neumann algebra \(\mathcal{M}\) and a normal, i.e. \(\sigma\)-weakly continuous, unital completely positive map \(\Phi : \mathcal{M} \to \mathcal{M}\).

In this work we will prove that it is possible to dilate any quantum dynamical system to a quantum dynamical system where the dynamics \(\Phi\) is a \(*\)-homomorphism of a larger von Neumann algebra.

The existence of a dilation for a quantum dynamical system has been proven by Muhly and Solel [8, Prop. 2.24] using the minimal isometric dilation of completely contractive covariant representations of particular \(W^*\)-correspondences over von Neumann algebras. In contrast, we prove the existence of a dilation for a quantum dynamical system using the Nagy dilations for linear contractions on Hilbert spaces (see [9]) and a particular representation obtained by the Stinespring theorem for completely positive maps (see [13]).

Throughout this paper we will use the abbreviation ucp-map for unital completely positive maps, and we denote by \(\mathfrak{B}(\mathcal{H})\) the C*-algebra of all bounded linear operators on a Hilbert space \(\mathcal{H}\).

In the present paper by a dilation of a quantum dynamical system \((\mathcal{M}, \Phi)\), with \(\mathcal{M}\) defined on a Hilbert space \(\mathcal{H}\) we mean a quadruple \((\mathcal{R}, \Theta, \mathcal{K}, Z)\) where \((\mathcal{R}, \Theta)\) is a quantum dynamical system with \(\mathcal{R}\) defined on Hilbert space \(\mathcal{K}\) and \(\Theta\) is a \(*\)-homomorphism of \(\mathcal{R}\); and \(Z : \mathcal{H} \to \mathcal{K}\) is an isometry satisfying the following properties (see [8]):

- \(Z\mathcal{M}Z^* \subset \mathcal{R}\);
- \(Z^*\mathcal{R}Z \subset \mathcal{M}\);
- \(\Phi^n(A) = Z^*\Theta^n(ZAZ^*)Z\) for \(A \in \mathcal{M}\) and \(n \in \mathbb{N}\);
- \(Z^*\Theta^n(X)Z = \Phi^n(Z^*XZ)\) for \(X \in \mathcal{R}\) and \(n \in \mathbb{N}\).

Hence, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\Theta^n} & \mathcal{R} \\
Z \cdot Z^* & \uparrow & \downarrow Z^* \cdot Z \\
\mathcal{M} & \xrightarrow{\Phi^n} & \mathcal{M}
\end{array}
\]

Notice that in the literature of dynamical systems the dilation problem has taken meanings different from that used here, see e.g. [2, 3, 4, 12].

By a representation of a quantum dynamical system \((\mathcal{M}, \Phi)\) we mean a triple \((\pi, \mathcal{H}, V)\), where \(\pi : \mathcal{M} \to \mathfrak{B}(\mathcal{H})\) is a normal faithful representation on the Hilbert space \(\mathcal{H}\) and \(V\) is an isometry on \(\mathcal{H}\) such that

\[\pi(\Phi(A)) = V^*\pi(A)V\text{ for } A \in \mathcal{M}.\]
Since $\pi$ is faithful and normal, we identify the quantum dynamical system $(\mathcal{M}, \Phi)$ with $(\pi(\mathcal{M}), \Phi_*)$ where $\Phi_*$ is the ucp-map $\Phi_*(\pi(A)) = V^*\pi(A)V$, for any $A \in \mathcal{M}$.

This this leads us to the study of invariant algebras under the action of isometries. In fact, in Section 3, we consider a concrete C*-algebra $\mathfrak{A}$ of $\mathcal{B}(H)$ and an isometry $V$ of $H$ such that

$$V^*\mathfrak{A}V \subset \mathfrak{A}.$$ 

If $(\hat{V}, \hat{\mathcal{H}}, Z)$ is the minimal unitary dilation of the isometry $V$, we will prove that there is a C*-algebra $\hat{\mathfrak{A}}$ of $\mathcal{B}(\hat{\mathcal{H}})$ with the following properties:

- $Z\mathfrak{A}Z^* \subset \hat{\mathfrak{A}}$;
- $Z^*\hat{\mathfrak{A}}Z \subset \hat{\mathfrak{A}}$;
- $\hat{V}^*\hat{\mathfrak{A}}\hat{V} \subset \hat{\mathfrak{A}}$;
- $Z^*\hat{V}^*X\hat{V}Z = V^*Z^*XZV$ for $X \in \hat{\mathfrak{A}}$;
- $Z^*\hat{V}^*(ZAZ^*)\hat{V}Z = V^*AV$ for $A \in \mathfrak{A}$.

A dilation of a quantum dynamical system $(\pi(\mathcal{M}), \Phi_*)$ is given by $(\pi(\mathcal{M}), \Theta, \hat{\mathcal{H}}, Z)$, where the $*$-homomorphism $\Theta$ is defined by

$$\Theta(X) := \hat{V}^*X\hat{V} \quad \text{for} \quad X \in \pi(\mathcal{M}).$$

In Section 4 we prove a Stinespring-type theorem for ucp-maps between C*-algebras with unit, fundamental for the proof of the main result of this paper.

In Section 5 we discuss the ergodic properties of the dilation of a quantum dynamical system. To this end it is worth recalling the notion of $\varphi$-adjointness. Let $(\mathcal{M}, \Phi)$ be a quantum dynamical system and let $\varphi$ be a faithful normal state on $\mathcal{M}$ with $\varphi \circ \Phi = \varphi$. The dynamics $\Phi$ admits a $\varphi$-adjoint (see [6]) if there is a normal ucp-map $\Phi_\sharp : \mathcal{M} \to \mathcal{M}$ such that for each $A, B \in \mathcal{M}$

$$\varphi(\Phi(A)B) = \varphi(A\Phi_\sharp(B)),$$

(see [1, 5, 7, 10] for the relation between reversible processes, modular operators and $\varphi$-adjointness). If $(\mathcal{M}, \Theta)$ is our dilation of the quantum dynamical system $(\mathcal{M}, \Phi)$, we shall prove that if the dynamics $\Phi$ admits a $\varphi$-adjoint and

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)| = 0 \quad \text{for} \quad A, B \in \mathcal{M},$$

then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(Z^*X\Theta^k(Y)Z) - \varphi(Z^*XZ)\varphi(Z^*YZ)| = 0 \quad \text{for} \quad X, Y \in \mathcal{M}.$$ 

Before proving the existence of a dilation of a quantum dynamical system, it is necessary to recall the fundamental Nagy dilation theorem. This is the subject of the next section.

2. Nagy dilation theorem

If $V$ is an isometry on a Hilbert space $\mathcal{H}$, there is a triple $(\hat{V}, \hat{\mathcal{H}}, Z)$ where $\hat{\mathcal{H}}$ is a Hilbert space, $Z : \mathcal{H} \to \hat{\mathcal{H}}$ is an isometry and $\hat{V}$ is a unitary operator on $\hat{\mathcal{H}}$ with

$$\hat{V}Z = ZV \quad \text{(2.1)}$$

satisfying the following minimal property:

$$\hat{\mathcal{H}} = \bigvee_{k \in \mathbb{Z}} \hat{V}^kZ\mathcal{H}, \quad \text{(2.2)}$$

see [9]. However, for our purposes it is still useful to recall here the structure of the unitary minimal dilation of an isometry.
For a Hilbert space $K$, recall that $l^2(K)$ denotes the Hilbert space \{ $\xi: \mathbb{N} \rightarrow K: \sum_{n \geq 0} |\xi(n)|^2 < \infty$ \}. Consider the Hilbert space

$$\hat{H} = \mathcal{H} \oplus l^2(\mathcal{H})$$

(2.3)

and the unitary operator on $\hat{H}$ defined as

$$\hat{V} = \begin{vmatrix} V & F\Pi_0 \\ 0 & W \end{vmatrix},$$

(2.4)

where $F = I - VV^*$ and $\Pi_j: l^2(\mathcal{H}) \rightarrow \mathcal{H}$ is the canonical projection

$$\Pi_j(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = \xi_j \quad \text{for} \quad j \in \mathbb{N},$$

while $W: l^2(\mathcal{H}) \rightarrow l^2(\mathcal{H})$ is the operator

$$W(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (\xi_1, \xi_2, \ldots), \quad (\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in l^2(\mathcal{H}).$$

If $Z: \mathcal{H} \rightarrow \hat{H}$ is the isometry defined by $Zh = h \oplus 0$ for all $h \in \mathcal{H}$, it is simple to prove that the relations (2.1) and (2.2) are verified.

