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Pierre BOUSQUET, Augusto C. PONCE, and Jean VAN SCHAFTINGEN

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DENSITY OF SMOOTH MAPS FOR FRACTIONAL SOBOLEV SPACES $W^{s,p}$ INTO ℓ SIMPLY CONNECTED MANIFOLDS WHEN $s \geqslant 1$

PIERRE BOUSQUET, AUGUSTO C. PONCE, AND JEAN VAN SCHAFTINGEN

Abstract. Given a compact manifold $N^n \subset \mathbb{R}^\nu$ and real numbers $s \geqslant 1$ and $1 \leqslant p < \infty$, we prove that the class $C^\infty(\overline{Q}^m; N^n)$ of smooth maps on the cube with values into N^n is strongly dense in the fractional Sobolev space $W^{s,p}(Q^m; N^n)$ when N^n is $\lfloor sp \rfloor$ simply connected. For sp integer, we prove weak sequential density of $C^\infty(\overline{Q}^m; N^n)$ when N^n is $sp - 1$ simply connected. The proofs are based on the existence of a retraction of $\mathbb{R}^\nu$ onto N^n except for a small subset of N^n and on a pointwise estimate of fractional derivatives of composition of maps in $W^{s,p} \cap W^{1,sp}$.

1. Introduction

In this paper we discuss results and open questions related to the density of smooth maps in Sobolev spaces with values into a manifold. For this purpose, let N^n be a compact manifold of dimension n imbedded in the Euclidean space $\mathbb{R}^\nu$. For any $s > 0$ and $1 \leqslant p < +\infty$, we define the class of Sobolev maps defined on the unit m dimensional cube Q^m with values into N^n,

$$W^{s,p}(Q^m; N^n) = \big\{ u \in W^{s,p}(Q^m; \mathbb{R}^\nu) : u \in N^n \text{ a.e.} \big\}.$$

When $s = k$ is an integer, $W^{s,p}(Q^m; \mathbb{R}^\nu)$ is the standard Sobolev space equipped with the norm

$$\|u\|_{W^{s,p}(Q^m)} = \|u\|_{L^p(Q^m)} + \sum_{j=1}^{k} \|D^j u\|_{L^p(Q^m)}.$$

When s is not an integer, $s = k + \sigma$ with $k \in \mathbb{N}$ and $0 < \sigma < 1$. In this case, by $u \in W^{s,p}(Q^m; \mathbb{R}^\nu)$ we mean that $u \in W^{k,p}(Q^m; \mathbb{R}^\nu)$ and

$$[D^k u]_{W^{\sigma,p}(Q^m)} = \bigg(\int\limits_{Q^m} \int\limits_{Q^m} \frac{|D^k u(x) - D^k u(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \bigg)^{1/p} < +\infty,$$

and the associated norm is given by

$$\|u\|_{W^{s,p}(Q^m)} = \|u\|_{L^p(Q^m)} + \sum_{j=1}^{k} \|D^j u\|_{L^p(Q^m)} + [D^k u]_{W^{\sigma,p}(Q^m)}.$$

The fractional Sobolev spaces $W^{s,p}(Q^m; \mathbb{R}^\nu)$ arise in the trace theory of Sobolev spaces of integer order. For example, the trace is a continuous linear operator from $W^{1,p}(Q^m; \mathbb{R}^\nu)$ onto $W^{1-\frac{1}{p},p}(\partial Q^m; \mathbb{R}^\nu)$ [12, Theorem 1.I].

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We first address the question of *strong density* of smooth maps: given $u \in W^{s,p}(Q^m; N^n)$, does there exist a sequence in $C^\infty(\overline{Q}^m; N^n)$ which converges to u with respect to the strong topology induced by the $W^{s,p}$ norm?

A naive approach consists in applying a standard regularization argument. This works well for maps in $W^{s,p}(Q^m; \mathbb{R}^\nu)$ and shows that $C^\infty(\overline{Q}^m; \mathbb{R}^\nu)$ is strongly dense in that space. When $\mathbb{R}^\nu$ is replaced by N^n, the conclusion is less clear since the convolution of a map $u \in W^{s,p}(Q^m; N^n)$ with a smooth kernel φ_t yields a map with values in the convex hull of N^n. In this case, one might try to project $\varphi_t * u$ into the manifold N^n. This is indeed possible for $sp \geqslant m$.

If $sp > m$, then by the Morrey-Sobolev imbedding, $W^{s,p}$ is continuously imbedded into C^0. Thus, every map $u \in W^{s,p}(Q^m; N^n)$ has a continuous representative and $\varphi_t * u$ converges uniformly as t tends to zero. In particular,

$$\lim_{t \to 0} \sup_{x \in Q^m} \operatorname{dist}(\varphi_t * u(x), N^n) = 0. \tag{1.1}$$

Hence, one may project $\varphi_t * u$ back to N^n since the nearest point projection Π is well defined and smooth on a neighborhood of N^n.

If $sp = m$, then the Morrey-Sobolev imbedding fails but property (1.1) remains true [31] since $W^{s,p}$ injects continuously into the space VMO of functions with vanishing mean oscillation. This fact has been observed by Brezis and Nirenberg [9].

We may summarize as follows:

Theorem 1.1. — *$C^\infty(\overline{Q}^m; N^n)$ is strongly dense in $W^{s,p}(Q^m; N^n)$ for $sp \geqslant m$.*

The case where $sp < m$ is more subtle and the answer depends on the topology of N^n. Even when N^n is the unit sphere $\mathbb{S}^n$ the approximation problem is not fully understood. For instance, consider the map $u : B^3 \to \mathbb{S}^2$ defined in the unit ball $B^3 \subset \mathbb{R}^3$ by

$$u(x) = \frac{x}{|x|}.$$

Then, $u \in W^{s,p}(B^3; \mathbb{S}^2)$ for every $s > 0$ and $p \geqslant 1$ such that $sp < 3$, but u cannot be strongly approximated in $W^{s,p}$ by smooth maps with values into $\mathbb{S}^2$ when $2 \leqslant sp < 3$. This example originally due to Schoen and Uhlenbeck [31] for $s = 1$ can be adapted to the case where $\mathbb{S}^2$ is replaced by any compact manifold N^n and for any value of s [10, Theorem 3; 23, Theorem 4.4]:

Theorem 1.2. — *If $sp < m$ and $\pi_{\lfloor sp \rfloor}(N^n) \neq \{0\}$, then $C^\infty(\overline{Q}^m; N^n)$ is not strongly dense in $W^{s,p}(Q^m; N^n)$.*

It seems that the topological condition $\pi_{\lfloor sp \rfloor}(N^n) \neq \{0\}$ is the only obstruction to the strong density of smooth maps in $W^{s,p}(Q^m; N^n)$. This is indeed true when s is an integer by a remarkable result of Bethuel [3, Theorem 1; 17] for $s = 1$ which has been recently generalized by the authors [6, Theorem 4] for any $s \in \mathbb{N}$ (see also [14]):

Theorem 1.3. — *For any $s \in \mathbb{N}_*$, if $sp < m$ and $\pi_{\lfloor sp \rfloor}(N^n) = \{0\}$, then $C^\infty(\overline{Q}^m; N^n)$ is strongly dense in $W^{s,p}(Q^m; N^n)$.*

Some cases of non-integer values have been investigated. For instance when $s = 1 - 1/p$ in the setting of trace spaces [4, 24] and also when $s \geqslant 1$ and $N^n = \mathbb{S}^n$

[5, 10]. Brezis and Mironescu [8] have announced in a personal communication a solution to the question of strong density for any $0 < s < 1$.

All these cases give an affirmative answer to the following:

Open Problem 1.4. — Let $s \notin \mathbb{N}_*$. When $sp < m$ and $\pi_{\lfloor sp \rfloor}(N^n) = \{0\}$, is it true that $C^\infty(\overline{Q}^m; N^n)$ is strongly dense in $W^{s,p}(Q^m; N^n)$?

In this paper, we investigate Open problem 1.4 for ℓ simply connected manifolds N^n:

$$\pi_0(N^n) = \cdots = \pi_\ell(N^n) = \{0\}. \tag{1.2}$$

We prove the following:

THEOREM 1.5. — *Let $s \geqslant 1$. If $sp < m$ and if N^n is $\lfloor sp \rfloor$ simply connected, then $C^\infty(\overline{Q}^m; N^n)$ is strongly dense in $W^{s,p}(Q^m; N^n)$.*

Even in the case where s is an integer — which is covered in full generality by Theorem 1.3 — the proof is simpler and has its own interest. We have been inspired by Hajłasz [15] who has proved Theorem 1.5 for $s = 1$.

Our proof of Theorem 1.5 is based on two main ingredients. The *geometric tool* (Proposition 2.1) gives a smooth retraction of the ambient space $\mathbb{R}^\nu$ onto N^n except for a small subset of N^n. The *analytic tool* (Proposition 2.6) gives a pointwise estimate of the fractional derivative of $\eta \circ u$, where η is a smooth map and u is a $W^{s,p}$ map.

The counterpart of Theorem 1.5 for $0 < s < 1$ requires different tools and will be investigated in a subsequent paper.

The second problem we adress in this paper concerns the *weak sequential density* of $C^\infty(\overline{Q}^m; N^n)$ in $W^{s,p}(Q^m; N^n)$: given $u \in W^{s,p}(Q^m; N^n)$, does there exist a sequence in $C^\infty(\overline{Q}^m; N^n)$ which is bounded in $W^{s,p}(Q^m; N^n)$ and converges to u in measure?

The case $sp \geqslant m$ has an affirmative answer due to the strong density of smooth maps. When $sp < m$, we find the same topological obstruction as for the strong density problem when sp is not an integer [3, Theorem 3]:

THEOREM 1.6. — *If $sp < m$ is such that $sp \notin \mathbb{N}$ and if $C^\infty(\overline{Q}^m; N^n)$ is weakly sequentially dense in $W^{s,p}(Q^m; N^n)$, then $\pi_{\lfloor sp \rfloor}(N^n) = \{0\}$.*

From Theorem 1.3, it follows that for every $s \in \mathbb{N}_*$ such that $sp \notin \mathbb{N}$ the problems of weak and strong density of smooth maps in $W^{s,p}(Q^m; N^n)$ are equivalent. We expect the same is true for $s \notin \mathbb{N}$; the missing ingredient would be an affirmative answer to Open Problem 1.4.