We observe that for each $n \in \mathbb{N}$ we have

$$\hat{V}^n = \begin{vmatrix} V^n & C(n) \\ 0 & W^n \end{vmatrix},$$

(2.5)

where $C(n): l^2(\mathcal{H}) \rightarrow \mathcal{H}$ are the following operators:

$$C(n) := \sum_{j=1}^{n} V^{n-j} F\Pi_{j-1} \quad \text{for} \quad n \geq 1.$$

Furthermore, for each $n, m \in \mathbb{N}$ we obtain:

$$\Pi_n W^m = \Pi_{n+m} \quad \text{and} \quad \Pi_n W^m = \begin{cases} \Pi_{n-m} & \text{if} \quad n \geq m \\ 0 & \text{if} \quad n < m \end{cases},$$

(2.6)

since

$$W^m(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (0, \ldots, 0, \xi_0, \xi_1, \ldots),$$

while for each $k, p \in \mathbb{N}$ we obtain:

$$\Pi_p C(k)^* = \begin{cases} FV^{(k-p-1)^*} & \text{if} \quad k > p \\ 0 & \text{elsewhere} \end{cases}$$

(2.7)

since for each $h \in \mathcal{H}$ we have:

$$C(k)^* h = \sum_{k \text{ times}} FV^{(k-1)^*} h \ldots FV^* h, Fh, 0, 0 \ldots.$$  

(2.8)

3. Isometric dilation and invariant algebras

In this section we consider a concrete unital C*-algebra $A$ of $\mathfrak{B}(\mathcal{H})$ and an isometry $V$ on the Hilbert space $\mathcal{H}$ such that

$$V^* A V \subseteq A.$$

If $(\hat{V}, \hat{H}, Z)$ denotes the minimal unitary dilation of the isometry $V$, we will prove the following proposition:

**Proposition 3.1.** — *There exists a unital C*-algebra $\hat{A} \subseteq \mathfrak{B}(\hat{H})$ such that:

(a) $ZA^* Z^* \subseteq \hat{A}$;
(b) $Z^* \hat{A} Z \subseteq A$;
(c) $\hat{V}^* A \hat{V} \subseteq \hat{A}$;
(d) $Z^* \hat{V} X \hat{V} Z = V^* Z^* X V$ for $X \in \hat{A}$;
(e) $Z^* \hat{V} (ZA^* Z^*) \hat{V} Z = V^* AV$ for $A \in A$.*
The statements (d) and (e) are straightforward consequences of (a) and (b) and of the relationship $\hat{V}Z = ZV$. In order to prove the other statements, we must study two classes of operators on the Hilbert space $H$, associated to the pair $(A, V)$ defined above, which we shall call the gamma and the napla operators.

3.1. Gamma operators. We consider the sequences

$$\alpha := (n_1, n_2, \ldots, n_r, A_1, A_2, \ldots, A_r),$$

with $n_j \in \mathbb{N}$ and $A_j \in \mathfrak{A}$ for $j = 1, 2, \ldots, r$. These elements $\alpha$ are called strings of $\mathfrak{A}$ of length $l(\alpha) := r$ and weight $\hat{\alpha} := \sum n_i$.

To any string $\alpha$ of $\mathfrak{A}$ we shall call the gamma and the napla operators.

$$\hat{\beta}, \tilde{\beta}$$

where

$$\dot{\beta}, \tilde{\alpha}$$

and $\check{\alpha}$ and $\check{\beta}$ are strings of $\mathfrak{A}$ with $\check{\alpha} = n$. The symbols $(n)$ and $\mathfrak{A}(n)$ have analogous meanings.

**Proposition 3.2.** — Let $\alpha$ and $\beta$ be strings of $\mathfrak{A}$. For each $R \in \mathfrak{A}$ we have:

$$\left(\alpha | R | \beta \right) \in \begin{cases} \mathfrak{A}(\hat{\alpha} - \check{\beta}) & \text{if } \hat{\alpha} \geq \check{\beta} \\ \{\beta - \alpha\} & \text{if } \hat{\alpha} < \check{\beta} \end{cases}$$

and

$$|\alpha| R |\beta| \in |\alpha + \check{\beta}|.$$  

**Proof.** — For each $m, n \in \mathbb{N}$ and $R \in \mathfrak{A}$ we have:

$$V^m R V^n \in \begin{cases} V^{(m-n)} \mathfrak{A} & \text{if } m \geq n \\ \mathfrak{A} V^{(n-m)} & \text{if } m < n \end{cases}$$

Given $\alpha = (m_1, m_2, \ldots, m_r, A_1, A_2, \ldots, A_r)$ and $\beta = (n_1, n_2, \ldots, n_s, B_1, B_2, \ldots, B_s)$ we have that

$$(\alpha | R | \beta) = V^{m_r} A_r \cdots V^{m_i} A_i RB_1 V^{n_1} \cdots B_s V^{n_s} = (\check{\alpha} | I | \check{\beta}),$$

where $\check{\alpha}$ and $\check{\beta}$ are strings of $\mathfrak{A}$ with $l(\check{\alpha}) = l(\check{\beta}) = l(\alpha) + l(\beta) - 1$. Moreover if $\hat{\alpha} \geq \check{\beta}$ then $\hat{\alpha} \geq \check{\beta}$, while if $\hat{\alpha} < \check{\beta}$ then $\hat{\alpha} < \check{\beta}$. In fact if $m_1 \geq n_1$ we obtain:

$$(\alpha | R | \beta) = V^{m_r} A_r \cdots V^{m_i} A_i R_1 B_2 V^{n_2} \cdots B_s V^{n_s} = (\check{\alpha} | I | \check{\beta}),$$

where

$$R_1 = V^{n_1} A_1 RB_1 V^{n_1},$$

$\check{\alpha} = (m_1 - n_1, m_2, \ldots, m_r, R_1, A_2, A_r),$ and

$\check{\beta} = (n_2, \ldots, n_s, B_2, \ldots, B_s).$

If $m_1 < n_1$ then we can write:

$$(\alpha | R | \beta) = V^{m_r} A_r \cdots V^{m_i} A_i R_1 V^{(n_1 - m_1)} B_2 \cdots B_s V^{n_s} = (\check{\alpha} | I | \check{\beta}),$$

where

$$R_1 = V^{m_i} A_1 RB_1 V^{m_1},$$

$\check{\alpha} = (m_2, \ldots, m_r, A_2, A_r)$ and

$\check{\beta} = (n_1 - m_1, n_2, \ldots, n_s, R_1, B_2, \ldots, B_s).$

The proof of (3.1) follows by induction on the number $\nu = l(\alpha) + l(\beta)$. The equation (3.2) follows by a direct calculation. ∎
Now, given the orthogonal projection $F = I - VV^*$ (see Section 2), for each string $\alpha$ of $\mathfrak{A}$ with $\hat{\alpha} \geq 1$ we define

$$\Gamma(\alpha) := (\alpha|F\Pi_{\alpha-1},$$

which we call the gamma operator associated to $(\mathfrak{A}, V)$. The linear space generated by all gamma operators $\Gamma(\alpha)$ for $\hat{\alpha} \geq 1$ will be denoted by $\mathfrak{G}(\mathfrak{A}, V)$.

**Proposition 3.3.** — For any strings $\alpha$ and $\beta$ of $\mathfrak{A}$ with $\hat{\alpha}, \hat{\beta} \geq 1$, we have

$$\Gamma(\alpha)\Gamma(\beta)^* \in \mathfrak{A}.$$

**Proof.** — Note that

$$\Gamma(\alpha)\Gamma(\beta)^* = (\alpha|F\Pi_{\alpha-1}\Pi_{\beta-1}^*F|\beta) = \left\{ \begin{array}{ll}
(\alpha|F|\beta) & \text{if } \hat{\alpha} = \hat{\beta} \\
0 & \text{if } \hat{\alpha} \neq \hat{\beta}.
\end{array} \right.$$  

In fact if $\hat{\alpha} = \hat{\beta}$ we have that

$$(\alpha|F|\beta) = (\alpha|(I - VV^*)|\alpha) = (\alpha|I|\alpha) - (\alpha|VV^*|\alpha) \in \mathfrak{A},$$

since $(\alpha|V| (\hat{\alpha} - 1) \text{ and } V^*|\alpha) \in (\hat{\alpha} - 1)$, and $(\hat{\alpha} - 1)|I|\hat{\alpha} - 1) \subset \mathfrak{A}$ by relationship (3.1)

The gamma operators associated to $(\mathfrak{A}, V)$ define an operator system $\Sigma$ of $\mathfrak{B}(l^2(F\mathcal{H}))$ by

$$\Sigma := \{ T \in \mathfrak{B}(l^2(F\mathcal{H})) : \Gamma_1 T\Gamma_2 \in \mathfrak{A} \text{ for all } \Gamma_1, \Gamma_2 \in \mathfrak{G}(\mathfrak{A}, V) \}. \quad (3.4)$$

We observe that the unit $I$ belongs to $\Sigma$ and that

$$\Gamma_i^*A\Gamma_j \in \Sigma \text{ for } A \in \mathfrak{A},$$

for any pair of gamma operators $\Gamma_1$, $\Gamma_2$. Furthermore, it is easy to prove that $\Sigma$ is norm closed, and it is weakly closed if $\mathfrak{A}$ is a $W^*$-algebra.

3.2. **Napla operators.** For strings $\alpha$ and $\beta$ of $\mathfrak{A}$, any $A \in \mathfrak{A}$ and $k \in \mathbb{N}$ we define

$$\Delta_k(A, \alpha, \beta) := \Pi_{\alpha+k}^*F|\alpha)A(\beta|F\Pi_{\beta+k}.$$

We call these operators of $\mathfrak{B}(l^2(F\mathcal{H}))$ the napla operators associated to the pair $(\mathfrak{A}, V)$.

In the next lines we show that the linear space generated by the napla operators form a $*$-algebra. To this end, it is easily seen that $\Delta_{h+k}(A, \alpha, \beta)^* = \Delta_k(A^*, \beta, \alpha)$ for any $h, k \geq 0$. Moreover we have the following two relationships: if $k + \hat{\beta} \neq h + \hat{\gamma}$, then

$$\Delta_{h+k} = 0,$$

while if $k + \hat{\beta} = h + \hat{\gamma}$, then there is $\theta$ and $R \in \mathfrak{A}$ with

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \left\{ \begin{array}{ll}
\Delta_k(R, \alpha, \theta) & \text{if } h - k \geq 0, \text{ where } \hat{\theta} = \hat{\delta} + h - k \\
\Delta_h(R, \theta, \delta) & \text{if } h - k < 0, \text{ where } \hat{\theta} = \hat{\delta} + k - h.
\end{array} \right. \quad (3.5)$$

In fact, notice that

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \Pi_{\alpha+k}^*F|\alpha)A(\beta|F\Pi_{\beta+k}^*F|\gamma)B(\delta|F\Pi_{\delta+h}.$$

If $k + \hat{\beta} \neq h + \hat{\gamma}$ it follows that $\Pi_{\beta+h}^*\Pi_{\gamma+k} = 0$, and this shows (3.5). If $k + \hat{\beta} = h + \hat{\gamma}$, without loss of generality we can assume that $h \geq k$. So $\hat{\delta} = \hat{\delta} + h - k \geq \hat{\gamma}$ and, by relationship (3.1), we have that $(\beta|F|\gamma) \in \mathfrak{A}(\hat{\beta} - \hat{\gamma})$. Consequently, $A(\beta|F|\gamma)B(\delta|\theta) \in \mathfrak{A}(\hat{\delta} + \hat{\beta} - \hat{\gamma})$, and there exists a $\theta'$ string of $\mathfrak{A}$ and an element $R \in \mathfrak{A}$ such that $\hat{\theta} = \hat{\delta} + \hat{\beta} - \hat{\gamma}$ and $A(\beta|F|\gamma)B(\delta|\theta \in \mathfrak{A}(\hat{\theta'} + \hat{\beta} - \hat{\gamma})$. Now, since $\hat{\theta} = \hat{\delta} + h - k$ we have:

$$\Delta_k(A, \alpha, \beta)\Delta_h(B, \gamma, \delta) = \Pi_{\alpha+k}^*F|\alpha)R(\theta|F\Pi_{\theta+h}$$

$$= \Pi_{\alpha+k}^*F|\alpha)R(\theta|F\Pi_{\theta+k} = \Delta_k(R, \alpha, \theta),$$

showing relationship (3.6).
Proposition 3.4. — The linear space $X_{o}$ generated by the apla operators is a $*$-subalgebra of $\mathfrak{B}(l^2(FH))$ included in the operator systems $\Sigma$ defined in (3.4).

Proof. — From relationships (3.5),(3.6) the linear space $X_{o}$ is a $*$-algebra. Furthermore for each pair $(\alpha, \beta)$ of gamma operators we obtain:

$$\Gamma(\alpha)\Delta_{k}(A, \gamma, \delta)\Gamma(\beta)^{*} = (\alpha|F\Pi_{\alpha-1}\Pi_{\gamma+k}^{*}F|\gamma)A(\delta|F\Pi_{\beta+k}\Pi_{\beta-1}^{*}F|\beta) \in \mathfrak{A},$$

since by the relationships (3.1) and (3.2) we have

$$(\alpha|F\Pi_{\alpha-1}\Pi_{\gamma+k}^{*}F|\gamma)A(\delta|F\Pi_{\beta+k}\Pi_{\beta-1}^{*}F|\beta) \in \begin{cases} (k+1)\mathfrak{A}|k+1\rangle & \text{if} \quad \hat{\alpha} - 1 = \hat{\gamma} + k, \\
0 & \text{elsewhere} \end{cases}$$

In fact, if $\hat{\alpha} = \hat{\gamma} + k + 1$ we can write

$$(\alpha|F\Pi_{\alpha-1}\Pi_{\gamma+k}^{*}F|\gamma) = (\alpha|I|\gamma) = (\alpha|VV^{*}|\gamma) \in \mathfrak{A}(k+1),$$

since $(\alpha|I|\gamma) \in \mathfrak{A}(k+1)$ and $(\alpha|VV^{*}|\gamma) \in \mathfrak{A}(k+1)$. If $\hat{\beta} = \hat{\delta} + k + 1$ we have

$$(\delta|F\Pi_{\beta+k}\Pi_{\beta-1}^{*}F|\beta) \in (k+1)\mathfrak{A}, \text{ completing the proof. } \square$$

The next result is concerned with $W$-invariance.