The conclusion of Theorem 1.6 need not be true when sp is an integer. For instance, by a result of Bethuel [2, Theorem 3], $C^\infty(\overline{Q}^3; \mathbb{S}^2)$ is weakly sequentially dense in $W^{1,2}(Q^3; \mathbb{S}^2)$, even though it is not strongly dense by Theorem 1.2.

As a byproduct of the tools we use to prove Theorem 1.5, we establish the following:

THEOREM 1.7. — *Let $s \geqslant 1$. If $sp < m$ is such that $sp \in \mathbb{N}$ and if N^n is $sp - 1$ simply connected, then $C^\infty(\overline{Q}^m; N^n)$ is weakly sequentially dense in $W^{s,p}(Q^m; N^n)$.*

This result is due to Hajłasz [15, Corollary 1] when $s = 1$; Hajłasz's argument still applies for $p = 1$ although it is not explicitly stated in his paper. More recently, Hang and Lin [18, Corollary 8.6] proved an analogue of Theorem 1.7 under a weaker topological assumption for $s = 1$. To our knowledge, the only result concerning weak sequential density of smooth maps for non-integer values of s deals with the case $s = 1/2$, $p = 2$ and $N = \mathbb{S}^1$ and is due to Rivière [29, Theorem 1.2].

Combining Theorem 1.2 and Theorem 1.7 we deduce that $C^\infty(\overline{Q}^m; \mathbb{S}^n)$ is weakly sequentially dense but not strongly dense in $W^{s,p}(Q^m; \mathbb{S}^n)$ for $n < m$ and $sp = n$.

When $sp \in \mathbb{N}$, we do not know whether $C^\infty(\overline{Q}^m; N^n)$ is weakly sequentially dense in $W^{s,p}(Q^m; N^n)$ with no extra assumption on the compact manifold N^n. The only results which are known in this sense concern $s = 1$: for $p = 1$ [16, Theorem 1.3; 27, Theorem I] and for $p = 2$ [28, Theorem I].

2. Main tools

2.1. **Geometric tool.** Our first tool is the construction of a retraction of $\mathbb{R}^\nu$ onto N^n except for a small subset of N^n. This is the only place where the topological assumption (1.2) concerning the ℓ simply connectedness of the manifold N^n comes into place.

Proposition 2.1. — *If N^n is ℓ simply connected, then for every $0 < \epsilon \leqslant 1$ there exist a smooth function $\eta : \mathbb{R}^\nu \to N^n$ and a compact set $K \subset N^n$ such that*

(*i*) *for every $x \in N^n \setminus K$, $\eta(x) = x$,*

(*ii*) *$\mathcal{H}^n(K) \leqslant C\epsilon^{\ell+1}$, for some constant $C > 0$ depending on N^n and ν,*

(*iii*) *for every $j \in \mathbb{N}_*$,*

$$\|D^j \eta\|_{L^\infty(\mathbb{R}^\nu)} \leqslant \frac{C'}{\epsilon^j},$$

for some constant $C' > 0$ depending on N^n, ν and j.

The set K is chosen as the ϵ neighborhood of an $n-\ell-1$ dimensional dual skeleton of N^n. This proposition is the smooth counterpart of Hajłasz's construction of a Lipschitz continuous map η [15, Section 4].

The proof of Proposition 2.1 relies on the existence of a triangulation of the manifold N^n. It is more convenient to use a variant of the triangulation based on the decomposition of N^n in terms of cubes rather than simplices.

A cubication $\mathcal{T}$ of N^n is a finite collection of closed sets covering N^n of the form $\Phi(\sigma)$ with $\sigma \in \mathcal{Q}$ such that

(*a*) $\Phi : \bigcup_{\sigma \in \mathcal{Q}} \sigma \to N^n$ is a biLipschitz map,

(*b*) $\mathcal{Q}$ is a finite collection of cubes of dimension m in some Euclidean space $\mathbb{R}^\mu$, such that two elements of $\mathcal{Q}$ are either disjoint or intersect along a common face of dimension ℓ for some $\ell \in \{0, \dots, n\}$.

Given $\ell \in \{0, \dots, n\}$, we denote by T^ℓ the union of all ℓ dimensional faces of elements of $\mathcal{T}$; we call T^ℓ the ℓ dimensional skeleton of $\mathcal{T}$.

We recall the following lemma [15, Lemma 1]:

Lemma 2.2. — *Let $\mathcal{T}$ be a cubication of N^n and let T^ℓ be the ℓ dimensional skeleton of $\mathcal{T}$. If N^n is ℓ simply connected, then there exists a Lipschitz continuous function $\underline{\eta} : \mathbb{R}^\nu \to N^n$ such that for every $x \in T^\ell$, $\underline{\eta}(x) = x$.*

Proof. — Let $CT^\ell \subset \mathbb{R} \times \mathbb{R}^\nu$ denote the cone

$$\big\{(\lambda, \lambda x) \in \mathbb{R} \times \mathbb{R}^\nu : \lambda \in [0, 1] \text{ and } x \in T^\ell\big\}.$$

Since CT^ℓ is contractible, there exists a continuous map $\xi : \mathbb{R}^\nu \to CT^\ell$ such that for every $x \in T^\ell$, $\xi(x) = (1, x)$.

We may choose ξ to be uniformly continuous. Indeed, if $p : \mathbb{R}^\nu \to \mathbb{R}^\nu$ is any Lipschitz function such that p coincides with the identity on T^ℓ and p is constant outside some ball containing T^ℓ, then for every $x \in T^\ell$, $\xi \circ p(x) = (1, x)$ and, in addition, $\xi \circ p$ is uniformly continuous. Replacing ξ by $\xi \circ p$ if necessary, we assume in the sequel that ξ itself is uniformly continuous.

Since N^n is ℓ simply connected, the identity map in N^n is homotopic to a continuous map in N^n which is constant on T^ℓ [33, Section 6]. More precisely, there exist a continuous map $H : [0, 1] \times N^n \to N^n$ and $a \in N^n$ such that

(a) for every $x \in T^\ell$, $H(0, x) = a$,
(b) for every $x \in N^n$, $H(1, x) = x$.

Since H is constant on $\{0\} \times T^\ell$, H induces a continuous quotient map $\overline{H} : CT^\ell \to N^n$ defined for every $(\lambda, \lambda x) \in CT^\ell$ by $\overline{H}(\lambda, \lambda x) = H(\lambda, x)$. Then, $\overline{H} \circ \xi$ is a uniformly continuous map with values into N^n which coincides with the identity map on T^ℓ.

Using a standard approximation argument, we may construct a Lipschitz map having the same properties. We present the argument for the sake of completeness.

Given $\iota > 0$, let $\theta : \mathbb{R}^\nu \to [0, 1]$ be a Lipschitz continuous function supported in a neighborhood of T^ℓ such that

(a') for every $x \in T^\ell$, $\theta(x) = 1$,
(b') for every $x \in \operatorname{supp} \theta$, $|x - \overline{H} \circ \xi(x)| \leqslant \iota$.

Since $\overline{H} \circ \xi$ is uniformly continuous, there exists a Lipschitz approximation $h : \mathbb{R}^\nu \to \mathbb{R}^\nu$ such that for every $x \in \mathbb{R}^\nu$,

$$|h(x) - \overline{H} \circ \xi(x)| \leqslant \iota.$$

Then, for every $x \in \mathbb{R}^\nu$,

$$\big|\overline{H} \circ \xi(x) - \big(\theta(x)x + (1 - \theta(x))h(x)\big)\big| \leqslant \iota.$$

Since $\overline{H} \circ \xi(x) \in N^n$, it follows that

$$\theta(x)x + (1 - \theta(x))h(x) \in N^n + \overline{B}^\nu_\iota,$$

where $\overline{B}^\nu_\iota$ is the closed ball in $\mathbb{R}^\nu$ of radius ι centered at 0. Choosing ι such that the nearest point projection $\Pi : N^n + \overline{B}^\nu_\iota \to N^n$ is well-defined and smooth, then we have the conclusion by taking $\underline{\eta} : \mathbb{R}^\nu \to N^n$ defined for $x \in \mathbb{R}^\nu$ by

$$\underline{\eta}(x) = \Pi\big(\theta(x)x + (1 - \theta(x))h(x)\big).$$

The proof is complete. □

We shall also use dual skeletons associated to a cubication $\mathcal{T}$ given by a map $\Phi : \bigcup_{\sigma \in \mathcal{Q}} \sigma \to N^n$. We first define dual skeletons for a cube in $\mathbb{R}^n$. Let $j \in \{0, \dots, n\}$. When the center of the cube is 0 and the faces are parallel to the coordinate axes, the dual skeleton of dimension j is the set of points in the cube which have at least $n - j$ components equal to zero. By using an isometry, we can define the dual skeleton of a cube of dimension n in $\mathbb{R}^\mu$ in general position. Then, the dual skeleton of dimension j of a family $\mathcal{Q}$ of cubes as above is simply the union of the dual skeletons of dimension j of each cube. Finally, the dual skeleton L^j of dimension j of the cubication $\mathcal{T}$ of N^n is the image by Φ of the j dimensional dual skeleton of $\mathcal{Q}$.

The following lemma implies the homotopy equivalence between the skeleton T^ℓ of the manifold N^n and the complement of the dual skeleton $L^{n-\ell-1}$ in N^n. We are particularly interested in the pointwise estimates of the homotopy f:

Lemma 2.3. — *Let $\ell \in \{0, \dots, n-1\}$, let $\mathcal{T}$ be a cubication of N^n and let $L^{n-\ell-1}$ be the $n-\ell-1$ dimensional dual skeleton of $\mathcal{T}$. Then, there exists a locally Lipschitz continuous function*

$$f : [0,1] \times (N^n \setminus L^{n-\ell-1}) \to N^n$$

such that

(i) *for every $t \in [0,1]$ and for every $x \in T^\ell$, $f(t,x) = x$,*
(ii) *for every $x \in N^n \setminus L^{n-\ell-1}$, $f(0,x) = x$ and $f(1,x) \in T^\ell$,*
(iii) *for every $t \in [0,1]$ and for every $x \in N^n \setminus L^{n-\ell-1}$,*

$$|\partial_t f(t,x)| \leqslant C,$$

and

$$|\partial_x f(t,x)| \leqslant \frac{C'}{\operatorname{dist}(x, L^{n-\ell-1})},$$

for some constants $C, C' > 0$ depending on n, ℓ, N^n and $\mathcal{T}$.