Proposition 3.5. — The $*$-algebra $X_{o}$ and the operator system $\Sigma$ are $W$-invariants:

$$W^{*}X_{o}W \subset X_{o} \quad \text{and} \quad W^{*}\Sigma W \subset \Sigma.$$

Proof. — The first inclusion follows by (2.6). Concerning the second one, let $T \in \Sigma$. For each pair $(\alpha, \beta)$ of gamma operators

$$\Gamma(\alpha)(W^{*}TW)\Gamma(\beta)^{*} = (\alpha|F\Pi_{\alpha-1}W^{*}TW\Pi_{\beta-1}^{*}F|\beta) = (\alpha|F\Pi_{\alpha-2}TW\Pi_{\beta-2}^{*}F|\beta) \in \mathfrak{A}W^{*}\Gamma_{1}(\alpha^{o})T\Gamma_{2}(\beta^{o})V\mathfrak{A},$$

where $\alpha_{o}$ and $\beta_{o}$ are strings of $\mathfrak{A}$ with $\hat{\alpha}_{o} = \hat{\alpha} - 1$ and $\hat{\beta}_{o} = \hat{\beta} - 1$. In fact if $\alpha = (m_{1}, m_{2} \ldots m_{r}, A_{1}, A_{2} \ldots A_{r})$, then, by definition of the gamma operator, there is $i \leq r$ with $m_{i} \geq 1$ such that

$$(\alpha|F\Pi_{\alpha-2}A_{1} \cdots A_{r}^{*}\alpha_{o}|F\Pi_{\alpha-2}^{*} = A_{1} \cdots A_{r}^{*}\Gamma(\alpha_{o}),$$

where

$$\alpha_{o} = (0, \ldots, 0, m_{i} - 1, m_{i+1} \ldots m_{r}, A_{1}, A_{2} \ldots A_{r})$$

with $\hat{\alpha}_{o} = \hat{\alpha} - 1$. Consequently

$$\Gamma(\alpha)(W^{*}TW)\Gamma(\beta)^{*} \subset V^{*}\mathfrak{A}W \subset \mathfrak{A},$$

completing the proof. $\square$

3.3. The algebra generated by the apla and gamma operators. Let $X$ be the closure in norm of the $*$-algebra $X_{o}$ of the apla operators previously defined. Since the operator system $\Sigma$ defined in (3.4) is a norm closed set, we have $X \subset \Sigma$. Notice that in case $\mathfrak{A}$ is a von Neumann algebra of $\mathfrak{B}(H)$, the operator system $\Sigma$ is weakly closed and $X_{o}^{w} \subset \Sigma$.

Proposition 3.6. — The set

$$S = \left\{ \begin{bmatrix} A & \Gamma_{1} \\ \Gamma_{2} & T \end{bmatrix} : A \in \mathfrak{A}, T \in X \text{ and } \Gamma_{1}, \Gamma_{2} \in \mathfrak{G}(\mathfrak{A}, V) \right\}$$

(3.7)

is an operator system of $\mathfrak{B}(\hat{H})$ such that:

$$\hat{V}^{*}S\hat{V} \subset S.$$

Furthermore

$$\hat{V}^{*}A^{*}(S)\hat{V} \subset A^{*}(S),$$

where $A^{*}(S)$ is the $*$-algebra generated by the set $S$. 
Proof. — From relationship (2.4) we obtain:
\[
\hat{V}^*S\hat{V} = \begin{vmatrix}
V^*AV & V^*AC(1) + V^*\Gamma_1W \\
C(1)^*AV + W^*\Gamma_2V & C(1)^*AC(1) + W^*\Gamma_2C(1) + C(1)^*\Gamma_1W + W^*TW
\end{vmatrix}
\]
We observe that \(V^*\Gamma(\alpha)W\) and \(V^*AC(1)\) are gamma operators associated to the pair \((\mathfrak{A}, V)\), while \(C(1)^*AC(1), C(1)^*\Gamma(\alpha)W\) and \(W^*TW\) are operators belonging to \(X\). In fact we have \(V^*AC(1) = V^*AF\Pi_0 = \Gamma(\vartheta)\) with \(\vartheta = (1, A)\); while if
\[
\alpha = (m_1, m_2 \ldots m_r, A_1, A_2 \ldots A_r),
\]
then \(V^*\Gamma(\alpha)W = V^*(\alpha|F\Pi_{a-1}W = \Gamma(\vartheta)),\)
with \(\vartheta = (m_1 + 1, m_2 \ldots m_r, A_1, A_2 \ldots A_r)
\]
since \(\Pi_{a-1}W = \Pi_a\). Furthermore
\[
C(1)^*AC(1) = \Pi_0^*F\Pi\Pi_0 = \Delta_0(A, \alpha, \beta),
\]
with \(\alpha = \beta = (0, I)\); while
\[
C(1)^*\Gamma(\alpha)W = \Pi_0^*F(\alpha|F\Pi_{a-1}W = \Pi_0^*F|\gamma)(\alpha|F\Pi_{a+0} = \Delta_0(I, \gamma, \alpha)
\]
with \(\gamma = (0, I)\), where the last statement follows from the fact that \(\hat{V}\) is unitary. \(\square\)

We observe that \(A^*(S)\), the *-algebra generated by the operator system \(S\) defined in (3.7), is the linear space generated by the following elements of \(\mathfrak{B}(\mathcal{H})\):
\[
\begin{array}{ccc}
A_1 & A_2\Gamma_jT_1 \\
T_2\Gamma_j^2A_3 & T_3
\end{array}
\]
with \(A_i \in \mathfrak{A}, \Gamma_j \in G(\mathfrak{A}, V)\) and \(T_k \in \mathfrak{A}\) for all \(i,k = 1,2,3\) and \(j = 1,2\). We list here some easy properties of the *-algebra \(A^*(S)\):

(a) \(Z\mathfrak{A}Z^* \subset A^*(S)\);
(b) \(Z^*A^*(S)Z \subset \mathfrak{A}\);
(c) \(\hat{V}^*A^*(S)\hat{V} \subset A^*(S)\).

Furthermore, since \(\hat{V}Z = ZV\) we have:

(d) \(Z^*\hat{V}^*X\hat{V}Z = V^*Z^*XZV\);
(e) \(Z^*\hat{V}^*(ZAZ^*)\hat{V}Z = V^*AV\).

Using these results we prove the Proposition 3.1.

Proof of Proposition 3.1. — Let \(\tilde{\mathfrak{A}}\) be the C*-subalgebra of \(\mathfrak{B}(\mathcal{H})\) generated by
\[
\bigcup_{k=0}^{\infty} \hat{V}^k XAZ^*\hat{V}^k \text{ for } A \in \mathfrak{A}. \tag{3.8}
\]
For each natural number \(k\) we have that \(\hat{V}^k XAZ^*\hat{V}^k \subset \hat{V}^k S\hat{W}^k \subset \mathcal{S}\), since \(ZAZ^* \subset \mathcal{S}\); so \(\tilde{\mathfrak{A}} \subset A^*(S)\), the norm closure of the *-algebra \(A^*(S)\). It is easily seen that \(\tilde{\mathfrak{A}}\) satisfies the conditions of Proposition 3.1, completing the proof. \(\square\)

Remark 3.7. — It is straightforward to show that if \(\mathfrak{A}\) is a von Neumann algebra of \(\mathfrak{B}(\mathcal{H})\), then the Proposition 3.1 still holds true, with \(\tilde{\mathfrak{A}}\) the von Neumann algebra of \(\mathfrak{B}(\mathcal{H})\) generated by the elements (3.8).

4. STINESPRING REPRESENTATION AND QUANTUM DYNAMICAL SYSTEMS

We consider a concrete C*-algebra \(\mathfrak{A}\) of \(\mathfrak{B}(\mathcal{H})\) with unit and a ucp-map \(\Phi : \mathfrak{A} \rightarrow \mathfrak{A}\).

On the algebraic tensor product \(\mathfrak{A} \otimes \mathcal{H}\) we can define a semi-inner product by
\[
(A_1 \otimes h_1, A_2 \otimes h_2)_\Phi := (h_1, \Phi(A_1^* A_2)h_2)_\mathcal{H},
\]
for all \(A_1, A_2 \in \mathfrak{A}\) and \(h_1, h_2 \in \mathcal{H}\). We denote by \(\mathfrak{A} \overline{\otimes}_\Phi \mathcal{H}\) the Hilbert space completion of the quotient space of \(\mathfrak{A} \otimes \mathcal{H}\) by the linear subspace \(\{ T \in \mathfrak{A} \otimes \mathcal{H} : \langle T, T \rangle_\Phi = 0 \}\).
with inner product induced by $\langle \cdot, \cdot \rangle_\Phi$. Furthermore, we denote the image of $A \otimes h \in \mathfrak{A} \otimes \mathcal{H}$ in $\mathfrak{A} \otimes_\Phi \mathcal{H}$ by $A \otimes_\Phi h$; so

$$\langle A \otimes_\Phi h_1, A_2 \otimes_\Phi h_2 \rangle_{\mathfrak{A} \otimes_\Phi \mathcal{H}} = \langle h_1, \Phi(A^*_1 A_2) h_2 \rangle_\mathcal{H}$$

for all $A_1, A_2 \in \mathfrak{A}$ and $h_1, h_2 \in \mathcal{H}$.

Moreover, we define a representation $\sigma_{\Phi} : \mathfrak{A} \to \mathcal{B}(\mathfrak{A} \otimes_\Phi \mathcal{H})$ by

$$\sigma_{\Phi}(A)(X \otimes_\Phi h) := AX \otimes_\Phi h \quad \text{for} \quad A \in \mathfrak{A} \text{ and } X \otimes_\Phi h \in \mathfrak{A} \otimes_\Phi \mathcal{H},$$

and a linear isometry $V_{\Phi} : \mathcal{H} \to \mathfrak{A} \otimes_\Phi \mathcal{H}$ by

$$V_{\Phi} h := 1 \otimes_\Phi h \quad \text{for} \quad h \in \mathcal{H},$$

satisfying the equation

$$\Phi(A) = V_{\Phi}^* \sigma_{\Phi}(A) V_{\Phi} \quad \text{for} \quad A \in \mathfrak{A}. \quad (4.1)$$

The triple $(V_{\Phi}, \sigma_{\Phi}, \mathfrak{A} \otimes_\Phi \mathcal{H})$ is the Stinespring representation of the ucp-map $\Phi$ (see [13]).

Our aim is to analyze the behaviour of the isometry $V_{\Phi}$ and of its adjoint $V_{\Phi}^*$ on the multiplicative domain of the ucp-map $\Phi$. To this end note that the adjoint $V_{\Phi}^*$ verifies $V_{\Phi}^* A \otimes_\Phi h = \Phi(A) h$ for any $A \in \mathfrak{A}$ and $h \in \mathcal{H}$. Furthermore, recall that the multiplicative domain of the ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$ is the C*-subalgebra with unit of $\mathfrak{A}$ defined as

$$D_\Phi = \{ A \in \mathfrak{A} : \Phi(A^*) \Phi(A) = \Phi(A^* A) \text{ and } \Phi(A) \Phi(A^*) = \Phi(A A^*) \},$$

see [11]. The multiplicative domain is characterized by the following relationship

$$A \in D_\Phi \iff \sigma_{\Phi}(A) V_{\Phi} V_{\Phi}^* = V_{\Phi} V_{\Phi}^* \sigma_{\Phi}(A). \quad (4.2)$$

In fact, we first note that

$$A \otimes_\Phi h = 1 \otimes_\Phi \Phi(A) h \quad \text{for all} \quad h \in \mathcal{H} \iff \Phi(A^* A) = \Phi(A^* \Phi(A),$$

since

$$|A \otimes_\Phi h - 1 \otimes_\Phi \Phi(A) h|^2 = \langle h, \Phi(A^* A) h \rangle - \langle h, \Phi(A^* \Phi(A) h \rangle. \quad (4.3)$$

Consequently, for any $A \in D_\Phi$ and $B \otimes_\Phi h \in \mathfrak{A} \otimes_\Phi \mathcal{H}$ we have

$$\sigma_{\Phi}(A) V_{\Phi} V_{\Phi}^* B \otimes_\Phi h = A \otimes_\Phi \Phi(B) h = 1 \otimes_\Phi \Phi(A) \Phi(B) h\quad (4.4)$$

$$= 1 \otimes_\Phi \Phi(AB) h = V_{\Phi} V_{\Phi}^* \sigma_{\Phi}(A) B \otimes_\Phi h,$n

where we have used the property of the multiplicative domain $\Phi(A) \Phi(B) = \Phi(AB)$ (see [13]). Conversely, if $\sigma_{\Phi}(A) V_{\Phi} V_{\Phi}^* = V_{\Phi} V_{\Phi}^* \sigma_{\Phi}(A)$ then

$$\Phi(A^* A) = V_{\Phi}^* \sigma_{\Phi}(A^* A) V_{\Phi} = V_{\Phi}^* \sigma_{\Phi}(A^*) \sigma_{\Phi}(A) V_{\Phi} V_{\Phi}^* V_{\Phi} = V_{\Phi}^* \sigma_{\Phi}(A^*) V_{\Phi} V_{\Phi}^* \sigma_{\Phi}(A) V_{\Phi} = \Phi(A^*) \Phi(A),$$

and this completes the proof of (4.2).

It is easily seen from (4.2) that $\Phi$ is a *-homomorphism if, and only if, $V_{\Phi}$ is a unitary operator.

The next steps provides some simple applications of the Stinespring representation of ucp-maps.

Let $\mathfrak{A}$ be a concrete C*-subalgebra with unit of $\mathcal{B}(\mathcal{H})$ and $\Phi : \mathfrak{A} \to \mathfrak{A}$ a ucp-map. By the Stinespring’s theorem we obtain a triple $(V_{\Phi}, \sigma_{\Phi}, \mathcal{H}_1)$, with $\mathcal{H}_1 = \mathfrak{A} \otimes_\Phi \mathcal{H}$ such that $\Phi(A) = V_{\Phi}^* \sigma_{\Phi}(A) V_{\Phi}$ for all $A \in \mathfrak{A}$. Moreover the application $\Phi_1 : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_1)$ defined by $\Phi_1(A) := \sigma_{\Phi}(A)$, for $A \in \mathfrak{A}$, is a ucp-map because it is a composition of ucp-maps. By applying the Stinespring’s theorem to $\Phi_1$, we have a new triple $(V_{\Phi}, \sigma_{\Phi}, \mathcal{H}_2)$, with $\mathcal{H}_2 = \mathfrak{A} \otimes_\Phi \mathcal{H}_1$, such that $\Phi_1(A) = V_{\Phi}^* \sigma_2(A) V_{\Phi}$ for all $A \in \mathfrak{A}$. So, iterating this procedure we obtain, for each natural number $n \geq 1$, a ucp-map $\Phi_n : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_n)$ such that

$$\Phi_n(A) = \sigma_n(\Phi(A)) \quad \text{for} \quad A \in \mathfrak{A},$$

(4.3)
and a new triple \((V_n, \sigma_{n+1}, H_{n+1})\), where \(H_{n+1} = \mathfrak{A} \otimes_{\Phi_n} H_n\), and an isometry \(V_n : H_n \to H_{n+1}\) such that \(\Phi_n(A) = V_n^* \sigma_{n+1}(A) V_n\) for all \(A \in \mathfrak{A}\).

Now we prove the following Stinespring-type theorem (see [14]):

**Proposition 4.1.** — Let \(\mathfrak{A}\) be a concrete \(C^*\)-algebra with unit of \(\mathcal{B}(H)\) and \(\Phi : \mathfrak{A} \to \mathfrak{A}\) a ucp-map. There exists an injective representation \((\pi_\infty, H_\infty)\) of \(\mathfrak{A}\) and a linear isometry \(V_\infty\) on the Hilbert space \(H_\infty\) such that

\[
\pi_\infty(\Phi(A)) = V_\infty^* \pi_\infty(A) V_\infty \quad \text{for} \quad A \in \mathfrak{A}.
\]

Furthermore, \(A \in \mathcal{D}_\Phi\) if, and only if, \(V_\infty V_\infty^* \pi_\infty(A) = \pi_\infty(A) V_\infty V_\infty^*\).