Proof. — We first establish the result when the manifold N^n is replaced by the cube $[-1,1]^n$ and $L^{n-\ell-1}$ is the dual skeleton of dimension $n-\ell-1$ of $[-1,1]^n$. Following [33], we consider for every $x \in [-1,1]^n$,

$$|x|_\ell = \min_{\substack{S \subset \{1,\dots,n\} \\ |S| = \ell+1}} \max_{i \in S} |x_i|.$$

In particular, for every $x \in [-1,1]^n$, $x \in L^{n-\ell-1}$ if and only if $|x|_\ell = 0$. The function $x \in [-1,1]^n \mapsto |x|_\ell$ is Lipschitz continuous of constant 1.

Let $\phi_\ell : [-1,1]^n \setminus L^{n-\ell-1} \to T^\ell$ be defined for every $x \in [-1,1]^n$ by

$$\phi_\ell(x) = (y_1, \dots, y_n),$$

where

$$y_i = \begin{cases} \operatorname{sgn} x_i & \text{if } |x_i| \geqslant |x|_\ell, \\ x_i/|x|_\ell & \text{if } |x_i| < |x|_\ell. \end{cases}$$

The homotopy $f : [0,1] \times ([-1,1]^n \setminus L^{n-\ell-1}) \to [-1,1]^n$ defined by

$$f(t,x) = (1-t)x + t\phi_\ell(x)$$

has the required properties.

In order to prove the existence of the homotopy f for a general compact manifold N^n, we perform the above construction in every cube of a given cubication $\Phi : \bigcup_{\sigma \in \mathcal{Q}} \sigma \to N^n$. If two cubes σ_1 and σ_2 in $\mathcal{Q}$ have a non empty intersection, then the corresponding maps $\phi_{\ell,1}$ and $\phi_{\ell,2}$ coincide on the common face $\sigma_1 \cap \sigma_2$. Hence, we can glue together the locally Lipschitz continuous maps obtained for each cube so as to obtain a global map f_0 which is defined on the entire collection of cubes in $\mathcal{Q}$. The conclusion follows by taking

$$f(t,x) = \Phi\big(f_0(t, \Phi^{-1}(x))\big). \qquad \square$$

We now prove a counterpart of Proposition 2.1 for a Lipschitz continuous map η:

LEMMA 2.4. — *Let $\ell \in \{0, \dots, n-1\}$, let $\mathcal{T}$ be a cubication of N^n and $L^{n-\ell-1}$ be the $n-\ell-1$ dimensional dual skeleton of $\mathcal{T}$ and let $\iota > 0$ be such that the nearest point projection Π onto N^n is smooth on $N^n + \overline{B}^\nu_{2\iota}$. If N^n is ℓ simply connected, then for every $0 < \epsilon \leqslant 1$ there exists a Lipschitz continuous map $\eta : \mathbb{R}^\nu \to N^n$ such that*

(*i*) *$\eta = \Pi$ on $(N^n + B^\nu_\iota) \setminus \Pi^{-1}(L^{n-\ell-1} + B^\nu_\epsilon)$,*
(*ii*) *for every $x \in \mathbb{R}^\nu$,*
$$|D\eta(x)| \leqslant \frac{C''}{\epsilon}.$$
for some constant $C'' > 0$ depending on N^n, $\mathcal{T}$ and ν.

Proof. — Let f be the map given by Lemma 2.3. The extension
$$\bar{f} : \big(\{0\} \times L^{n-\ell-1}\big) \cup \big([0,1] \times (N^n \setminus L^{n-\ell-1})\big) \to N^n$$
defined by
$$\bar{f}(t,x) = \begin{cases} x & \text{if } t = 0, \\ f(t,x) & \text{if } 0 < t \leqslant 1, \end{cases}$$
is continuous.

Let Π be the nearest point projection onto N^n and denote by $\overline{\Pi} : \mathbb{R}^\nu \to \mathbb{R}^\nu$ a smooth extension of Π. The image of $\overline{\Pi}$ need not be contained in the manifold N^n.

Let $\theta : \mathbb{R}^\nu \to [0,2]$ be a Lipschitz continuous function such that

(a) for every $x \in N^n + B^\nu_\iota$, $\theta(x) = 2$,
(b) for every $x \in \mathbb{R}^\nu \setminus (N^n + B^\nu_{2\iota})$, $\theta(x) = 0$.

Given $0 < \epsilon \leqslant 1$, let $d_\epsilon : N^n + B^\nu_{2\iota} \to \mathbb{R}$ be defined by
$$d_\epsilon(x) = \frac{1}{\epsilon} \operatorname{dist}\,(\Pi(x), L^{n-\ell-1}).$$

Let $\lambda : [0, +\infty) \to [0,1]$ be a Lipschitz continuous function such that

(a') for every $t \leqslant \frac{1}{2}$ and for every $t \geqslant 2$, $\lambda(t) = 0$,
(b') $\lambda(1) = 1$.

Denote by $\underline{\eta} : \mathbb{R}^\nu \to N^n$ the function given by Lemma 2.2. Let $\eta : \mathbb{R}^\nu \to N^n$ be the map defined by
$$\eta(x) = \begin{cases} \overline{f}\big(\lambda(\theta(x) d_\epsilon(x)), \Pi(x)\big) & \text{if } x \in N^n + B^\nu_{2\iota} \text{ and } \theta(x) d_\epsilon(x) > 1, \\ \underline{\eta} \circ \overline{f}\big(\lambda(\theta(x) d_\epsilon(x)), \Pi(x)\big) & \text{if } x \in N^n + B^\nu_{2\iota} \text{ and } \theta(x) d_\epsilon(x) \leqslant 1, \\ \underline{\eta}\big(\overline{\Pi}(x)\big) & \text{if } x \notin N^n + B^\nu_{2\iota}. \end{cases}$$

We first check that η is continuous. For this purpose we only need to consider the borderline cases:

(1) $x \in N^n + B^\nu_{2\iota}$ and $\theta(x) d_\epsilon(x) = 1$,
(2) $x \in \partial(N^n + B^\nu_{2\iota})$.

In the first case, since $\lambda(1) = 1$, $\overline{f}(1, \cdot) \in T^\ell$ and $\underline{\eta}$ is the identity map on T^ℓ, we have
$$\begin{aligned} \overline{f}\big(\lambda(\theta(x) d_\epsilon(x)), \Pi(x)\big) &= \overline{f}(1, \Pi(x)) \\ &= \underline{\eta}\big(\overline{f}(1, \Pi(x))\big) = \underline{\eta} \circ \overline{f}\big(\lambda(\theta(x) d_\epsilon(x)), \Pi(x)\big). \end{aligned}$$

In the second case, $\theta(x) = 0$. Since $\lambda(0) = 0$ and $\overline{f}(0, \cdot)$ is the identity map on N^n,

$$\overline{f}\big(\lambda(\theta(x)d_\epsilon(x)), \Pi(x)\big) = \Pi(x) = \overline{\Pi}(x),$$

whence

$$\underline{\eta} \circ \overline{f}\big(\lambda(\theta(x)d_\epsilon(x)), \Pi(x)\big) = \underline{\eta}\big(\overline{\Pi}(x)\big).$$

We now check that property (i) holds. Indeed, if

$$x \in (N^n + B^\nu_\iota) \setminus \Pi^{-1}(L^{n-\ell-1} + B^\nu_\epsilon),$$

then $\theta(x) = 2$ and $d_\epsilon(x) \geqslant 1$. Thus,

$$\lambda(\theta(x)d_\epsilon(x)) = 0.$$

We then have

$$\eta(x) = \overline{f}(0, \Pi(x)) = \Pi(x).$$

It remains to establish property (ii). Indeed, if $x \notin N^n + B^\nu_{2\iota}$, then $\eta(x) = \underline{\eta}\big(\overline{\Pi}(x)\big)$ and the conclusion follows since $\underline{\eta}$ and $\overline{\Pi}(x)$ are both Lipschitz continuous, with Lipschitz constants independent of ϵ. If $x \in N^n + B^\nu_{2\iota}$ and $\theta(x)d_\epsilon(x) < \frac{1}{2}$, then $\eta(x) = \underline{\eta} \circ \Pi(x)$ and the estimate follows similarly. Finally, if $x \in N^n + B^\nu_{2\iota}$ and $\theta(x)d_\epsilon(x) \geqslant \frac{1}{2}$, then

$$\operatorname{dist}(\Pi(x), L^{n-\ell-1}) \geqslant \frac{\epsilon}{4}.$$

By the chain rule and the estimates given by Lemma 2.3,

$$|D\eta(x)| \leqslant C_1 \left(\frac{1}{\epsilon} + \frac{1}{\operatorname{dist}(\Pi(x), L^{n-\ell-1})}\right).$$

Combining both estimates, we get the conclusion. The proof of the lemma is complete. □

We now have all tools to prove Proposition 2.1.

Proof of Proposition 2.1. — Let $\varphi : \mathbb{R}^\nu \to \mathbb{R}$ be a smooth map supported in the unit ball B^ν_1. For every $t > 0$, let $\varphi_t : \mathbb{R}^\nu \to \mathbb{R}$ be the function defined for $x \in \mathbb{R}^\nu$ by $\varphi_t(x) = \frac{1}{t^\nu}\varphi(\frac{x}{t})$. Let $\iota > 0$ as in the previous lemma.

Given $0 < \epsilon \leqslant 1$, let $\zeta : \mathbb{R}^\nu \to [0, 1]$ be a smooth function such that

(a) for every $x \in N^n \setminus (L^{n-\ell-1} + B^\nu_{2\epsilon})$, $\zeta(x) = 1$,
(b) for every $x \notin (N^n + B^\nu_\iota) \setminus \Pi^{-1}(L^{n-\ell-1} + B^\nu_\epsilon)$, $\zeta(x) = 0$,
(c) for every $j \in \mathbb{N}_*$,

$$\|D^j\zeta\|_{L^\infty(\mathbb{R}^\nu)} \leqslant \frac{C_1}{\epsilon^j},$$

where $C_1 > 0$ depends on j.

Let $\eta_\epsilon : \mathbb{R}^\nu \to N^n$ be the Lipschitz continuous map given by the previous lemma and let $t > 0$ to be chosen below.

By property (b) and by Lemma 2.4 (i), the function

$$\zeta\eta_\epsilon + (1 - \zeta)\varphi_t * \eta_\epsilon$$

is smooth in $\mathbb{R}^\nu$ and for every $j \in \mathbb{N}_*$ there exists $C_2 > 0$ such that

$$\big\|D^j\big(\zeta\eta_\epsilon + (1 - \zeta)\varphi_t * \eta_\epsilon\big)\big\|_{L^\infty(\mathbb{R}^\nu)} \leqslant C_2\Big(1 + \frac{1}{t^j} + \frac{1}{\epsilon^j}\Big). \tag{2.1}$$

Moreover, by property (a) and by Lemma 2.4 (i), for every $x \in N^n \setminus (L^{n-\ell-1} + B^\nu_{2\epsilon})$,

$$\big(\zeta\eta_\epsilon + (1 - \zeta)\varphi_t * \eta_\epsilon\big)(x) = \eta_\epsilon(x) = \Pi(x) = x.$$

By Lemma 2.4 (ii) we have for every $t > 0$,

$$\|\varphi_t * \eta_\epsilon - \eta_\epsilon\|_{L^\infty(\mathbb{R}^\nu)} \leqslant t\|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^\nu)} \leqslant t\frac{C_3}{\epsilon}.$$

Taking

$$t = \frac{\iota\epsilon}{C_3},$$

it follows from the previous estimate that the image of $\zeta\eta_\epsilon+(1-\zeta)\varphi_t*\eta_\epsilon$ is contained in $N^n + B^\nu_\iota$. Hence, the function $\eta : \mathbb{R}^\nu \to N^n$,

$$\eta = \Pi \circ \bigl(\zeta\eta_\epsilon + (1-\zeta)\varphi_t * \eta_\epsilon\bigr),$$

is well-defined and smooth. Property (i) holds with

$$K = N^n \cap (L^{n-\ell-1} + B^\nu_{2\epsilon}).$$

Property (ii) also holds since K is a neighborhood of $L^{n-\ell-1}$ in N^n whose radius is of the order of ϵ. By estimate (2.1), property (iii) is also satisfied. This completes the proof of the proposition. □

2.2. **Analytic tool.** In this section we establish pointwise estimates of derivatives and fractional derivatives of the map $\eta \circ u$, where η is a smooth function and u belongs to $W^{s,p} \cap L^\infty$. In the case where s is an integer, this estimate follows from the classical chain rule for higher order derivatives:

Proposition 2.5. — *Let $k \in \mathbb{N}_*$. If $u \in W^{k,p}(Q^m;\mathbb{R}^\nu) \cap W^{1,kp}(Q^m;\mathbb{R}^\nu)$, then for every $j \in \{1,\dots,k\}$ there exists a measurable function $G_j \in L^p(Q^m)$ such that for every smooth map $\eta : \mathbb{R}^\nu \to \mathbb{R}^\nu$,*

$$|D^j(\eta \circ u)| \leqslant [\eta]_{C^j(\mathbb{R}^\nu)} G_j.$$

Moreover, $\|G_j\|_{L^p(Q^m)}$ is bounded from above by a constant depending on k, p, m, $\|u\|_{W^{k,p}(Q^m)}$ and $\|u\|_{W^{1,kp}(Q^m)}$.

We use the following notation:

$$[\eta]_{C^j(\mathbb{R}^\nu)} = \sum_{i=1}^{j} \|D^i\eta\|_{L^\infty(\mathbb{R}^\nu)}.$$

Proof. — We first observe that $\eta \circ u \in W^{k,p}(Q^m;\mathbb{R}^\nu)$. By the chain rule,

$$\begin{aligned}|D^j(\eta \circ u)(x)| &\leqslant C_1 \sum_{i=1}^{j} |D^i\eta(u(x))| \sum_{\substack{1\leqslant t_1 \leqslant \cdots \leqslant t_i \leqslant j,\\ t_1+\cdots+t_i=j}} |D^{t_1}u(x)|\cdots|D^{t_i}u(x)|\\ &\leqslant C_1[\eta]_{C^j(\mathbb{R}^\nu)} \sum_{i=1}^{j} \sum_{\substack{1\leqslant t_1 \leqslant \cdots \leqslant t_i \leqslant j,\\ t_1+\cdots+t_i=j}} |D^{t_1}u(x)|\cdots|D^{t_i}u(x)|.\end{aligned}$$

Let

$$G_j = C_1 \sum_{i=1}^{j} \sum_{\substack{1\leqslant t_1 \leqslant \cdots \leqslant t_i \leqslant j,\\ t_1+\cdots+t_i=j}} |D^{t_1}u|\cdots|D^{t_i}u|$$

Since the map u in the statement belongs to $W^{k,p}(Q^m;\mathbb{R}^\nu)\cap W^{1,kp}(Q^m;\mathbb{R}^\nu)$, it follows from the Gagliardo-Nirenberg interpolation inequality [13, 25] that

$$D^i u \in L^{\frac{jp}{i}}(Q^m).$$

By Hölder's inequality, we deduce that $G_j \in L^p(Q^m)$. □

We now establish a counterpart of the previous proposition for the fractional derivative introduced by Maz′ya and Shaposhnikova [22]. More precisely, given $0<\sigma<1$, $1\leqslant p<+\infty$, a domain $\Omega\subset\mathbb{R}^m$ and a measurable function $u:\Omega\to\mathbb{R}^\nu$, define for $x\in\Omega$,

$$D^{\sigma,p}u(x)=\biggl(\int\limits_{\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\biggr)^{1/p}.$$

We extend this definition for any $s>0$ such that $s\notin\mathbb{N}$ as follows:

$$D^{s,p}u=D^{\sigma,p}(D^k u),$$

where $k=\lfloor s\rfloor$ is the integral part of s and $\sigma=s-\lfloor s\rfloor$ is the fractional part of s. Using this notation, we have

$$[D^k u]_{W^{\sigma,p}(\Omega)}=\|D^{s,p}u\|_{L^p(\Omega)}.$$

Proposition 2.6. — *Let $s>1$ be such that $s\notin\mathbb{N}$. If $u\in W^{s,p}(Q^m;\mathbb{R}^\nu)\cap W^{1,sp}(Q^m;\mathbb{R}^\nu)$, then there exists a measurable function $H\in L^p(Q^m)$ such that for every smooth map $\eta:\mathbb{R}^\nu\to\mathbb{R}^\nu$,*

$$|D^{s,p}(\eta\circ u)|\leqslant[\eta]^{\sigma}_{C^{k+1}(\mathbb{R}^\nu)}[\eta]^{1-\sigma}_{C^{k}(\mathbb{R}^\nu)}H.$$

Moreover, $\|H\|_{L^p(Q^m)}$ is bounded from above by some constant depending on s, p, m, $\|u\|_{W^{s,p}(Q^m)}$ and $\|u\|_{W^{1,sp}(Q^m)}$.

This proposition implies a theorem of Brezis and Mironescu [8, Theorem 1.1] concerning the boundedness of the composition operator from $W^{s,p}\cap W^{1,sp}$ into $W^{s,p}$. A more elementary proof of the same result has been provided by Maz′ya and Shaposhnikova [22]; our proof of Proposition 2.6 is based on their strategy.

We begin with the following pointwise estimate of Maz′ya and Shaposhnikova [22, Lemma]:

Lemma 2.7. — *Let $q\geqslant 1$. If $v\in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{R}^\nu)$, then for $x\in\mathbb{R}^m$,*

$$\bigl(D^{\sigma,q}v(x)\bigr)^q\leqslant C\bigl(\mathcal{M}|Dv|^q(x)\bigr)^{\sigma}\bigl(\mathcal{M}|v|^q(x)\bigr)^{1-\sigma},$$

for some constant $C>0$ depending on m and q.

The maximal function associated to a nonnegative function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ is defined for $x\in\mathbb{R}^m$ by

$$\mathcal{M}f(x)=\sup_{\rho>0}\frac{1}{|B^m_\rho|}\int\limits_{B^m_\rho(x)}f.$$

For completeness, we prove Lemma 2.7 using a property of the maximal function due to Hedberg [20, Lemma]:

Lemma 2.8. — *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ be a nonnegative function and let $\delta > 0$. For every $x \in \mathbb{R}^m$ and $\rho > 0$,*

$$\int\limits_{B^m_\rho(x)} \frac{f(y)}{|y-x|^{m-\delta}}\,\mathrm{d}y \leqslant C\rho^\delta \mathcal{M}f(x),$$

$$\int\limits_{\mathbb{R}^m\setminus B^m_\rho(x)} \frac{f(y)}{|y-x|^{m+\delta}}\,\mathrm{d}y \leqslant \frac{C}{\rho^\delta} \mathcal{M}f(x),$$

for some constant $C > 0$ depending on m and δ.

Proof. — We briefly sketch the proof of Hedberg for the first estimate. The proof of the second one is similar. One has

$$\begin{aligned}\int\limits_{B^m_\rho(x)} \frac{f(y)}{|y-x|^{m-\delta}}\,\mathrm{d}y &= \sum_{i=0}^{\infty} \int\limits_{B^m_{\rho 2^{-i}}(x)\setminus B^m_{\rho 2^{-i-1}}(x)} \frac{f(y)}{|x-y|^{m-\delta}}\,\mathrm{d}y \\ &\leqslant C_1\rho^\delta \sum_{i=0}^{\infty} 2^{-\delta i}\mathcal{M}f(x) \leqslant C_2\rho^\delta \mathcal{M}f(x).\end{aligned}$$ □

Proof of Lemma 2.7. — Let $\rho > 0$. By Hardy's inequality [21, Section 1.3],

$$\int\limits_{B^m_\rho(x)} \frac{|v(x)-v(y)|^q}{|x-y|^{m+\sigma q}}\,\mathrm{d}y \leqslant C_1 \int\limits_{B^m_\rho(x)} \frac{|Dv(y)|^q}{|x-y|^{m-(1-\sigma)q}}\,\mathrm{d}y.$$

Thus, by Hedberg's lemma,

$$\int\limits_{B^m_\rho(x)} \frac{|v(x)-v(y)|^q}{|x-y|^{m+\sigma q}}\,\mathrm{d}y \leqslant C_2\rho^{(1-\sigma)q}\mathcal{M}|Dv|^q(x).$$