**Proof.** — We consider for each natural number \(n\) the ucp-map \(\Phi_n : \mathfrak{A} \to \mathfrak{B}(H_n)\) defined in (4.3) and its Stinespring representation \((V_n, \sigma_{n+1}, H_{n+1})\) with \(H_0 = H\) and \(\sigma_0 = id\). Then, we obtain a faithful representation \(\pi_\infty : \mathfrak{A} \to \mathfrak{B}(H_\infty)\) on the Hilbert space \(H_\infty = \bigoplus H_n\) by defining

\[
\pi_\infty(A) := \bigoplus_{n \geq 0} \sigma_n(A) \quad \text{for} \quad A \in \mathfrak{A}.
\]

Now, let \(V_\infty : H_\infty \to H_\infty\) be the isometry defined by

\[
V_\infty(h_0, h_1, \ldots, h_n, \ldots) := (0, V_0 h_0, V_1 h_1, \ldots, V_n h_n, \ldots),
\]

for all \(h_n \in H_n\) and \(n \in \mathbb{N}\). Note that the adjoint of \(V_\infty\) is

\[
V_\infty^*(h_0, h_1, \ldots, h_n, \ldots) = (V_0^* h_0, V_1^* h_1, \ldots, V_n^* h_n, \ldots)
\]

for all \(h_n \in H_n\) and \(n \in \mathbb{N}\). Hence, for any \(n\) and \(h_n \in H_n\) we have

\[
V_\infty^* \pi_\infty(A) V_\infty \bigoplus_{n \geq 0} h_n = \bigoplus_{n \geq 0} \Phi_n(A) h_n = \bigoplus_{n \geq 0} \sigma_n(\Phi(A)) h_n = \pi_\infty(\Phi(A)) \bigoplus_{n \geq 0} h_n.
\]

Finally, the last statement easily follows by 4.2.

In fact if \(A \in \mathcal{D}_\Phi\) then \(A \in \mathcal{D}_{\Phi_n}\) for all natural number \(n\), where \(\mathcal{D}_{\Phi_n}\) is the multiplicative domain of the ucp-map (4.3), then

\[
V_\infty V_\infty^* \in \pi_\infty(\bigcap_{n \geq 0} \mathcal{D}_{\Phi_n})' \subset \pi_\infty(\mathcal{D}_\Phi)'\]

We have the following remark on the existence of a representation of a quantum dynamical system:

**Remark 4.2.** — Let \((\mathfrak{M}, \Phi)\) be a quantum dynamical system. The injective representation \(\pi_\infty(\Phi) : \mathfrak{M} \to \mathfrak{B}(H_\infty)\) defined in proposition 4.1 is normal, since the Stinespring representation \(\sigma_\Phi : \mathfrak{A} \to \mathfrak{B}(L_\Phi)\) is a normal map. Then \((\pi_\infty, H_\infty, V_\infty)\) is a representation of the quantum dynamical system \((\mathfrak{M}, \Phi)\).

### 4.1. Dilation of a quantum dynamical system

We use the results of the previous section to analyze the problem of dilation of quantum dynamical systems.

Consider a ucp-map \(\Phi : \mathfrak{A} \to \mathfrak{A}\) with \(\mathfrak{A}\) a concrete \(C^*\)-algebra with unit of \(\mathfrak{B}(H)\). If \((H_\infty, \pi_\infty, V_\infty)\) is the Stinespring representation of Proposition 4.1, then

\[
V_\infty^* \pi_\infty(\mathfrak{A}) V_\infty \subset \pi_\infty(\Phi(\mathfrak{A})) \subset \pi_\infty(\mathfrak{A}).
\]

Hence, we can define a normal ucp-map \(\Phi_\infty : \pi_\infty(\mathfrak{A})' \to \pi_\infty(\mathfrak{A})''\) as

\[
\Phi_\infty(B) := V_\infty^* B V_\infty \quad \text{for} \quad B \in \pi_\infty(\mathfrak{A})'.
\]

Clearly we have that \(\Phi_\infty(\pi_\infty(A)) = \pi_\infty(\Phi(A))\) for all \(A \in \mathfrak{A}\).

Now, if \((\tilde{V}, \tilde{H}, \tilde{Z})\) is minimal unitary dilation of the isometry \(V_\infty : H_\infty \to H_\infty\), then by Proposition 3.1 there is a \(C^*\)-algebra with unit \(\tilde{\mathfrak{A}}\) of \(\mathfrak{B}(\tilde{H})\) such that:

(a) \(Z \pi_\infty(\mathfrak{A}) Z^* \subset \tilde{\mathfrak{A}}\),
(b) \(Z^* \tilde{\mathfrak{A}} Z = \pi_\infty(\mathfrak{A})\),
(c) \(\tilde{V}^* \tilde{\mathfrak{A}} \tilde{V} \subset \tilde{\mathfrak{A}}\).
Furthermore, we have a $^*$-homomorphism $\hat{\Phi} : \hat{\mathfrak{A}} \to \hat{\mathfrak{A}}$ defined by
\[
\hat{\Phi}(X) = \hat{V}^*X\hat{V} \quad \text{for} \quad X \in \hat{\mathfrak{A}},
\] (4.6)
such that for any $A \in \mathfrak{A}$, $X \in \hat{\mathfrak{A}}$ and any natural number $n$ we have:
\[
\pi_\infty(\Phi^n(A)) = Z^*\hat{\Phi}^n(ZAZ^*)Z,
\]
and
\[
\pi_\infty(\Phi^n(Z^*XZ)) = Z^*\hat{\Phi}^n(X)Z.
\]
In conclusion, it is straightforward to prove that $(\hat{\mathfrak{A}}'', \Theta, \hat{\mathfrak{H}}, Z)$, with $\Theta : \hat{\mathfrak{A}}'' \to \hat{\mathfrak{A}}''$ the normal $^*$-homomorphism
\[
\Theta(X) := \hat{V}^*X\hat{V} \quad \text{for} \quad X \in \hat{\mathfrak{A}}'',
\]
is a dilation of the quantum dynamical system $(\pi_\infty(\mathfrak{M})'', \Phi_\infty)$ above defined.

Summarizing, the quantum dynamical system $(\mathfrak{M}, \Phi)$ can be identified with its associated quantum dynamical system $(\pi_\infty(\mathfrak{M}), \Phi_\infty)$ which admits the dilation $(\pi_\infty(\mathfrak{M}), \Theta, \hat{\mathfrak{H}}, Z)$.

4.2. The deterministic part of a quantum dynamical system and its dilations. In this section we study which relationships there are between the dilations and the deterministic part of a quantum dynamical system.

Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a ucp-map as described in previous section and $C^*(S)$ the $C^*$-algebra generated by the operator systems $S$ defined in (3.7).

We recall that $S \subset A^*(S) \subset C^*(S) \subset \mathfrak{B}(\hat{\mathfrak{H}})$ where $\hat{\mathfrak{H}} = H_\infty \oplus l^2(FH_\infty)$ with $F = I - V_\infty V_\infty^*$. By relationships (a), (b) and (c) of Section 3.3, we can define a $^*$-homomorphism $\Lambda : C^*(S) \to C^*(S)$ as follows:
\[
\Lambda(X) = \hat{V}^*X\hat{V} \quad \text{for} \quad X \in C^*(S). 
\] (4.7)
Furthermore, we have a ucp-map $\mathcal{E} : C^*(S) \to \mathfrak{A}$ such that
\[
\pi_\infty(\mathcal{E}(X)) = Z^*XZ \quad \text{for} \quad X \in C^*(S)
\]
and for any natural number $n \in \mathbb{N}$
\[
\mathcal{E} \circ \Lambda^n = \Phi^n \circ \mathcal{E}.
\]

Hence, we have the following diagram:
\[
\begin{array}{cccc}
C^*(S) & \xrightarrow{\Lambda^n} & C^*(S) \\
\mathcal{E} \downarrow & & \downarrow \mathcal{E} \\
\mathfrak{A} & \xrightarrow{\Phi^n} & \mathfrak{A}
\end{array}
\]
where $\mathcal{E}(ZAZ^*) = A$ for all $A \in \mathfrak{A}$.