Since

$$|v(x)-v(y)|^q \leqslant C_3\big(|v(x)|^q + |v(y)|^q\big),$$

by an explicit integral computation and by Hedberg's lemma,

$$\int\limits_{\mathbb{R}^m\setminus B^m_\rho(x)} \frac{|v(x)-v(y)|^q}{|x-y|^{m+\sigma q}}\,\mathrm{d}y \leqslant \frac{C_4}{\rho^{\sigma q}}\big(|v(x)|^q + \mathcal{M}|v|^q(x)\big) \leqslant \frac{C_5}{\rho^{\sigma q}}\mathcal{M}|v|^q(x).$$

We conclude that

$$\big(D^{\sigma,q}v(x)\big)^q \leqslant C_2\rho^{(1-\sigma)q}\mathcal{M}|Dv|^q(x) + \frac{C_5}{\rho^{\sigma q}}\mathcal{M}|v|^q(x).$$

Minimizing the right-hand side with respect to ρ, we deduce the pointwise estimate. □

The following lemma is implicitly proved in [22, Section 2]:

Lemma 2.9. — *Let $0 < \sigma < 1$, $1 \leqslant p < +\infty$ and $i \in \mathbb{N}_*$. If for every $\alpha \in \{1,\dots,i\}$, $v_\alpha \in L^{q_\alpha}(\mathbb{R}^m)$ and $Dv_\alpha \in L^{r_\alpha}(\mathbb{R}^m)$, where $1 < r_\alpha < q_\alpha$ and*

$$\frac{1-\sigma}{q_\alpha} + \frac{\sigma}{r_\alpha} + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{i} \frac{1}{q_\beta} = \frac{1}{p},$$

then

$$\Big[\prod_{\alpha=1}^{i} v_\alpha\Big]_{W^{\sigma,p}(\mathbb{R}^m)} \leqslant C \sum_{\alpha=1}^{i} \bigg(\|v_\alpha\|_{L^{q_\alpha}(\mathbb{R}^m)}^{1-\sigma} \|Dv_\alpha\|_{L^{r_\alpha}(\mathbb{R}^m)}^{\sigma} \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{i} \|v_\beta\|_{L^{q_\beta}(\mathbb{R}^m)} \bigg),$$

for some constant $C > 0$ *depending on* m, σ, $r_1, \dots, r_i$, $q_1, \dots, q_i$.

Proof. — We first consider the case of dimension $m = 1$. Note that

$$\Big|\prod_{\alpha=1}^{i} v_\alpha(x) - \prod_{\alpha=1}^{i} v_\alpha(y)\Big| \leqslant \sum_{\alpha=1}^{i} \big|v_1(x)\cdots v_{\alpha-1}(x)\big(v_\alpha(x) - v_\alpha(y)\big)v_{\alpha+1}(y)\cdots v_i(y)\big|. \tag{2.2}$$

Thus, the left-hand side is bounded from above by a sum of functions of the form

$$\underline{f}_\alpha(x)\,|v_\alpha(x) - v_\alpha(y)|\,\overline{f}_\alpha(y).$$

By the Fundamental theorem of Calculus, for every $x, y \in \mathbb{R}$,

$$|v_\alpha(x) - v_\alpha(y)| \leqslant 2|x-y|\mathcal{M}|v_\alpha'|(x).$$

Thus, for every $\rho > 0$,

$$\int\limits_{B^1_\rho(x)} \frac{|v_\alpha(x) - v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}y \leqslant C_1\big(\mathcal{M}|v_\alpha'|(x)\big)^p \int\limits_{B^1_\rho(x)} \frac{(\overline{f}_\alpha(y))^p}{|x-y|^{1-(1-\sigma)p}}\,\mathrm{d}y.$$

By Hedberg's lemma, we get

$$\int\limits_{B^1_\rho(x)} \frac{|v_\alpha(x) - v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}y \leqslant C_2 \rho^{(1-\sigma)p}\big(\mathcal{M}|v_\alpha'|(x)\big)^p \mathcal{M}(\overline{f}_\alpha)^p(x).$$

Next, we write

$$|v_\alpha(x) - v_\alpha(y)|^p (\overline{f}_\alpha(y))^p \leqslant C_3\big(|v_\alpha(x)|^p(\overline{f}_\alpha(y))^p + |v_\alpha(y)|^p(\overline{f}_\alpha(y))^p\big).$$

By Hedberg's lemma, we also have

$$\int\limits_{\mathbb{R}\setminus B^1_\rho(x)} \frac{|v_\alpha(x) - v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}y \leqslant \frac{C_4}{\rho^{\sigma p}}\big(|v_\alpha(x)|^p\mathcal{M}(\overline{f}_\alpha)^p(x) + \mathcal{M}|v_\alpha\overline{f}_\alpha|^p(x)\big).$$

We conclude that

$$\begin{aligned}\int\limits_{\mathbb{R}} \frac{|v_\alpha(x) - v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}y \\ \leqslant C_2 \rho^{(1-\sigma)p}\big(\mathcal{M}|v_\alpha'|(x)\big)^p \mathcal{M}(\overline{f}_\alpha)^p(x) \\ + \frac{C_4}{\rho^{\sigma p}}\big(|v_\alpha(x)|^p\mathcal{M}(\overline{f}_\alpha)^p(x) + \mathcal{M}|v_\alpha\overline{f}_\alpha|^p(x)\big).\end{aligned}$$

Minimizing the right hand side with respect to ρ, we then get

$$\begin{aligned}\int\limits_{\mathbb{R}} \frac{|v_\alpha(x) - v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}y \\ \leqslant C_5\big(\mathcal{M}|v_\alpha'|(x)\big)^{\sigma p}\big(\mathcal{M}(\overline{f}_\alpha)^p(x)\big)^{\sigma}\big(|v_\alpha(x)|^p\mathcal{M}(\overline{f}_\alpha)^p(x) + \mathcal{M}|v_\alpha\overline{f}_\alpha|^p(x)\big)^{1-\sigma}.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (\underline{f}_\alpha(x))^p \frac{|v_\alpha(x) - v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}x\,\mathrm{d}y \leqslant C_5 \int_{\mathbb{R}} (\underline{f}_\alpha)^p \big(\mathcal{M}|v_\alpha'|\big)^{\sigma p} \big(\mathcal{M}(\overline{f}_\alpha)^p\big)^\sigma \big(|v_\alpha|^p \mathcal{M}(\overline{f}_\alpha)^p + \mathcal{M}|v_\alpha \overline{f}_\alpha|^p\big)^{1-\sigma}. \tag{2.3}$$

Let $\frac{1}{\underline{q}_\alpha} = \sum_{\beta=1}^{\alpha-1} \frac{1}{q_\beta}$ and $\frac{1}{\overline{q}_\alpha} = \sum_{\beta=\alpha+1}^{i} \frac{1}{q_\beta}$, so that by assumption,

$$\frac{1}{\underline{q}_\alpha} + \frac{\sigma}{r_\alpha} + \frac{1-\sigma}{q_\alpha} + \frac{1}{\overline{q}_\alpha} = \frac{1}{p}.$$

By Hölder's inequality,

$$\int_{\mathbb{R}} (\underline{f}_\alpha)^p \big(\mathcal{M}|v_\alpha'|\big)^{\sigma p} |v_\alpha|^{(1-\sigma)p} \mathcal{M}(\overline{f}_\alpha)^p \leqslant \|\underline{f}_\alpha\|^p_{L^{\underline{q}_\alpha}(\mathbb{R})} \|\mathcal{M}|v_\alpha'|\|^{\sigma p}_{L^{r_\alpha}(\mathbb{R})} \|v_\alpha\|^{(1-\sigma)p}_{L^{q_\alpha}(\mathbb{R})} \|(\mathcal{M}(\overline{f}_\alpha)^p)^{1/p}\|^p_{L^{\overline{q}_\alpha}(\mathbb{R})}.$$

We estimate the right hand side as follows. By Hölder's inequality,

$$\|\underline{f}_\alpha\|_{L^{\underline{q}_\alpha}(\mathbb{R})} \leqslant \prod_{\beta=1}^{\alpha-1} \|v_\beta\|_{L^{q_\beta}(\mathbb{R})}.$$

Since $r_\alpha > 1$, by the Maximal theorem [32, Chapter 1, Theorem 1],

$$\|\mathcal{M}|v_\alpha'|\|_{L^{r_\alpha}(\mathbb{R})} \leqslant C_6 \|v_\alpha'\|_{L^{r_\alpha}(\mathbb{R})}.$$

Since $\overline{q}_\alpha/p > 1$, by the Maximal theorem and by Hölder's inequality,

$$\begin{aligned}
\|(\mathcal{M}(\overline{f}_\alpha)^p)^{1/p}\|_{L^{\overline{q}_\alpha}(\mathbb{R})} &= \|\mathcal{M}(\overline{f}_\alpha)^p\|^{1/p}_{L^{\overline{q}_\alpha/p}(\mathbb{R})} \\
&\leqslant C_7 \|(\overline{f}_\alpha)^p\|^{1/p}_{L^{\overline{q}_\alpha/p}(\mathbb{R})} \\
&= C_7 \|\overline{f}_\alpha\|_{L^{\overline{q}_\alpha}(\mathbb{R})} \leqslant C_7 \prod_{\beta=\alpha+1}^{i} \|v_\beta\|_{L^{q_\beta}(\mathbb{R})}.
\end{aligned}$$

Combining these estimates we get

$$\int_{\mathbb{R}} (\underline{f}_\alpha)^p \big(\mathcal{M}|v_\alpha'|\big)^{\sigma p} |v_\alpha|^{(1-\sigma)p} \mathcal{M}(\overline{f}_\alpha)^p \leqslant C_8 \|v_\alpha\|^{(1-\sigma)p}_{L^{q_\alpha}(\mathbb{R})} \|v_\alpha'\|^{\sigma p}_{L^{r_\alpha}(\mathbb{R})} \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{i} \|v_\beta\|^p_{L^{q_\beta}(\mathbb{R})}.$$

Similarly,

$$\int_{\mathbb{R}} (\underline{f}_\alpha)^p \big(\mathcal{M}|v_\alpha'|\big)^{\sigma p} \big(\mathcal{M}(\overline{f}_\alpha)^p\big)^\sigma \big(\mathcal{M}|v_\alpha \overline{f}_\alpha|^p\big)^{1-\sigma} \leqslant C_9 \|v_\alpha\|^{(1-\sigma)p}_{L^{q_\alpha}(\mathbb{R})} \|v_\alpha'\|^{\sigma p}_{L^{r_\alpha}(\mathbb{R})} \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{i} \|v_\beta\|^p_{L^{q_\beta}(\mathbb{R})}.$$

Therefore, by (2.3),

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (\underline{f}_\alpha(x))^p \frac{|v_\alpha(x)-v_\alpha(y)|^p}{|x-y|^{1+\sigma p}} (\overline{f}_\alpha(y))^p \,\mathrm{d}x\,\mathrm{d}y \leqslant C_{10} \|v_\alpha\|_{L^{q_\alpha}(\mathbb{R})}^{(1-\sigma)p} \|v_\alpha'\|_{L^{r_\alpha}(\mathbb{R})}^{\sigma p} \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{i} \|v_\beta\|_{L^{q_\beta}(\mathbb{R})}^p.$$

In view of the triangle inequality (2.2), we have the conclusion in dimension $m = 1$.

When $m > 1$, we reduce the problem to the one dimensional case using the estimate [1, Lemma 7.44]

$$\Big[\prod_{\alpha=1}^{i} v_\alpha\Big]_{W^{\sigma,p}(\mathbb{R}^m)} \leqslant C_1 \sum_{j=1}^{m} \bigg(\int_{\mathbb{R}^m}\int_{\mathbb{R}} \frac{\big|\prod_{\alpha=1}^{i} v_\alpha(x+te_j) - \prod_{\alpha=1}^{i} v_\alpha(x)\big|^p}{t^{1+\sigma p}} \,\mathrm{d}t\,\mathrm{d}x\bigg)^{1/p},$$

where $(e_1,\dots,e_m)$ is the canonical basis of $\mathbb{R}^m$.

We only estimate the first term of the sum in the right hand side. We write any $x \in \mathbb{R}^m$ as $x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{m-1}$. For $x' \in \mathbb{R}^{m-1}$, we apply the case $m = 1$ to the function $x_1 \in \mathbb{R} \mapsto v_\alpha(x_1, x')$. This gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\big|\prod_{\alpha=1}^{i} v_\alpha(x_1+t,x') - \prod_{\alpha=1}^{i} v_\alpha(x_1,x')\big|^p}{t^{1+\sigma p}} \,\mathrm{d}t\,\mathrm{d}x_1 \leqslant C_1 \sum_{\alpha=1}^{i} \bigg(\|v_\alpha(\cdot,x')\|_{L^{q_\alpha}(\mathbb{R})}^{(1-\sigma)p} \|\partial_1 v_\alpha(\cdot,x')\|_{L^{r_\alpha}(\mathbb{R})}^{\sigma p} \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{i} \|v_\beta(\cdot,x')\|_{L^{q_\beta}(\mathbb{R})}^p \bigg).$$

Integrating both sides with respect to x' over $\mathbb{R}^{m-1}$, we obtain by Fubini's theorem,

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}} \frac{\big|\prod_{\alpha=1}^{i} v_\alpha(x+te_1) - \prod_{\alpha=1}^{i} v_\alpha(x)\big|^p}{t^{1+\sigma p}} \,\mathrm{d}t\,\mathrm{d}x \leqslant C_2 \sum_{\alpha=1}^{i} \int_{\mathbb{R}^{m-1}} \|v_\alpha(\cdot,x')\|_{L^{q_\alpha}(\mathbb{R})}^{(1-\sigma)p} \|\partial_1 v_\alpha(\cdot,x')\|_{L^{r_\alpha}(\mathbb{R})}^{\sigma p} \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{i} \|v_\beta(\cdot,x')\|_{L^{q_\beta}(\mathbb{R})}^p \,\mathrm{d}x'.$$

Using Hölder's inequality with exponents $\frac{q_\alpha}{(1-\sigma)p}$, $\frac{r_\alpha}{\sigma p}$ and $\frac{q_\beta}{p}$ for $\beta \neq \alpha$, we get the desired result. □

There is an alternative proof of Lemma 2.9 using the Triebel-Lizorkin spaces $F^\sigma_{t,p}(\mathbb{R}^m)$, based on the imbedding of the product of functions in such spaces. By the Gagliardo-Nirenberg interpolation inequality [7, Lemma 3.1; 26],

$$\|v_\alpha\|_{F^\sigma_{s_\alpha,p}} \leqslant C \|v_\alpha\|_{L^{q_\alpha}}^{1-\sigma} \|v_\alpha\|_{W^{1,r_\alpha}}^{\sigma},$$

with

$$\frac{1}{s_\alpha} = \frac{1-\sigma}{q_\alpha} + \frac{\sigma}{r_\alpha}.$$

Since for every $\alpha \in \{1, \dots, i\}$,

$$\frac{1}{s_\alpha} + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{i} \frac{1}{q_\beta} = \frac{1-\sigma}{q_\alpha} + \frac{\sigma}{r_\alpha} + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{i} \frac{1}{q_\beta} = \frac{1}{p},$$

then it follows that [30, p. 345]

$$\prod_{\alpha=1}^{i} v_\alpha \in F^{\sigma}_{p,p}(\mathbb{R}^m) = W^{\sigma,p}(\mathbb{R}^m).$$

Proof of Proposition 2.6. — By continuous extension of functions in Sobolev spaces to the whole space, it suffices to establish the estimate on $\mathbb{R}^m$ instead of Q^m. By the chain rule and by the triangle inequality, we have for $x, y \in \mathbb{R}^m$,

$$\begin{aligned}
&|D^k(\eta \circ u)(x) - D^k(\eta \circ u)(y)| \\
&\qquad \leqslant C_1 \sum_{i=1}^{k} \sum_{\substack{1 \leqslant t_1 \leqslant \dots \leqslant t_i \leqslant k, \\ t_1 + \dots + t_i = k}} \big| D^i\eta(u(x))[D^{t_1}u(x), \dots, D^{t_i}u(x)] \\
&\qquad\qquad - D^i\eta(u(y))[D^{t_1}u(y), \dots, D^{t_i}u(y)] \big|.
\end{aligned}$$

Given $1 \leqslant t_1 \leqslant \dots \leqslant t_i \leqslant k$ such that $t_1 + \dots + t_i = k$, by the triangle inequality we have

$$\begin{aligned}
&\big| D^i\eta(u(x))[D^{t_1}u(x), \dots, D^{t_i}u(x)] - D^i\eta(u(y))[D^{t_1}u(y), \dots, D^{t_i}u(y)] \big| \\
&\qquad \leqslant F_{t_1,\dots,t_i}(x,y) + G_{t_1,\dots,t_i}(x,y)
\end{aligned}$$

with

$$F_{t_1,\dots,t_i}(x,y) = \big| D^i\eta(u(x)) - D^i\eta(u(y)) \big| |D^{t_1}u(x)| \cdots |D^{t_i}u(x)|$$

and

$$G_{t_1,\dots,t_i}(x,y) = |D^i\eta(u(y))| \big| D^{t_1}u(x) \otimes \dots \otimes D^{t_i}u(x) - D^{t_1}u(y) \otimes \dots \otimes D^{t_i}u(y) \big|.$$

The notation $\otimes$ is used in the following sense: if $f_\alpha = (f^1_\alpha, \dots, f^\nu_\alpha) : (\mathbb{R}^m)^{t_\alpha} \to \mathbb{R}^\nu$ is a t_α-linear transformation for $\alpha \in \{1, \dots, i\}$, then $f_1 \otimes \dots \otimes f_i$ is the $(\sum_{\alpha=1}^{i} t_\alpha)$-linear transformation

$$(X_1, \dots, X_i) \in \prod_{\alpha=1}^{i} (\mathbb{R}^m)^{t_\alpha} \mapsto \big(f_1^{j_1}(X_1) \dots f_i^{j_i}(X_i) \big)_{1 \leqslant j_1, \dots, j_i \leqslant \nu} \in \mathbb{R}^{\nu^i}.$$

Thus,

$$\begin{aligned}
&D^{s,p}(\eta \circ u)(x) \\
&\leqslant C_1 \sum_{i=1}^{k} \sum_{\substack{1 \leqslant t_1 \leqslant \dots \leqslant t_i \leqslant k \\ t_1 + \dots + t_i = k}} \bigg(\int_{\mathbb{R}^m} \frac{F_{t_1,\dots,t_i}(x,y)^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \bigg)^{1/p} + \bigg(\int_{\mathbb{R}^m} \frac{G_{t_1,\dots,t_i}(x,y)^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \bigg)^{1/p}.
\end{aligned}$$

We have

$$\int_{\mathbb{R}^m} \frac{F_{t_1,\dots,t_i}(x,y)^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y = \big(D^{\sigma,p}(D^i\eta \circ u)(x) \big)^p |D^{t_1}u(x)|^p \cdots |D^{t_i}u(x)|^p.$$

By Lemma 2.7,

$$\big(D^{\sigma,p}(D^i\eta\circ u)(x)\big)^p \leqslant C_2\big(\mathcal{M}|D(D^i\eta\circ u)|^p(x)\big)^\sigma\big(\mathcal{M}|D^i\eta\circ u|^p(x)\big)^{1-\sigma}.$$

Moreover, for every $i\in\{1,\dots,k\}$,

$$|D(D^i\eta\circ u)|\leqslant[\eta]_{C^{k+1}(\mathbb{R}^\nu)}|Du| \quad\text{and}\quad |D^i\eta\circ u|\leqslant[\eta]_{C^k(\mathbb{R}^\nu)}.$$

Hence,

$$\bigg(\int\limits_{\mathbb{R}^m}\frac{F_{t_1,\dots,t_i}(x,y)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\bigg)^{1/p} \\ \leqslant C_2[\eta]^\sigma_{C^{k+1}(\mathbb{R}^\nu)}[\eta]^{1-\sigma}_{C^k(\mathbb{R}^\nu)}\big(\mathcal{M}|Du|^p(x)\big)^{\frac{\sigma}{p}}|D^{t_1}u(x)|\cdots|D^{t_i}u(x)|.$$

Since $Du\in L^{sp}(\mathbb{R}^m)$ and $s>1$, by the Maximal Theorem we have

$$\mathcal{M}|Du|^p\in L^s(\mathbb{R}^m).$$

By Hölder's inequality it follows that

$$\big(\mathcal{M}|Du|^p\big)^{\frac{\sigma}{p}}|D^{t_1}u|\cdots|D^{t_i}u|\in L^p(\mathbb{R}^m).$$

Next,

$$\bigg(\int\limits_{\mathbb{R}^m}\frac{G_{t_1,\dots,t_i}(x,y)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\bigg)^{1/p}\leqslant[\eta]_{C^k(\mathbb{R}^\nu)}D^{\sigma,p}(D^{t_1}u\otimes\cdots\otimes D^{t_i}u)(x).$$

If $t_i=k$, then $i=1$ and this estimate becomes

$$\bigg(\int\limits_{\mathbb{R}^m}\frac{G_k(x,y)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\bigg)^{1/p}\leqslant[\eta]_{C^k(\mathbb{R}^\nu)}D^{s,p}u(x).$$

By assumption on u, the right-hand side belongs to $L^p(\mathbb{R}^m)$.