We consider now the $C^*$-algebra $\mathcal{D} := \bigcap_{n \geq 0} D_{\Phi^n}$ where the set $D_{\Phi^n}$ is the multiplicative domain of the ucp-map $\Phi^n : \mathfrak{A} \to \mathfrak{A}$ for all natural numbers $n$. The restriction of $\Phi$ to $\mathcal{D}$ is a $^*$-homomorphism $\Phi_\mathcal{D} : \mathcal{D} \to \mathcal{D}$ of $C^*$-algebras. It is said to be the deterministic part of the ucp-map $\Phi : \mathfrak{A} \to \mathfrak{A}$.

The $^*$-homomorphism $\Lambda$ defined above is related to the deterministic part of $\Phi$ in the following way:

**Proposition 4.3.** There is an injective $^*$-homomorphism $i : \mathcal{D} \to C^*(S)$ such that for each natural number $n$ and $D \in \mathcal{D}$ we have:
\[
\mathcal{E}(\Lambda^n(i(D))) = \Phi^n(D)
\]
and
\[
\Lambda^n(i(D)) = i(\Phi^n(D)).
\]
Proof. — Since $F \in \pi_{\infty}(D_{\Phi})'$ by Proposition 4.1, the map $\Xi : \mathcal{D} \to \mathfrak{B}(l^2(F \mathcal{H}_{\infty}))$ defined by

$$
\Xi(D) = \sum_{k \geq 0} \Pi_k^* F \pi_{\infty}(\Phi_r^{-k-1}(D)) F \Pi_k \quad D \in \mathcal{D}
$$

is a representation. Furthermore for any $D \in \mathcal{D}$ we have that $\Xi(D)$ belongs to $\mathfrak{X}_0$, that is, the linear space generated by the napla operators defined in Proposition 3.4, since $\Pi_k^* F \pi_{\infty}(\Phi_r^{-k-1}(D)) F \Pi_k$ is the napla operator $\Delta_k(\pi_{\infty}(\Phi_r^{-k-1}(D)), \alpha, \beta)$ with the strings $\alpha = \beta = (0, I)$.

We define a *-homomorphism $i : \mathcal{D} \to C^*(S)$ as follows

$$
i(D) = \pi_{\infty}(D) \oplus \Xi(D) \quad \text{for} \quad D \in \mathcal{D},
$$

and by relationship (2.5) we obtain that

$$
\Lambda^n(i(D)) = \begin{vmatrix}
V^n \pi_{\infty}(D)V^*_n, & V^n \pi_{\infty}(D)C_n, & C_n \pi_{\infty}(D)V^*_n, & C_n \pi_{\infty}(D)C_n + W^n \Xi(D)W^n
\end{vmatrix}.
$$

It is straightforward to prove that

$$
C_n \pi_{\infty}(D)C_n + W^n \Xi(D)W^n = \Xi(\Phi^n(D))
$$

and $C_n \pi_{\infty}(D)V^n = 0$, since by relationship (2.8) we have

$$
FV^{(n-k)} \pi_{\infty}(D)V^n = \pi_{\infty}(\Phi^{(n-k)}(D))FV^k = 0
$$

for all $1 \leq k \leq n$, completing the proof.

Finally, we observe that there is the following relationship between dilations and the deterministic part of a quantum dynamical system:

If $(\mathfrak{M}, \Theta, K, Z)$ is any dilation of quantum dynamical system $(\mathfrak{M}, \Phi)$, then for any natural number $n$ and $D \in \mathcal{D}$ we have :

$$
\Theta^n(ZDZ^*)Z = Z\Phi^n_c(D),
$$

since if $Y = \Theta^n(ZDZ^*)Z - Z\Phi^n(D)$, then $Y^*Y = 0$.

5. Ergodic Properties

Let $\mathfrak{A}$ be a concrete $C^*$-algebra of $\mathcal{B}(\mathcal{H})$ with unit, $\Phi : \mathfrak{A} \to \mathfrak{A}$ a ucp-map and $\varphi$ a state on $\mathfrak{A}$ such that $\varphi \circ \Phi = \varphi$. We recall that $\varphi$ is an ergodic state, relative to the ucp-map $\Phi$ (see [10]), if for each $A, B \in \mathfrak{A}$

$$
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^n (\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)) = 0,
$$

and that $\varphi$ is weakly mixing if for each $A, B \in \mathfrak{A}$

$$
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^n |\varphi(A\Phi^k(B)) - \varphi(A)\varphi(B)| = 0.
$$

By Proposition 4.1 we can assume that $\mathfrak{A}$ is a concrete $C^*$-algebra of $\mathfrak{B}(\mathcal{H})$, and that there is an isometry $V$ on $\mathcal{H}$ such that:

$$
\Phi(A) = V^*AV \quad \text{for} \quad A \in \mathfrak{A}.
$$

Let $(\hat{V}, \hat{\mathcal{H}}, Z)$ be the minimal unitary dilation of $(V, \mathcal{H})$ defined in (2.4), let $\hat{\mathfrak{A}}$ be the $C^*$-algebra included in $\mathfrak{B}(\hat{\mathcal{H}})$ defined in Proposition 3.1, and let $\hat{\Phi} : \hat{\mathfrak{A}} \to \hat{\mathfrak{A}}$ be the ucp-map defined in (4.6).

Proposition 5.1. — If the ucp-map $\Phi$ admits a $\varphi$-adjoint and $\varphi$ is an ergodic state, then:

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^N |\varphi(Z^*X\hat{\Phi}^k(Y)Z) - \varphi(Z^*XZ)\varphi(Z^*YZ)| = 0
$$
for all \( X, Y \in \mathfrak{A} \), while if \( \varphi \) is weakly mixing, then:

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} |\varphi(Z^kX\hat{H}^k(Y)Z) - \varphi(Z^kZ)\varphi(Z^kY)| = 0
\]

for all \( X, Y \in \mathfrak{A} \).

The proof of this proposition is a straightforward consequence of the next lemma. To this purpose, we make a preliminary observation. Recall that \( \hat{\mathcal{H}} = \mathcal{H} \oplus l^2(F\mathcal{H}) \) and that, writing an element \( X \) of \( B(\hat{\mathcal{H}}) \) in matrix representation

\[
X = \begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix},
\]

the following relationship holds:

\[
\varphi(Z^kX\hat{H}^k(Y)Z) = \varphi(X_{1,1}\Phi^k(Y_{1,1})) + \varphi(X_{1,2}C(k)^*Y_{1,1}V^k) + \varphi(X_{1,2}W^kY_{2,1}V^k).
\]

**Lemma 5.2.** — Let \( X \in A^*(S) \), the *-algebra generated by the operator system \( S \) defined in (3.7) and \( Y \in \mathfrak{A} \). The following relations hold:

(a) If \( \varphi \) is an ergodic state then we have:

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^*Y_{1,1}V^k + X_{1,2}W^kY_{2,1}V^k) = 0,
\]

(b) If \( \varphi \) is weakly mixing then we have:

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} |\varphi(X_{1,2}C(k)^*Y_{1,1}V^k + X_{1,2}W^kY_{2,1}V^k)| = 0.
\]