If $t_i<k$, then each component of $D^{t_1}u\otimes\cdots\otimes D^{t_i}u$ is the product of i functions $v_{t_1},\dots,v_{t_i}$ with $v_{t_\alpha}\in L^{\frac{sp}{t_\alpha}}(\mathbb{R}^m)$ and $Dv_{t_\alpha}\in L^{\frac{sp}{t_\alpha+1}}(\mathbb{R}^m)$. Then, by Lemma 2.9, we get

$$D^{\sigma,p}(D^{t_1}u\otimes\cdots\otimes D^{t_i}u)\in L^p(\mathbb{R}^m).$$

The proof of the proposition is complete. □

3. Strong density

We rely on an averaging argument due to Federer and Fleming [11, 19] based on the following observation:

Lemma 3.1. — *Let $u: Q^m\to\mathbb{R}^\nu$ be a measurable function. For every measurable function $f: Q^m\to\mathbb{R}$ and for every Borel measurable set $E\subset\mathbb{R}^\nu$,*

$$\int\limits_{\mathbb{R}^\nu}\int\limits_{u^{-1}(E+\xi)}|f(x)|\,\mathrm{d}x\,\mathrm{d}\xi=\mathcal{H}^\nu(E)\|f\|_{L^1(Q^m)}.$$

We shall apply this lemma with $E=\Pi^{-1}(K)$ where $K\subset N^n$ is a compact set and $\Pi: N^n+\overline{B}^\nu_\iota\to N^n$ is the nearest point projection. In this case, by the coarea formula we have

$$\mathcal{H}^\nu(E)\leqslant C\mathcal{H}^n(K).$$

Proof. — We may assume that f is a nonnegative function. For every $\xi \in \mathbb{R}^\nu$,

$$\int_{u^{-1}(E+\xi)} f(x)\,\mathrm{d}x = \int_{Q^m} f(x)\chi_E(u(x)-\xi)\,\mathrm{d}x.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^\nu}\int_{u^{-1}(E+\xi)} f(x)\,\mathrm{d}x\,\mathrm{d}\xi = \int_{Q^m} f(x)\biggl(\int_{\mathbb{R}^\nu} \chi_E(u(x)-\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}x.$$

Using the change of variable $z = u(x) - \xi$ with respect to ξ, we get

$$\begin{aligned}\int_{\mathbb{R}^\nu}\int_{u^{-1}(E+\xi)} f(x)\,\mathrm{d}x\,\mathrm{d}\xi &= \int_{Q^m} f(x)\biggl(\int_{\mathbb{R}^\nu} \chi_E(z)\,\mathrm{d}z\biggr)\,\mathrm{d}x\\ &= \int_{Q^m} f(x)\mathcal{H}^\nu(E)\,\mathrm{d}x = \mathcal{H}^\nu(E)\int_{Q^m} f(x)\,\mathrm{d}x.\end{aligned}$$

This gives the conclusion. □

Proof of Theorem 1.5. — Given $u \in W^{s,p}(Q^m; N^n)$, the restriction to Q^m of the maps $u_\gamma \in W^{s,p}(Q^m_{1+\gamma}; N^n)$ defined for $x \in Q^m_{1+\gamma}$ by $u_\gamma(x) = u(x/(1+\gamma))$ converge strongly to u in $W^{s,p}(Q^m; N^n)$ when γ tends to zero. We may thus assume that $u \in W^{s,p}(Q^m_{1+\gamma}; N^n)$.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a smooth mollifier such that $\operatorname{supp}\varphi \subset Q^m$. For every $0 < t \leqslant \gamma$, the convolution $\varphi_t * u$ is well-defined and converges to u in $W^{s,p}(Q^m; \mathbb{R}^\nu)$ as t tends to zero.

The nearest point projection Π onto N^n is well-defined and smooth on $N^n + \overline{B}^\nu_\iota$ for some $\iota > 0$. Let $\overline{\Pi} : \mathbb{R}^\nu \to \mathbb{R}^\nu$ be a smooth extension of the projection Π to $\mathbb{R}^\nu$. The image of this map $\overline{\Pi}$ need not be contained in N^n.

For every $\xi \in B^\nu_\iota$, we consider the map $P_\xi : \mathbb{R}^\nu \to \mathbb{R}^\nu$ defined for every $x \in \mathbb{R}^\nu$ by

$$P_\xi(x) = \overline{\Pi}(x - \xi).$$

There exists $0 < \delta \leqslant \iota$ such that for every $\xi \in B^\nu_\delta$, the map $P_\xi|_{N^n} : N^n \to N^n$ is a smooth diffeomorphism. Given a smooth map $\eta : \mathbb{R}^\nu \to N^n$ and $\xi \in B^\nu_\delta$, let

$$\eta_\xi = (P_\xi|_{N^n})^{-1} \circ \eta \circ P_\xi.$$

Our goal is to approximate u by a family of maps of the form

$$\eta_\xi \circ (\varphi_t * u),$$

for some $\xi \in B^\nu_\delta$ and $0 < t \leqslant \gamma$. By the triangle inequality,

$$\begin{aligned}&\|\eta_\xi \circ (\varphi_t * u) - u\|_{W^{s,p}(Q^m)}\\ &\qquad\leqslant \|\eta_\xi \circ (\varphi_t * u) - \eta_\xi \circ u\|_{W^{s,p}(Q^m)} + \|\eta_\xi \circ u - u\|_{W^{s,p}(Q^m)}. \quad (3.1)\end{aligned}$$

Since η_ξ is a smooth map and $\varphi_t * u$ converges to u in $W^{s,p}(Q^m; \mathbb{R}^\nu)$, by the property of continuity of maps in $W^{s,p} \cap L^\infty$ under composition [7, Theorem 1.1'; 22, Theorem], for every $\xi \in B^\nu_\delta$,

$$\lim_{t\to 0} \|\eta_\xi \circ (\varphi_t * u) - \eta_\xi \circ u\|_{W^{s,p}(Q^m)} = 0. \quad (3.2)$$

In view of (3.1), we need to control the quantity

$$\|\eta_\xi \circ u - u\|_{W^{s,p}(Q^m)}$$

for some suitable $\xi \in B^\nu_\delta$. We start with the following:

CLAIM. — *There exists a nonnegative function $F \in L^1(Q^m)$ depending on s, p, m and u such that for every $\xi \in B^\nu_\delta$,*

$$\|\eta_\xi \circ u - u\|^p_{W^{s,p}(Q^m)} \leqslant [\eta_\xi]^{\sigma p}_{C^{k+1}(\mathbb{R}^\nu)} [\eta_\xi]^{(1-\sigma)p}_{C^k(\mathbb{R}^\nu)} \int\limits_{\{\eta_\xi \circ u \neq u\}} F.$$

Proof of the claim. — By definition of the $W^{s,p}$ norm,

$$\|\eta_\xi \circ u - u\|_{W^{s,p}(Q^m)} = \sum_{j=0}^{k} \|D^j(\eta_\xi \circ u) - D^j u\|_{L^p(Q^m)} + \|D^{s,p}(\eta_\xi \circ u - u)\|_{L^p(Q^m)};$$

when s is an integer, we disregard the last term.

Since the map u is bounded,

$$\|\eta_\xi \circ u - u\|_{L^p(Q^m)} = \|\eta_\xi \circ u - u\|_{L^p(\{\eta_\xi \circ u \neq u\})} \leqslant C_1 \mathcal{H}^m(\{\eta_\xi \circ u \neq u\}). \tag{3.3}$$

Moreover, for every $j \in \{1, \ldots, k\}$,

$$\begin{aligned}\|D^j(\eta_\xi \circ u) - D^j u\|_{L^p(Q^m)} &= \|D^j(\eta_\xi \circ u) - D^j u\|_{L^p(\{\eta_\xi \circ u \neq u\})} \\ &\leqslant \|D^j(\eta_\xi \circ u)\|_{L^p(\{\eta_\xi \circ u \neq u\})} + \|D^j u\|_{L^p(\{\eta_\xi \circ u \neq u\})}.\end{aligned}$$

Since $u \in W^{s,p}(Q^m; \mathbb{R}^\nu) \cap L^\infty(Q^m; \mathbb{R}^\nu)$, by the Gagliardo-Nirenberg interpolation inequality [7, Corollary 3.2], $u \in W^{1,sp}(Q^m; \mathbb{R}^\nu)$. By Proposition 2.5, there exists a function $G_j \in L^p(Q^m)$ independent of η_ξ such that

$$\|D^j(\eta_\xi \circ u) - D^j u\|_{L^p(Q^m)} \leqslant [\eta_\xi]_{C^j(\mathbb{R}^\nu)} \|G_j\|_{L^p(\{\eta_\xi \circ u \neq u\})} + \|D^j u\|_{L^p(\{\eta_\xi \circ u \neq u\})}. \tag{3.4}$$

If s is non-integer, then

$$\begin{aligned}\|D^{s,p}(\eta_\xi \circ u - u)\|_{L^p(Q^m)} &\leqslant 2^{1/p} \|D^{s,p}(\eta_\xi \circ u - u)\|_{L^p(\{\eta_\xi \circ u \neq u\})} \\ &\leqslant 2^{1/p} \big(\|D^{s,p}(\eta_\xi \circ u)\|_{L^p(\{\eta_\xi \circ u \neq u\})} + \|D^{s,p} u\|_{L^p(\{\eta_\xi \circ u \neq u\})}\big).\end{aligned}$$

By Proposition 2.6, there exists $H \in L^p(Q^m)$ independent of η_ξ such that

$$\|D^{s,p}(\eta_\xi \circ u - u)\|_{L^p(Q^m)} \leqslant 2^{1/p} \big([\eta_\xi]^{\sigma}_{C^{k+1}(\mathbb{R}^\nu)} [\eta_\xi]^{1-\sigma}_{C^k(\mathbb{R}^\nu)} \|H\|_{L^p(\{\eta_\xi \circ u \neq u\})} + \|D^{s,p} u\|_{L^p(\{\eta_\xi \circ u \neq u\})}\big). \tag{3.5}$$

Combining estimates (3.3), (3.4) and (3.5), we conclude that

$$\|\eta_\xi \circ u - u\|^p_{W^{s,p}(Q^m)} \leqslant [\eta_\xi]^{\sigma p}_{C^{k+1}(\mathbb{R}^\nu)} [\eta_\xi]^{(1-\sigma)p}_{C^k(\mathbb{R}^\nu)} \int\limits_{\{\eta_\xi \circ u \neq u\}} F,$$

with

$$F = C_2 \bigg(1 + \sum_{j=1}^{k} (G_j^p + |D^j u|^p) + H^p + (D^{s,p} u)^p\bigg).$$

This proves the claim. □

Let $K \subset N^n$ be a compact set such that for every $x \in N^n \setminus K$,

$$\eta(x) = x. \tag{3.6}$$

If $x \in N^n$ is such that $\eta_\xi(u(x)) \neq u(x)$ for some $\xi \in B^\nu_\delta$, then

$$P_\xi(u(x)) = \Pi(u(x) - \xi) \in K,$$

whence $x \in u^{-1}(\Pi^{-1}(K) + \xi)$. In other words, for every $\xi \in B^\nu_\delta$,

$$\{\eta_\xi \circ u \neq u\} \subset u^{-1}(\Pi^{-1}(K) + \xi). \tag{3.7}$$

Thus, from the previous claim,

$$\begin{aligned}\|\eta_\xi \circ u - u\|^p_{W^{s,p}(Q^m)} &\leqslant [\eta_\xi]^{\sigma p}_{C^{k+1}(\mathbb{R}^\nu)} [\eta_\xi]^{(1-\sigma)p}_{C^k(\mathbb{R}^\nu)} \int\limits_{u^{-1}(\Pi^{-1}(K)+\xi)} F\\ &\leqslant C_3 [\eta]^{\sigma p}_{C^{k+1}(\mathbb{R}^\nu)} [\eta]^{(1-\sigma)p}_{C^k(\mathbb{R}^\nu)} \int\limits_{u^{-1}(\Pi^{-1}(K)+\xi)} F,\end{aligned}$$

for some constant $C_3 > 0$ independent of ξ. By Lemma 3.1, we get

$$\int\limits_{B^\nu_\delta} \|\eta_\xi \circ u - u\|^p_{W^{s,p}(Q^m)} \,\mathrm{d}\xi \leqslant C_3 [\eta]^{\sigma p}_{C^{k+1}(\mathbb{R}^\nu)} [\eta]^{(1-\sigma)p}_{C^k(\mathbb{R}^\nu)} \mathcal{H}^\nu(\Pi^{-1}(K)) \|F\|_{L^1(Q^m)}.$$

Since

$$\mathcal{H}^\nu(\Pi^{-1}(K)) \leqslant C_4 \mathcal{H}^n(K),$$

we conclude that

$$\int\limits_{B^\nu_\delta} \|\eta_\xi \circ u - u\|^p_{W^{s,p}(Q^m)} \,\mathrm{d}\xi \leqslant C_5 [\eta]^{\sigma p}_{C^{k+1}(\mathbb{R}^\nu)} [\eta]^{(1-\sigma)p}_{C^k(\mathbb{R}^\nu)} \mathcal{H}^n(K) \|F\|_{L^1(Q^m)}.$$

Let $0 < \epsilon \leqslant 1$. Since N^n is $\lfloor sp \rfloor$ simply connected, by Proposition 2.1 there exists a smooth map $\eta = \eta_\epsilon$ satisfying (3.6) for some compact set $K = K_\epsilon \subset N^n$ such that

$$\mathcal{H}^n(K_\epsilon) \leqslant C_6 \epsilon^{\lfloor sp \rfloor + 1} \tag{3.8}$$

and for every $j \in \{1, \dots, k+1\}$,

$$\|D^j \eta_\epsilon\|_{L^\infty(\mathbb{R}^\nu)} \leqslant \frac{C_7}{\epsilon^j}.$$

Thus,

$$\int\limits_{B^\nu_\delta} \|\eta_{\epsilon,\xi} \circ u - u\|^p_{W^{s,p}(Q^m)} \,\mathrm{d}\xi \leqslant C_8 \epsilon^{\lfloor sp \rfloor + 1 - sp}. \tag{3.9}$$

Since $sp < \lfloor sp \rfloor + 1$, there exists $\xi = \xi_\epsilon \in B^\nu_\delta$ such that

$$\lim_{\epsilon \to 0} \|\eta_{\epsilon,\xi_\epsilon} \circ u - u\|_{W^{s,p}(Q^m)} = 0.$$

By (3.2),

$$\lim_{t \to 0} \|\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_t * u) - \eta_{\epsilon,\xi_\epsilon} \circ u\|_{W^{s,p}(Q^m)} = 0,$$

whence for every $0 < \epsilon \leqslant 1$ there exists $0 < t_\epsilon \leqslant \gamma$ such that

$$\|\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_{t_\epsilon} * u) - \eta_{\epsilon,\xi_\epsilon} \circ u\|_{W^{s,p}(Q^m)} \leqslant \epsilon.$$

In particular,

$$\lim_{\epsilon\to 0} \|\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_{t_\epsilon} * u) - \eta_{\epsilon,\xi_\epsilon} \circ u\|_{W^{s,p}(Q^m)} = 0.$$

Thus, by the triangle inequality (3.1),

$$\lim_{\epsilon\to 0} \|\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_{t_\epsilon} * u) - u\|_{W^{s,p}(Q^m)} = 0.$$

This completes the proof of Theorem 1.5. □

4. Weak sequential density

We prove a more precise version of Theorem 1.7:

Theorem 4.1. — *Let $s \geqslant 1$. If $sp < m$ is such that $sp \in \mathbb{N}$ and if N^n is $sp-1$ simply connected, then for every $u \in W^{s,p}(Q^m; N^n)$ there exists a sequence $(u_i)_{i\in\mathbb{N}}$ in $C^\infty(\overline{Q}^m; N^n)$ such that*

(i) $(u_i)_{i\in\mathbb{N}}$ converges in measure to u,
(ii) for every $j \in \{1, \dots, k\}$, $(D^j u_i)_{i\in\mathbb{N}}$ converges in measure to $D^j u$,
(iii) for every $i \in \mathbb{N}$,

$$\|u_i\|_{W^{s,p}(Q^m)} \leqslant C,$$

for some constant $C > 0$ depending on s, p, m, $\|u\|_{W^{s,p}(Q^m)}$ and N^n.

Proof. — We explain what should be changed in the proof of Theorem 1.5. Since N^n is now merely $sp-1$ simply connected, the map η may be chosen so that $\eta(x) = x$ on $N^n \setminus K$, where the compact set K satisfies

$$\mathcal{H}^n(K) \leqslant C_1 \epsilon^{sp}. \tag{4.1}$$

By inclusion (3.7), by Lemma 3.1 and by property (4.1),

$$\begin{aligned}\int_{B^\nu_\delta} \mathcal{H}^m\big(\{\eta_\xi \circ u \neq u\}\big)\, \mathrm{d}\xi &\leqslant \int_{B^\nu_\delta} \mathcal{H}^m\big(u^{-1}(\Pi^{-1}(K) + \xi)\big)\, \mathrm{d}\xi \\ &\leqslant C_2 \mathcal{H}^n(K) \mathcal{H}^m(Q^m) \leqslant C_3 \epsilon^{sp}.\end{aligned}$$

Replacing (3.8) by (4.1), estimate (3.9) becomes

$$\int_{B^\nu_\delta} \|\eta_\xi \circ u - u\|^p_{W^{s,p}(Q^m)}\, \mathrm{d}\xi \leqslant C_4.$$

Thus, for every $0 < \epsilon \leqslant 1$ there exists a smooth map $\eta = \eta_\epsilon$ and $\xi = \xi_\epsilon \in B^\nu_\delta$ such that

$$\mathcal{H}^m\big(\{\eta_{\epsilon,\xi_\epsilon} \circ u \neq u\}\big) \leqslant C_5 \epsilon^{sp}$$

and

$$\|\eta_{\epsilon,\xi_\epsilon} \circ u - u\|^p_{W^{s,p}(Q^m)} \leqslant C_6.$$

As in the proof of Theorem 1.5, for every $0 < \epsilon \leqslant 1$ we find $0 < t_\epsilon \leqslant \gamma$ such that

$$\lim_{\epsilon\to 0} \|\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_{t_\epsilon} * u) - \eta_{\epsilon,\xi_\epsilon} \circ u\|_{W^{s,p}(Q^m)} = 0. \tag{4.2}$$

Thus, by the triangle inequality,

$$\|\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_{t_\epsilon} * u)\|_{W^{s,p}(Q^m)} \leqslant C_7.$$

Note that $\eta_{\epsilon,\xi_\epsilon} \circ u$ and u as well as their derivatives up to order k coincide almost everywhere on $\{\eta_{\xi_\epsilon,\epsilon} \circ u = u\}$. Combining

$$\lim_{\epsilon \to 0} \mathcal{H}^m\bigl(\{\eta_{\epsilon,\xi_\epsilon} \circ u \neq u\}\bigr) = 0$$

and (4.2), we deduce the convergence in measure of $\eta_{\epsilon,\xi_\epsilon} \circ (\varphi_{t_\epsilon} * u)$ and its derivatives as ϵ tends to zero. This completes the proof of Theorem 1.7. □

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Pierre BOUSQUET
Aix-Marseille Université, Laboratoire d'analyse, topologie, probabilités UMR7353, CMI 39, Rue Frédéric Joliot Curie, 13453 Marseille Cedex 13, France
Pierre.Bousquet@univ-amu.fr (Corresponding author)

Augusto C. PONCE & Jean VAN SCHAFTINGEN
Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du cyclotron 2, bte L7.01.02, 1348 Louvain-la-Neuve, Belgium
Augusto.Ponce@uclouvain.be
Jean.VanSchaftingen@uclouvain.be