**Proof.** — Since \( X \in A^*(S) \), we can assume without loss of generality that \( X_{1,2} = A\Gamma(\gamma)\Delta_m(B,\alpha,\beta) \) with \( A, B \in \mathfrak{A} \) and \( \alpha, \beta, \gamma \) strings of \( \mathfrak{A} \). Then we can write

\[
X_{1,2} = \begin{cases} A(\gamma|\alpha)B(\beta|F\Pi_{\gamma-1}B\Pi_{\alpha+m}F|\alpha)B(\beta|F\Pi_{\gamma+m} & \text{if } \gamma - 1 = \alpha + m \\ 0 & \text{elsewhere} \end{cases}
\]

since

\[
X_{1,2} = A(\gamma|\alpha)B(\beta|F\Pi_{\gamma-1}B\Pi_{\alpha+m}F|\alpha)B(\beta|F\Pi_{\gamma+m}.
\]

Observe that we can find a natural number \( k_0 \) such that the relation

\[
X_{1,2}W^{k_m}Y_{2,1}V^k = 0
\]

holds for each \( k > k_0 \). In fact

\[
W^{k_m}(\xi_0, \xi_1, \ldots, \xi_n, \ldots) = (0, \ldots, 0, \xi_0, \xi_1, \ldots),
\]

for all vectors \( (\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in l^2(F\mathcal{H}) \); so \( \Pi_{\beta+m}W^{k_m} = 0 \) for all \( k > \beta + m \). Then by equation (5.4) it follows that

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^*Y_{1,1}V^k + X_{1,2}W^kY_{2,1}V^k)
\]

\[
= \lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^*Y_{1,1}V^k).
\]

Hence we have to compute only \( \varphi(X_{1,2}C(k)^*Y_{1,1}V^k) \). Notice that

\[
X_{1,2}C(k)^*Y_{1,1}V^k = A(\gamma|\alpha)B(\beta|F\Pi_{\gamma-1}B\Pi_{\alpha+m}C(k)^*Y_{1,1}V^k
\]

by relationship (5.3), and that

\[
\Pi_{\beta+m}C(k)^* = FV^{(k-\beta-m-1)} \quad \text{for } k > \beta + m,
\]
by relationship (2.7). It follows that
\[
X_{1,2}C(k)^*Y_{1,1}V^k = A(\gamma|F|\alpha)B(\beta|FV(k-\hat{\beta}-m-1)^*Y_{1,1}V^k
= A(\gamma|F|\alpha)B(\beta|F\Phi(k-\hat{\beta}-1)(Y_{1,1})V_{\hat{\beta}+m+1}.
\]
Since \(\hat{\gamma} = \hat{\alpha} + m + 1\), we have \(A(\gamma|F|\alpha)B(\beta) \in \mathfrak{A}(\hat{\beta} + m + 1)\) by relationship (3.1).
Hence there is a string \(\vartheta\) of \(\mathfrak{A}\) with \(\vartheta = \hat{\beta} + m + 1\) and an operator \(R \in \mathfrak{A}\), such that \(A(\gamma|F|\alpha)B(\beta) = R(\vartheta)\). So we can write
\[
X_{1,2}C(k)^*Y_{1,1}V^k = R(\vartheta|F\Phi(k-\hat{\beta}-1)(Y_{1,1})V_{\hat{\beta}+m+1}.
\]
If we set \(\vartheta = (n_1, n_2, \ldots, n_r, A_1, A_2, \ldots, A_r)\) then we have \(n_1 + n_2 + \ldots + n_r = \hat{\beta} + m + 1\) and
\[
R(\vartheta|F\Phi(k-\hat{\beta}-1)(Y_{1,1})V_{\hat{\beta}+m+1} = RV^r|A_1V^r|A_2V^r|A_r|A_1F\Phi(k-\hat{\beta}-1)(Y_{1,1})V_{\hat{\beta}+m+1})
= R\Phi^n(A_1\Phi^{n-1}(A_{r-1} \cdots \Phi^n(A_2R_k)),
\]
where
\[
R_k = \Phi^n(A_1\Phi^{n-1}(Y_{1,1})) - \Phi^{n-1}(A_1)\Phi^{n}(Y_{1,1}) \in \mathfrak{A}.
\]
Using the \(\varphi\)-adjoint, we have
\[
\varphi(X_{1,2}C(k)^*Y_{1,1}V^k) = \varphi(\Phi^n(\varphi \cdots \Phi_{r-1}(\varphi_{r}(R)A_r) \cdots A_3)A_2R_k). \quad (5.5)
\]
In fact,
\[
\varphi(X_{1,2}C(k)^*Y_{1,1}V^k) = \varphi(R\Phi^n(A_1\Phi^{n-1}(A_{r-1} \cdots \Phi^n(A_2R_k)))
= \varphi(\Phi^n(R)A_1\Phi^{n-1}(A_{r-1} \cdots \Phi^n(A_2R_k)))
= \varphi(\Phi^n(R)A_rA_{r-1} \cdots A_3R^n(A_2R_k))
= \varphi(\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_3)A_2R_k),
\]
and replacing \(R_k\) we obtain that
\[
\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_3)A_2R_k
= \Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_3)A_2\Phi^{n}(A_1\Phi^{(k-\beta)}(Y_{1,1})))
\]
Therefore
\[
\varphi(X_{1,2}C(k)^*Y_{1,1}V^k)
= \varphi(\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_2)A_1\Phi^{(k-\beta)}(Y_{1,1})))
\]
Now, assume that \(\varphi\) is ergodic. Then we have that
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_2)A_1\Phi^{(k-\beta)}(Y_{1,1}))
= \varphi(\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_2)A_1)\varphi(Y_{1,1}),
\]
and that
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_2)A_1\Phi^{(k-\beta)}(Y_{1,1}))
= \varphi(\Phi^n(\varphi \cdots \Phi^{r-1}(\varphi_{r}(R)A_r) \cdots A_2)A_1)\varphi(Y_{1,1}).
\]
Thus

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(X_{1,2}C(k)^*Y_{1,1}V^k) = 0,
\]

completing the proof of item (a).

In the weakly mixing case, using relationship (5.5) we obtain:

\[
|\varphi(X_{1,2}C_{1}^*Y_{1,1}V^k)| = |\varphi (T\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1}) - \varphi(T\Phi^{n_1-1}(\Phi(A_1))\Phi^{k-\beta})(Y_{1,1}))|,
\]

where \(T = \Phi^{n_2} \cdots \Phi^{n_r-1}(\Phi^{n_r}(R)A_r) \cdots A_2\).

Adding and subtracting the element \(\varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})\) we can write:

\[
|\varphi(X_{1,2}C_{1}^*Y_{1,1}V^k)| \leq |\varphi(T\Phi^{n_1}(A_1)\Phi^{k-\beta-1}(Y_{1,1})) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})| \\
+ |\varphi(T\Phi^{n_1-1}(\Phi(A_1))\Phi^{(k-\beta)}(Y_{1,1}))) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})|.
\]

Moreover

\[
|\varphi(T\Phi^{n_1-1}(\Phi(A_1))\Phi^{(k-\beta)}(Y_{1,1}))) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})| \\
= |\varphi(\Phi^{n_1-1}(T)\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})) - \varphi(\Phi^{n_1-1}(T)\Phi(A_1)\varphi(Y_{1,1})|,
\]

and by the weakly mixing properties we obtain:

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(T\Phi^{n_1}(A_1)\Phi^{(k-\beta-1)}(Y_{1,1})) - \varphi(T\Phi^{n_1}(A_1))\varphi(Y_{1,1})| = 0,
\]

and

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(\Phi^{n_1-1}(T)\Phi(A_1)\Phi^{(k-\beta)}(Y_{1,1})) - \varphi(\Phi^{n_1-1}(T)\Phi(A_1))\varphi(Y_{1,1})| = 0
\]

completing the proof of item (b). \(\square\)

Finally, the proof of proposition Proposition 5.1 is a simple consequence of this lemma since the \(C^*\)-algebra \(\mathfrak{A}\) is included in \(C^*(S)\), the norm closure of \(*\)-algebra \(\mathcal{A}(S)\).

It is clear that Proposition 5.1 can be extended to a quantum dynamical system \((\mathfrak{M}, \Phi)\) with \(\varphi\) a normal faithful state on \(\mathfrak{M}\).

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